Appl. Set-Valued Anal. Optim. 1 (2019), No. 1, pp. 39-52
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.1.2019.1.04

STRONG CONVERGENCE THEOREMS FOR δ-INVERSE STRONGLY ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. By using an iteration method, we study some variational inequalities including an δ-inverse strongly accretive operator in Banach spaces. We present a strong convergence theorem for the production sequence method.

Keywords. δ-inverse strongly accretive operator, Monotone mapping, Strong convergence, Variational inequality problems.

2010 Mathematics Subject Classification. 47H05, 47J20.

1. INTRODUCTION

Let D be a nonempty subset of a real Hilbert space H and P_D the metric projection of H onto D. A mapping $B : D \to H$ is said to be monotone if

$$\langle Bu - Bv, u - v \rangle \geq 0, \quad \forall u, v \in D.$$

A mapping $B : D \to H$ is said to be L-Lipschitz if there exits $L \geq 0$ such that

$$\|Bu - Bv\| \leq L\|u - v\|, \quad \forall u, v \in D.$$

If $L = 1$ then B is called a nonexpansive mapping. We denote by $F(B)$ the set of fixed points of B. B is said to be δ-inverse strongly monotone, if there exists a positive real number δ such that

$$\langle Bu - Bv, u - v \rangle \geq \delta\|Bu - Bv\|^2, \quad \forall u, v \in D.$$

It is obvious that any δ-inverse strongly monotone B is Lipschitz and $\|Bu - Bv\| \leq \frac{1}{\delta}\|u - v\|$. Let D be a nonempty closed convex subset of H and let $B : D \to H$ be a nonlinear mapping. The variational inequality problem is to find a point $\tilde{x} \in D$ such that

$$\langle x - \tilde{x}, B\tilde{x} \rangle \geq 0, \quad \forall x \in D. \tag{1.1}$$

We denote the solution set of variational inequality problem by $VI(D,B)$. We note that, if $D = H$ then $VI(H,B) = B^{-1}(0)$, where $B^{-1}(0) = \{x \in H : Bx = 0\}$. Let I be identity mapping and T be a nonexpansive mapping of D into itself. If $B = I - T$, then B is $\frac{1}{2}$-inverse strongly monotone and $F(T) = VI(D,B)$; see [10]. A mapping $T : D \to D$ is said to be strictly pseudocontractive if there exists $0 \leq k < 1$ such that

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 + k\|(I - T)u - (I - T)v\|^2, \quad \forall u, v \in D.$$

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Received February 8, 2019; Accepted March 13, 2019.

Let $T : D \to D$ be a strictly pseudocontractive. If $B = I - T$, then B is $\frac{1-k}{2}$-inverse strongly monoton and $F(T) = VI(D,B)$; see [20]. Since Stampacchia introduced variational inequality 1.1 in 1964, many authors extensively studied it from viewpoint of existence, stability, solution methods etc; see, e.g., [1, 2, 5, 6, 7, 8, 9, 11, 12, 13, 16, 17, 19, 20] and the references therein. In 2004, Iiduka, Takahashi and Toyoda [10] introduced the following iterative algorithm

$$x_1 \in D, \quad x_{n+1} = P_D(\lambda_n x_n + (1-\lambda_n)P_D(x_n - \alpha_n B x_n)), \quad \forall n \geq 1, \tag{1.2}$$

P_D is a the metric projection from H onto D and $B : D \to H$ is δ-inverse strongly monotone, $\lambda_n \in [a,b]$ for some $a,b \in (-1,1)$ and $\alpha_n \in [c,d]$ for some $c,d \in (0, 2(1+a)\delta)$. They proved that the sequence $\{x_n\}$ generated by (1.2) converges weakly to some element of $VI(D,B)$.

In 2005, Iiduka and Takahashi [11] introduced the following iterative algorithm for finding solutions to a variational inequality involving a δ-inverse strongly monotone mapping and obtained the strongly convergence of the following iterative process

$$x_1 \in D, \quad x_{n+1} = \lambda_n x + (1-\lambda_n)P_D(x_n - \alpha_n B x_n), \quad \forall n \geq 1, \tag{1.3}$$

where $B : D \to H$ is a δ-inverse strongly monotone mapping, $\lambda_n \in [0,1)$ with $\lim_{n\to\infty} \lambda_n = 0, \sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ and $\alpha_n \in [c,d]$ for some $c,d \in (0,2\delta)$ with $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. They proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{VI(D,B)}x$. Recently, Alghamdi, Shahzad and Zegeye [1] suggested the following iterative algorithm for finding a common solution of two variational inequality problems involving Lipschitz monotone mappings B_1, B_2

$$\begin{cases} x_1 \in D, \\ z_n = P_D(x_n - \alpha_n B_2 x_n), \\ y_n = P_D(x_n - \alpha_n B_1 x_n), \\ x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n)(a_n x_n + b_n P_D(x_n - \alpha_n B_1 y_n) + c_n P_D(x_n - \alpha_n B_2 z_n)), \quad \forall n \geq 1, \end{cases} \tag{1.4}$$

where $f : D \to D$ is a contraction mapping, $\alpha_n \subset [a,b] \subset (0, \frac{1}{L})$ for $L := \max\{L_1, L_2\}$ (L_1, L_2 Lipschitz constants). They proved that the sequence $\{x_n\}$ generated by (1.4) converge to a point $q \in VI(D,B_1) \cap VI(D,B_2)$.

Next, we turn our attention to the following generalized variational inequality problem in the framework of Banach spaces. Let X be a Banach space and let X^* the dual space of X. Let D be a nonempty closed convex subset of X. The generalized variational inequality problem is to find a point $\tilde{x} \in D, j(x-\tilde{x}) \in J(x-\tilde{x})$ such that

$$\langle j(x-\tilde{x}), B\tilde{x} \rangle \geq 0, \quad \forall x \in D. \tag{1.5}$$

We use $S(D,B)$ to denote the solution set of this problem, that is

$$S(D,B) = \{\tilde{x} \in D : \langle j(x-\tilde{x}), B\tilde{x} \rangle \geq 0, x \in D\}.$$

In 2006, Aoyama, Iiduka and Takahashi [2] investigated the following iterative algorithm

$$x_1 = x \in D, \quad x_{n+1} = \lambda_n x_n + (1-\lambda_n)Q_D(x_n - \alpha_n B x_n), \quad \forall n \geq 1, \tag{1.6}$$

where D is a nonempty closed convex subset of a uniformly covex and 2-uniformly smooth Banach space X and $B : D \to X$ is an δ-inverse strongly accretive operator, Q_D is a sunny nonexpansive retraction from X onto D, $\lambda_n \in [a,b]$ for some $a,b \in (0,1)$ and $\alpha_n \in [c, \frac{\delta}{k^2}]$ for some $c > 0$. They showed that if $S(D,B) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.6) converges weakly to some element of $S(D,B)$.

In 2011, Yao *et al.* [24] obtained a strong theorem in a uniformly convex and 2-uniformly smooth Banach space. They suggested the following iterative algorithm

$$x_1 \in D, \quad x_{n+1} = \lambda_n x_n + (1-\lambda_n) Q_D[\gamma_n v + (1-\gamma_n) Q_D(x_n - \alpha_n B x_n)], \quad \forall n \geq 0, \tag{1.7}$$

where D is a nonempty closed convex subset of uniformly covex and 2-uniformly smooth Banach space X and $B : D \to X$ is an δ-inverse strongly accretive operator with $S(D,B) \neq \emptyset$, Q_D is a sunny nonexpansive retraction from X onto D, $\lambda_n \in [a,b]$ for some $a,b \in (0,1)$ with $\lim_{n\to\infty} \lambda_n = 0, \sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ and $\alpha_n \in [c, \frac{\delta}{k^2}]$ for some $c > 0$ with $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $v \in X$ is fixed. They proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to $\tilde{Q}_D v$, where $\tilde{Q}_D$ is a sunny nonexpansive retraction of X onto $S(D,B)$.

Assume that B is an δ-inverse strongly accretive operator from D a nonempty, closed and convex subset of a uniformly convex and 2-uniformly smooth Banach space X, into X with $S(D,B) \neq \emptyset$ and Q_D is a sunny nonexpansive retraction from X onto D. In this paper, for solving the generalized variational inequality problem involving an δ-inverse strongly accretive operator, we introduce the following iterative algorithm in Banach spaces

$$\begin{cases} x_1 \in D, \\ z_n = Q_D(x_n - \alpha_n B x_n), \\ \tilde{x}_n = \lambda_n x_n + (1-\lambda_n) Q_D[\gamma_n z_n + (1-\gamma_n) Q_D(z_n - \alpha_n B z_n)], \\ x_{n+1} = \eta_n v + (1-\eta_n)(a_n x_n + b_n z_n + c_n \tilde{x}_n), \ n \geq 0, \end{cases}$$

where v is a fixed element in D, $\{\eta_n\}, \{\lambda_n\}, \{\gamma_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} \eta_n = 0$, $\sum_{n=0}^{\infty} \eta_n = \infty$ and $\{\alpha_n\}$ is a sequence in $[w, \frac{\delta}{k^2}]$ for some $0 < w \leq \frac{\delta}{k^2}$, $\{a_n\}, \{b_n\}, \{c_n\} \in [d,e] \subset (0,1)$ such that $a_n + b_n + c_n = 1$. We will prove that $\{x_n\}$ converges convergence strongly to $\tilde{Q}(v)$, where $\tilde{Q}$ is a sunny nonexpansive retracttion of X onto $S(D,B)$ in a uniformly convex and 2-uniformly smooth Banach space.

2. Preliminaries

Let D be a nonempty closed convex subset of Banach space X and let X^* be the dual space of X. We denote the pairing between X and X^* by $\langle ., . \rangle$. For $q > 1$, the generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q = \{ j \in X^* : \langle u, j \rangle = \|u\|^q, \ \|j\| = \|u\|^{q-1} \}, \quad \forall u \in X.$$

Recall that $J = J_2$ is called the normalized duality mapping as well as $J_q(u) = \|u\|^{q-2} J(u)$ for all $u \in X$. Let

$$S = \{ u \in X : \|u\| = 1 \}.$$

A Banach space X is said to be uniformly convex if for any $\varepsilon \in (0,2]$ and $u, v \in S$ and $\|u - v\| \geq \varepsilon$ imply there exists $\delta > 0$ such that $\|\frac{u+v}{2}\| \leq 1 - \delta$. It is clear that every uniformly convex Banach space is reflexive. A function $\rho : [0,\infty) \to [0,\infty)$ is said to be the modulus of smoothness of X if

$$\rho(t) = \sup\{ \frac{\|u + tv\| + \|u - tv\|}{2} - 1 : \|u\| = \|v\| = 1 \}, \quad t \geq 0$$

A Banach spaceb X is said to be uniformly smooth if and only if $\lim_{t\to 0}\frac{\rho(t)}{t}=0$. For example the l_p spaces $(1<p\le 2)$ are uniformly smooth. Let q be a fixed real number with $1<q\le 2$. A Banach space X is said to be q-uniformly smooth if there exists a constant $a>0$ such that $\rho(t)\le at^q$ for all $t>0$; see [3, 22] and the references therein.

Lemma 2.1. [3] *Let $1<q\le 2$ be real number. A Banach space X is said to be q-uniformly smooth if and only if there exists a constant $k\ge 1$ such that*

$$\frac{1}{2}(\|u+v\|^q+\|u-v\|^q)\le \|u\|^q+\|kv\|^q,\ \forall u,v\in X.$$

The best constant k is called the q-uniformly smoothness constant of X.

Lemma 2.2. [22] *Let q be a given real number with $1<q\le 2$ and let X be a q-uniformly smooth Banach space. Then*

$$\|u+v\|^q\le \|u\|^q+q\langle v,J_q(u)\rangle+2\|kv\|^q,\quad \forall u,v\in X,$$

where k is the q-uniformly smoothness constant of X.

Let D be a nonempty closed convex subset of a Banach space X. An operator $B:D\to X$ is said to be accretive if

$$\langle Bu-Bv,J(u-v)\rangle\ge 0,\quad \forall u,v\in D.$$

An operator B of D into X is said to be δ-inverse strongly accretive if there exists a constant $\delta>0$ such that

$$\langle Bu-Bv,J(u-v)\rangle\ge \delta\|Bu-Bv\|^2,\quad \forall u,v\in D.$$

Let C be a subset of D and Q_D a mapping of D into C. Then Q_D is said to be sunny if

$$Q_D(Q_Du+t(u-Q_Du))=Q_Du$$

whenever $Q_Du+t(u-Q_Du)\in D$ for $u\in D$ and $t\ge 0$. A mapping Q_D of D into itself is called a retraction if $Q_D^2=Q_D$. A subset C of D is called a sunny nonexpansive retract of D if there exists a sunny nonexpansive retraction from D onto C.

Lemma 2.3. [18] *Let D be a nonempty closed convex subset of a smooth Banach space X and let $Q_D:X\to D$ be a retraction. Then the following are equivalent:*
(a) Q_D is sunny and nonexpansive;
(b) $\langle u-Q_Du,J(v-Q_Du)\rangle\le 0$, for all $u\in X$ and $v\in D$.
(c) $\|Q_Du-Q_Dv\|^2\le\langle u-v,J(Q_Du-Q_Dv)\rangle$, for all $u.v\in X$.

Lemma 2.4. [2] *Let D be a nonempty closed convex subset of a smooth Banach space X. Let Q_D be a sunny nonexpansive retraction from X onto D and let B be an accretive operator of D into X. Thus, $S(D,B)=F(Q_D(I-\alpha B))$ for all $\alpha>0$.*

Let B is an δ-inverse strongly accretive operator of D into X. If $0<\alpha\le\frac{\delta}{k^2}$ then $I-\alpha B:D\to X$ is a nonexpansive mapping; see [2] and the references therein.

Lemma 2.5. [21] *Let X be Banach space. Then*

$$\|u+v\|^2\le\|v\|^2+2\langle u,J(u+v)\rangle,\ \forall u,v\in X.$$

Lemma 2.6. [15] *Let X be a uniformly convex Banach space. Thus, there exists a continuous strictly increasing convex function $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(0) = 0$ and*

$$\|ax+bz+c\tilde{x}\|^2 \leq a\|x\|^2 + b\|z\|^2 + c\|\tilde{x}\|^2 - ab\psi(\|x-y\|)$$

for all $x, z, \tilde{x} \in B_r(0) = \{u \in X : \|u\| \leq r\}$ and $a, b, c \in [0,1]$ with $a+b+c = 1$.

Lemma 2.7. [14] *Let $\{x_n\}$ be sequences of such that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \leq x_n$ for all $k \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{t_i\} \subset \mathbb{N}$ such that $t_i \to \infty$ and*

$$x_{t_i} \leq x_{t_i+1} \text{ and } x_i \leq x_{t_i+1}$$

for all $i \in \mathbb{N}$. In fact $t_i = \max\{k \leq i : x_k \leq x_{k+1}\}$.

Lemma 2.8. [4] *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and $\tilde{T} : C \to C$ a nonexpansive mapping. If $\{w_k\}$ is a sequence of C such that $w_k \rightharpoonup w_0$ and let $\lim_{k\to\infty} \|w_k - \tilde{T}w_k\| = 0$, then w_0 is a fixed point of $\tilde{T}$.*

Lemma 2.9. [23] *Let $\{x_n\}$ be a sequence of nonnegative real numbers satisfying*

$$x_{n+1} \leq (1-\lambda_n)x_n + \gamma_n$$

where $\{\lambda_n\} \subset (0,1)$ and $\{\gamma_n\}$ is a sequence such that $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \gamma_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n \lambda_n| < \infty$. Then $\lim_{n\to\infty} x_n = 0$.

3. Main Results

Theorem 3.1. *Let D be a nonempty, closed and convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let $Q_D : X \to D$ be a sunny nonexpansive retraction and let $B : D \to X$ be an δ-inverse strongly accretive operator with $S(D,B) \neq \emptyset$. Let $\{x_n\}, \{z_n\}$ and $\{\tilde{x}_n\}$ be a sequences generated by*

$$\begin{cases} x_1 \in D, \\ z_n = Q_D(x_n - \alpha_n B x_n), \\ \tilde{x}_n = \lambda_n x_n + (1-\lambda_n) Q_D[\gamma_n z_n + (1-\gamma_n) Q_D(z_n - \alpha_n B z_n)], \\ x_{n+1} = \eta_n v + (1-\eta_n)(a_n x_n + b_n z_n + c_n \tilde{x}_n), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where v is a fixed element in D, $\{\eta_n\}, \{\lambda_n\}, \{\gamma_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} \eta_n = 0$, $\sum_{n=0}^{\infty} \eta_n = \infty$ and $\{\alpha_n\}$ is a sequence in $[w, \frac{\delta}{k^2}]$ for some $0 < w \leq \frac{\delta}{k^2}$, $\{a_n\}, \{b_n\}, \{c_n\} \in [d,e] \subset (0,1)$ such that $a_n + b_n + c_n = 1$. Then $\{x_n\}$ converges strongly to $\tilde{Q}(v)$, where $\tilde{Q}$ is a sunny nonexpansive retracttion of X onto $S(D,B)$.

Proof. Let $q \in S(D,B)$. From Lemma 2.4, we have

$$q = Q_D(q - \alpha_n Bq).$$

Put

$$y_n = Q_D(z_n - \alpha_n B z_n), \; w_n = Q_D[\gamma_n z_n + (1-\gamma_n) Q_D(z_n - \alpha_n B z_n)].$$

Since $I-\alpha_n B$ is nonexpancive, we find from (3.1) that

$$\begin{aligned}\|z_n-q\| &= \|Q_D(x_n-\alpha_n Bx_n)-Q_D(q-\alpha_n Bq)\| \\ &\le \|(x_n-\alpha_n Bx_n)-(q-\alpha_n Bq)\| \\ &\le \|x_n-q\|.\end{aligned} \tag{3.2}$$

From $\alpha_n \in [w, \frac{\delta}{k^2}]$ and Lemma 2.2, we obtain that

$$\begin{aligned}\|y_n-q\|^2 &= \|Q_D(z_n-\alpha_n Bz_n)-Q_D(q-\alpha_n Bq)\|^2 \\ &\le \|(z_n-q)-\alpha_n(Bz_n-Bq)\|^2 \\ &\le \|z_n-q\|^2-2\alpha_n\langle Bz_n-Bq, J(z_n-q)\rangle+2k^2\alpha_n^2\|Bz_n-Bq\|^2 \\ &\le \|z_n-q\|^2-2\alpha_n\delta\|Bz_n-Bq\|^2+2k^2\alpha_n^2\|Bz_n-Bq\|^2 \\ &= \|z_n-q\|^2+2\alpha_n(\alpha_n k^2-\delta)\|Bz_n-Bq\|^2 \\ &\le \|z_n-q\|^2.\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have that

$$\|y_n-q\| \le \|x_n-q\|.$$

From $q=Q_D(q)$, we get

$$q=Q_D[\gamma_n q+(1-\gamma_n)Q_D(q-\alpha_n Bq)].$$

It follows from (3.2) that

$$\begin{aligned}\|w_n-q\| &= \|Q_D[\gamma_n z_n+(1-\gamma_n)Q_D(z_n-\alpha_n Bz_n)]-Q_D[\gamma_n q+(1-\gamma_n)Q_D(q-\alpha_n Bq)]\| \\ &\le \|[\gamma_n z_n+(1-\gamma_n)Q_D(z_n-\alpha_n Bz_n)]-[\gamma_n q+(1-\gamma_n)Q_D(q-\alpha_n Bq)]\| \\ &\le \gamma_n\|z_n-q\|+(1-\gamma_n)\|(z_n-\alpha_n Bz_n)-(q-\alpha_n Bq)\| \\ &\le \gamma_n\|x_n-q\|+(1-\gamma_n)\|x_n-q\| \\ &= \|x_n-q\|.\end{aligned}$$

This implies that

$$\begin{aligned}\|\tilde{x}_n-q\| &= \|\lambda_n(x_n-q)+(1-\lambda_n)(w_n-q)\| \\ &\le \lambda_n\|x_n-q\|+(1-\lambda_n)\|w_n-q\| \\ &\le \lambda_n\|x_n-q\|+(1-\lambda_n)\|x_n-q\| \\ &= \|x_n-q\|.\end{aligned} \tag{3.4}$$

We have from (3.1), (3.2) and (3.4) that

$$\begin{aligned}\|x_{n+1}-q\| &= \|\eta_n(v-q)+(1-\eta_n)(a_n x_n+b_n z_n+c_n\tilde{x}_n-q)\| \\ &\le \eta_n\|v-q\|+(1-\eta_n)\|a_n x_n+b_n z_n+c_n\tilde{x}_n-q\| \\ &\le \eta_n\|v-q\|+(1-\eta_n)(a_n\|x_n-q\|+b_n\|z_n-q\|+c_n\|\tilde{x}_n-q\|) \\ &\le \eta_n\|v-q\|+(1-\eta_n)(a_n\|x_n-q\|+b_n\|x_n-q\|+c_n\|x_n-q\|) \\ &\le \eta_n\|v_n-q\|+(1-\eta_n)\|x_n-q\| \\ &\le \max\{\|v-q\|, \|x_0-q\|\}.\end{aligned}$$

Thus, $\{x_n\}$ is bounded, and so are $\{z_n\},\{\tilde{x}_n\},\{Bx_n\},\{Bz_n\},\{y_n\},\{w_n\}$. Let $\tilde{Q}$ be a sunny nonexpansive retraction of X onto $S(D,B)$. Observe that

$$\begin{aligned} \|z_n-\tilde{Q}v\| &= \|Q_D(x_n-\alpha_nBx_n)-Q_D(\tilde{Q}v)\| \\ &\le \|(x_n-\alpha_nBx_n)-\tilde{Q}v\| \\ &\le \|x_n-\tilde{Q}v\|. \end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned} \|y_n-\tilde{Q}v\| &= \|Q_D(z_n-\alpha_nBz_n)-Q_D(\tilde{Q}v)\| \\ &\le \|(z_n-\alpha_nBz_n)-\tilde{Q}v\| \\ &\le \|z_n-\tilde{Q}v\| \\ &\le \|x_n-\tilde{Q}v\|. \end{aligned} \tag{3.6}$$

From (3.1) and (3.6) we have

$$\begin{aligned} \|w_n-\tilde{Q}v\| &= \|Q_D(\gamma_nz_n+(1-\gamma_n)Q_D(z_n-\alpha_nBz_n))-Q_D(\tilde{Q}v)\| \\ &\le \|\gamma_n(z_n-\tilde{Q}v)+(1-\gamma_n)(Q_D(z_n-\alpha_nBz_n)-\tilde{Q}v)\| \\ &\le \gamma_n\|z_n-\tilde{Q}v\|+(1-\gamma_n)\|Q_D(z_n-\alpha_nBz_n)-\tilde{Q}v\| \\ &\le \|x_n-\tilde{Q}v\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|\tilde{x}_n-\tilde{Q}v\| &= \|\lambda_n(x_n-\tilde{Q}v)+(1-\lambda_n)(w_n-\tilde{Q}v)\| \\ &\le \lambda_n\|x_n-\tilde{Q}v\|+(1-\lambda_n)\|w_n-\tilde{Q}v\| \\ &\le \lambda_n\|x_n-\tilde{Q}v\|+(1-\lambda_n)\|x_n-\tilde{Q}v\| \\ &= \|x_n-\tilde{Q}v\|. \end{aligned} \tag{3.7}$$

From Lemma 2.5, Lemma 2.6, (3.1), (3.5) and (3.7), we have

$$\begin{aligned} \|x_{n+1}-\tilde{Q}v\|^2 &= \|\eta_n(v-\tilde{Q}v)+(1-\eta_n)(a_nx_n+b_nz_n+c_n\tilde{x}_n-\tilde{Q}v)\|^2 \\ &\le (1-\eta_n)\|(a_nx_n+b_nz_n+c_n\tilde{x}_n-\tilde{Q}v)\|^2+2\eta_n\langle v-\tilde{Q}v,J(x_{n+1}-\tilde{Q}v)\rangle \\ &\le (1-\eta_n)a_n\|x_n-\tilde{Q}v\|^2+(1-\eta_n)b_n\|z_n-\tilde{Q}v\|^2+(1-\eta_n)c_n\|\tilde{x}-\tilde{Q}v\|^2 \\ &\quad -(1-\eta_n)a_nb_n\psi(\|x_n-z_n\|)+2\eta_n\langle v-\tilde{Q}v,J(x_{n+1}-\tilde{Q}v)\rangle \\ &\le (1-\eta_n)[a_n\|x_n-\tilde{Q}v\|^2+b_n\|x_n-\tilde{Q}v\|^2+c_n\|x_n-\tilde{Q}v\|^2 \\ &\quad -(1-\eta_n)a_nb_n\psi(\|x_n-z_n\|)+2\eta_n\langle v-\tilde{Q}v,J(x_{n+1}-\tilde{Q}v)\rangle \\ &\le \|x_n-\tilde{Q}v\|^2-(1-\eta_n)a_nb_n\psi(\|x_n-z_n\|)+2\eta_n\|v-\tilde{Q}v\|\|x_{n+1}-\tilde{Q}v\|. \end{aligned} \tag{3.8}$$

Similarly, we have

$$\|x_{n+1}-\tilde{Q}v\|^2 \le \|x_n-\tilde{Q}v\|^2-(1-\eta_n)a_nc_n\psi(\|x_n-\tilde{x}_n\|)+2\eta_n\|v-\tilde{Q}v\|\|x_{n+1}-\tilde{Q}v\|. \tag{3.9}$$

Next, we split the proof into two cases.

Case 1. Assume that there exists $m_0\in\mathbb{N}$ such that $\{\|x_n-\tilde{Q}v\|\}$ is decreasing for all $n\ge m_0$. Therefore, we obtain that $\lim_{n\to\infty}\|x_n-\tilde{Q}v\|=d$. Since $\lim_{n\to\infty}\gamma_n=0$ and $\lim_{n\to\infty}\eta_n=0$, we conclude from (3.8), (3.9) that

$$\lim_{n\to\infty}a_nc_n\psi(\|x_n-\tilde{x}_n\|)=\lim_{n\to\infty}a_nb_n\psi(\|x_n-z_n\|)=0.$$

Using the property of ψ, we conclude that

$$\lim_{n\to\infty}\|x_n - z_n\| = \lim_{n\to\infty}\|x_n - \tilde{x}_n\| = 0. \tag{3.10}$$

Since

$$\|z_n - \tilde{x}_n\| \le \|z_n - x_n\| + \|x_n - \tilde{x}_n\|,$$

we have $\|z_n - \tilde{x}_n\| \to 0$. Observe that

$$\begin{aligned}\|z_n - y_n\| &= \|Q_D(x_n - \alpha_n Bx_n) - Q_D(z_n - \alpha_n Bz_n)\|\\ &\le \|(x_n - \alpha_n Bx_n) - (z_n - \alpha_n Bz_n)\|\\ &\le \|x_n - z_n\|.\end{aligned}$$

From (3.10), we have $\lim_{n\to\infty}\|z_n - y_n\| = 0$. It follows that

$$\begin{aligned}\|y_n - w_n\| &= \|Q_D(Q_D(z_n - \alpha_n Bz_n)) - Q_D(\gamma_n z_n + (1-\gamma_n)Q_D(z_n - \alpha_n Bz_n))\|\\ &\le \|(Q_D(z_n - \alpha_n Bz_n)) - (\gamma_n z_n + (1-\gamma_n)Q_D(z_n - \alpha_n Bz_n))\|\\ &\le \gamma_n\|z_n - Q_D(z_n - \alpha_n Bz_n)\|\\ &\le \|z_n - y_n\| \to 0\end{aligned}$$

as $n\to\infty$. Since

$$\|z_n - w_n\| \le \|z_n - y_n\| + \|y_n - w_n\|,$$

we have $\|z_n - w_n\| \to 0$ $n\to\infty$. Since

$$\|x_n - y_n\| \le \|z_n - y_n\| + \|x_n - z_n\|,$$

we have $\|x_n - y_n\| \to 0$ as $n\to\infty$. From (3.1) and (3.10), we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|\eta_n(v - x_n) + (1-\eta_n)(b_n(z_n - x_n) + c_n(\tilde{x}_n - x_n))\| = 0. \tag{3.11}$$

Now, we prove that

$$\limsup_{n\to\infty}\langle v - \tilde{Q}v, J(x_n - \tilde{Q}v)\rangle \le 0. \tag{3.12}$$

Observe that $\{x_n\}$ is bounded and X is uniformly convex. We can select a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that converges weakly to u and

$$\limsup_{n\to\infty}\langle v - \tilde{Q}v, J(x_n - \tilde{Q}v)\rangle = \limsup_{n_k\to\infty}\langle v - \tilde{Q}v, J(x_{n_k} - \tilde{Q}v)\rangle \le 0. \tag{3.13}$$

We first show $u \in S(D,B)$. Since α_n is in $[w, \frac{\delta}{k^2}]$ for some $w > 0$, we have that $\{\alpha_{n_k}\}$ is bounded. Thus, there exists a subsequence $\{\alpha_{n_{k_i}}\}$ of $\{\alpha_{n_k}\}$ converges to $\alpha_0 \in [w, \frac{\delta}{k^2}]$. We may assume, without loss of generality, that $\alpha_{n_k} \to \alpha_0$. Since Q_D is nonexpansive, we have

$$\begin{aligned}\|Q_D(x_{n_k} - \alpha_0 Bx_{n_k}) - x_{n_k}\| &\le \|Q_D(x_{n_k} - \alpha_0 Bx_{n_k}) - z_{n_k}\| + \|z_{n_k} - x_{n_k}\|\\ &\le \|Q_D(x_{n_k} - \alpha_0 Bx_{n_k}) - Q_D(x_{n_k} - \alpha_{n_k} Bx_{n_k})\| + \|z_{n_k} - x_{n_k}\|\\ &\le \|(x_{n_k} - \alpha_0 Bx_{n_k}) - (x_{n_k} - \alpha_{n_k} Bx_{n_k})\| + \|z_{n_k} - x_{n_k}\|\\ &\le \|Bx_{n_k}\|\|\alpha_0 - \alpha_{n_k}\| + \|z_{n_k} - x_{n_k}\|.\end{aligned}$$

Since $\|z_n - x_n\| \to 0$ as $n\to\infty$, we have

$$\lim_{k\to\infty}\|Q_D(x_{n_k} - \alpha_0 Bx_{n_k}) - x_{n_k}\| = 0.$$

Since $Q_D(I-\alpha_0 B)$ is nonexpansive, we find from Lemma 2.8 that $u \in F(Q_D(I-\alpha_0 B))$. From Lemma 2.4, we also have $u \in S(D,B)$. Using Lemma 2.3 and (3.13) yields that

$$\begin{aligned}\limsup_{n\to\infty}\langle v-\tilde{Q}v, J(x_n-\tilde{Q}v)\rangle &= \limsup_{k\to\infty}\langle v-\tilde{Q}v, J(x_{n_k}-\tilde{Q}v)\rangle \\ &= \limsup_{k\to\infty}\langle v-\tilde{Q}v, J(u-\tilde{Q}v)\rangle \\ &\le 0.\end{aligned} \tag{3.14}$$

Set

$$t_n = \gamma_n z_n + (1-\gamma_n)Q_D(z_n-\alpha_n B z_n)$$

for all $n \ge 0$. Since $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$, this along with (3.14) shows that

$$\limsup_{k\to\infty}\langle v-\tilde{Q}v, J(x_{n+1}-\tilde{Q}v)\rangle \le 0. \tag{3.15}$$

From (3.1), (3.5), we have that

$$\begin{aligned}\|x_{n+1}-\tilde{Q}v\|^2 &= \langle \eta_n(v-\tilde{Q}v)+(1-\eta_n)((a_n x_n+b_n z_n+c_n\tilde{x})-\tilde{Q}v), J(x_{n+1}-\tilde{Q}v)\rangle \\ &= \eta_n\langle v-\tilde{Q}v, J(x_{n+1}-\tilde{Q}v)\rangle + (1-\eta_n)\langle (a_n x_n+b_n z_n+c_n\tilde{x})-\tilde{Q}v), J(x_{n+1}-\tilde{Q}v)\rangle \\ &\le \eta_n\langle v-\tilde{Q}v, J(x_{n+1}-\tilde{Q}v)\rangle + (1-\eta_n)(a_n\|x_n-\tilde{Q}v\|\|x_{n+1}-\tilde{Q}v\| \\ &\quad + b_n\|z_n-\tilde{Q}v\|\|x_{n+1}-\tilde{Q}v\| + c_n\|\tilde{x}_n-\tilde{Q}v\|\|x_{n+1}-\tilde{Q}v\|) \\ &\le \eta_n\langle v-\tilde{Q}v, J(x_{n+1}-\tilde{Q}v)\rangle + (1-\eta_n)((\frac{a_n}{2}(\|x_n-\tilde{Q}v\|^2+\|x_{n+1}-\tilde{Q}v\|^2) \\ &\quad + \frac{b_n}{2}(\|x_n-\tilde{Q}v\|^2+\|x_{n+1}-\tilde{Q}v\|^2) + \frac{c_n}{2}(\|x_n-\tilde{Q}v\|^2+\|x_{n+1}-\tilde{Q}v\|^2)) \\ &\le (1-\eta_n)\|x_n-\tilde{Q}v\|^2 + 2\eta_n\langle v-\tilde{Q}v, J(x_{n+1}-\tilde{Q}v)\rangle.\end{aligned} \tag{3.16}$$

Since $\lim_{n\to\infty}\eta_n = 0, \sum_{n=0}^{\infty}\eta_n = \infty$, we obtain from Lemma 2.9, (3.15) and (3.16) that

$$\lim_{n\to\infty}\|x_n-\tilde{Q}v\| = 0.$$

Case 2. Assume that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\|x_{n_k}-\tilde{Q}v\| \le \|x_{n_k+1}-\tilde{Q}v\|$$

for all $k \in \mathbb{N}$. From Lemma 2.7, there exists a nondecreasing sequence $\{t_i\} \subset \mathbb{N}$ such that $t_i \to \infty$ and

$$\|x_{t_i}-\tilde{Q}v\| \le \|x_{t_i+1}-\tilde{Q}v\| \textit{ and } \|x_i-\tilde{Q}v\| \le \|x_{t_i+1}-\tilde{Q}v\| \tag{3.17}$$

for all $i \in \mathbb{N}$. Since $\lim_{n\to\infty}\eta_n = 0$, we can obtain from (3.8) and (3.9) that

$$\lim_{i\to\infty}\|x_{t_i}-z_{t_i}\| = \lim_{i\to\infty}\|z_{t_i}-y_{t_i}\| = \lim_{i\to\infty}\|x_{t_i}-\tilde{x}_{t_i}\| = \lim_{i\to\infty}\|x_{t_i}-z_{t_i}\| = \lim_{i\to\infty}\|x_{t_i+1}-x_{t_i}\| = 0.$$

From Case 1, we also have

$$\limsup_{i\to\infty}\langle v-\tilde{Q}v, J(x_{t_i}-\tilde{Q}v)\rangle \le 0$$

and

$$\limsup_{i\to\infty}\langle v-\tilde{Q}v, J(x_{t_i+1}-\tilde{Q}v)\rangle \le 0. \tag{3.18}$$

From (3.16), we have

$$\|x_{t_i+1}-\tilde{Q}v\|^2 \le (1-\eta_{t_i})\|x_{t_i}-\tilde{Q}v\|^2 + 2\eta_{t_i}\langle v-\tilde{Q}v, J(x_{t_i+1}-\tilde{Q}v)\rangle \tag{3.19}$$

This implies that

$$\eta_{t_i}\|x_{t_i}-\tilde{Q}v\|^2 \le \|x_{t_i}-\tilde{Q}v\|^2 - \|x_{t_i+1}-\tilde{Q}v\|^2 + 2\eta_{t_i}\langle v-\tilde{Q}v, J(x_{t_i+1}-\tilde{Q}v)\rangle.$$

Since $\eta_{t_i} > 0$, we get from (3.17) that

$$\|x_{t_i}-\tilde{Q}v\|^2 \le 2\langle v-\tilde{Q}v, J(t_{t_i}-\tilde{Q}v)\rangle.$$

Thus, (3.18) implies that $\lim_{i\to\infty}\|x_{t_i}-\tilde{Q}v\| = 0$. From (3.19), we have that $\|x_{t_i+1}-\tilde{Q}v\| \to 0$ as $i\to\infty$. Using (3.17), we obtain that $\lim_{i\to\infty}\|x_i-\tilde{Q}v\| = 0$. This completes the proof. □

If $v = 0$ in Theorem 3.1, then we have the following result.

Corollary 3.1. *Let D be a nonempty, closed and convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let $Q_D : X \to D$ be a sunny nonexpansive retraction and let $B : D \to X$ be an δ-inverse strongly accretive operator with $S(D,B) \neq \emptyset$. Let $\{x_n\}, \{z_n\}$ and $\{\tilde{x}_n\}$ be a sequences generated by*

$$\begin{cases} x_1 \in D, \\ z_n = Q_D(x_n - \alpha_n B x_n) \\ \tilde{x}_n = \lambda_n x_n + (1-\lambda_n) Q_D[\gamma_n z_n (1-\gamma_n) Q_D(z_n - \alpha_n B z_n)], \\ x_{n+1} = \eta_n v + (1-\eta_n)(a_n x_n + b_n z_n + c_n \tilde{x}_n), \quad \forall n \ge 1, \end{cases}$$

where v is a fixed element in D, $\{\eta_n\}, \{\lambda_n\}, \{\gamma_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty}\eta_n = 0$, $\sum_{n=0}^{\infty}\eta_n = \infty$ and α_n is a sequence in $[w, \frac{\delta}{k^2}]$, and $\{a_n\}, \{b_n\}, \{c_n\} \in [d,e] \subset (0,1)$ such that $a_n + b_n + c_n = 1$. Then $\{x_n\}$ converges strongly to the minimum norm element in $S(D,B)$.

Proof. Using Theorem 3.1, we obtain that $\{x_n\}$ converges strongly to $\tilde{Q}(0)$, which is the minimum norm element in $S(D,B)$. □

4. APPLICATIONS

Using Theorem 3.1, we prove some strong convergence theorems in Banach spaces.

Theorem 4.1. *Let D be a nonempty, closed and convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let $B : X \to X$ be an δ-inverse strongly accretive operator with $B^{-1}(0) \neq \emptyset$. Let $\{x_n\}, \{z_n\}$ and $\{\tilde{x}_n\}$ be a sequences generated by*

$$\begin{cases} x_1 \in D, \\ z_n = (x_n - \alpha_n B x_n), \\ \tilde{x}_n = \lambda_n x_n + (1-\lambda_n)(\gamma_n z_n + (1-\gamma_n)(z_n - \alpha_n B z_n)), \\ x_{n+1} = \eta_n v + (1-\eta_n)(a_n x_n + b_n z_n + c_n \tilde{x}_n), \ n \ge 1, \end{cases}$$

where v is a fixed element in D, $\{\eta_n\}, \{\lambda_n\}, \{\gamma_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty}\eta_n = 0$, $\sum_{n=0}^{\infty}\eta_n = \infty$ and α_n is a sequence in $[w, \frac{\delta}{k^2}]$ for some $0 < w \le \frac{\delta}{k^2}$, $\{a_n\}, \{b_n\}, \{c_n\} \in [d,e] \subset (0,1)$ such that $a_n + b_n + c_n = 1$. Then $\{x_n\}$ converges strongly to $\tilde{Q}(v)$, where $\tilde{Q}$ is a sunny nonexpansive retracttion of X onto $B^{-1}(0)$.

Proof. Note that $B^{-1}(0) = S(X,B)$ and $Q_D = I$, where I is the identity mapping. Using Theorem 3.1, we get the desired result immediately. □

Let $0 \le k < 1$. A mapping $T : D \to D$ is said to be k-strictly pseudocontractive if there exists $j(u-v) \in J(u-v)$ such that

$$\langle Tu - Tv, j(u-v)\rangle \le \|u-v\|^2 - \frac{1-k}{2}\|(I-T)u-(I-T)v\|^2$$

for all $u, v \in D$. The above inequality can be rewritten as follows:

$$\langle (I-T)u-(I-T)v, j(u-v)\rangle \ge \frac{1-k}{2}\|(I-T)u-(I-T)v\|^2. \tag{4.1}$$

Theorem 4.2. *Let D be a nonempty, closed and convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let $Q_D : X \to D$ be a sunny nonexpansive retraction and let $T : D \to D$ be a k-strictly pseudocontractive mapping with $F(T) \ne \emptyset$. Let $\{x_n\}, \{z_n\}$ and $\{\tilde{x}_n\}$ be a sequences generated by*

$$\begin{cases} x_1 \in D, \\ z_n = Q_D(x_n - \alpha_n(x_n - Tx_n)), \\ \tilde{x}_n = \lambda_n x_n + (1-\lambda_n)Q_D[\gamma_n z_n + (1-\gamma_n)Q_D(z_n - \alpha_n(z_n - Tz_n))], \\ x_{n+1} = \eta_n v + (1-\eta_n)(a_n x_n + b_n z_n + c_n \tilde{x}_n),\ n \ge 1, \end{cases}$$

where v is a fixed element in D, $\{\eta_n\}, \{\lambda_n\}, \{\gamma_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty}\eta_n = 0$, $\sum_{n=0}^{\infty}\eta_n = \infty$ and α_n is a sequence in $[w, \frac{\delta}{k^2}]$ for some $0 < w \le \frac{\delta}{k^2}$, $\{a_n\}, \{b_n\}, \{c_n\} \in [d,e] \subset (0,1)$ such that $a_n + b_n + c_n = 1$. Then $\{x_n\}$ converges strongly to $\tilde{Q}(v)$, where $\tilde{Q}$ is a sunny nonexpansive retracttion of X onto $F(T)$.

Proof. Let $B = I - T$. From (4.1), we obtain that B is $\frac{1-k}{2}$-inverse strongly accretive. It is easy to show that $F(T) = S(D,B) = S(D, I-T)$ (see [2]). By using Theorem 3.1, $\{x_n\}$ converges strongly to $\tilde{Q}(v)$, where $\tilde{Q}$ is a sunny nonexpansive retracttion of X onto $F(T)$. □

Let D be a subset of a smooth Banach space X. Let $\delta > 0$. An operator $B : D \to X$ is said to be δ-strongly accretive if

$$\langle Bu - Bv, J(u-v)\rangle \ge \delta\|u-v\|^2, \quad \forall u, v \in D.$$

Theorem 4.3. *Let D be a nonempty, closed and convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let $Q_D : X \to D$ be a sunny nonexpansive retraction and let $B : D \to X$ be an δ-strongly accretive and L-Lipschitz continuous operator with $S(D,B) \ne \emptyset$. Let $\{x_n\}, \{z_n\}$ and $\{\tilde{x}_n\}$ be a sequences generated by*

$$\begin{cases} x_1 \in D, \\ z_n = Q_D(x_n - \alpha_n B x_n), \\ \tilde{x}_n = \lambda_n x_n + (1-\lambda_n)Q_D[\gamma_n z_n + (1-\gamma_n)Q_D(z_n - \alpha_n B z_n)], \\ x_{n+1} = \eta_n v + (1-\eta_n)(a_n x_n + b_n z_n + c_n \tilde{x}_n),\ n \ge 1, \end{cases}$$

where v is a fixed element in D, $\{\eta_n\}, \{\lambda_n\}, \{\gamma_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty}\eta_n = 0$, $\sum_{n=0}^{\infty}\gamma_n = \infty$ and α_n is a sequence in $[w, \frac{\delta}{k^2L^2}]$ for some $0 < w \le \frac{\delta}{k^2L^2}$, $\{a_n\}, \{b_n\}, \{c_n\} \in [d,e] \subset (0,1)$ such that $a_n + b_n + c_n = 1$. Then $\{x_n\}$ converges strongly to $\tilde{Q}(v)$, where $\tilde{Q}$ is a sunny nonexpansive retracttion of X onto $S(D,B)$.

Proof. Since $B : D \to X$ is δ-strongly accretive and L-Lipschitz continuous, we have

$$\langle Bu - Bv, J(u-v)\rangle \geq \delta \|u-v\|^2 \geq \frac{\delta}{L^2}\|Bu - Bv\|^2$$

for $u, v \in D$. Hence, B is $\frac{\delta}{L^2}$-inverse strongly accretive. By using Theorem 3.1, $\{x_n\}$ converges strongly to $\tilde{Q}(v)$, where $\tilde{Q}$ is a sunny nonexpansive retracttion of X onto $S(D,B)$. □

Example 4.1. Let $X = \mathbb{R}$ with Euclidean norm and usual Euclidean inner product. Let $D := [-2,1]$ and $Q : \mathbb{R} \to D$ be defined by

$$Q_D x := \begin{cases} -2, & -\infty < x < -2, \\ x, & -2 \leq x \leq 1, \\ 1, & 1 \leq x < \infty, \end{cases}$$

and $B : D \to \mathbb{R}$ be defined by

$$Bx := \begin{cases} 0, & x \in [-2,0], \\ \sin x, & x \in (0,1]. \end{cases}$$

Clearly, Q_D is a sunny nonexpansive retracttion, $S(D,B) = [-2,0]$, and B is a accretive. Next, we prove that B is a δ-inverse strongly accretive for $\delta \leq 1$. If $x, y \in [-2,0]$, then

$$\langle Bx - By, x - y\rangle = 0 \geq \|Bx - By\|^2 = 0.$$

If $x, y \in (0,1]$, then there exist $t \in (0,1]$ such that (let $x \leq y$)

$$\begin{aligned} \sin y - \sin x = \cos t(y-x) &\Leftrightarrow \sin y - \sin x \leq y - x \\ &\Leftrightarrow 0 \geq \sin x - \sin y \geq x - y \\ &\Leftrightarrow (\sin x - \sin y)(\sin x - \sin y) \leq (x-y)(\sin x - \sin y) \\ &\Leftrightarrow |\sin x - \sin y|^2 \leq \langle x - y, \sin x - \sin y\rangle. \end{aligned}$$

If $x \in [-2,0]$ and $y \in (0,1]$, then

$$\sin y < y \Rightarrow \sin y < y - x \Leftrightarrow \sin^2 y < (y-x)\sin y,$$

$$\langle Bx - By, x - y\rangle = \langle -\sin y, x - y\rangle = \langle \sin y, y - x\rangle \geq |Bx - By|^2 = |0 - \sin y|^2 = \sin^2 y.$$

Hence, $\langle Bx - By, x - y\rangle \geq \|Bx - By\|^2$ for all $x \in [-2,1]$, B is a δ-inverse strongly accretive for $\delta \leq 1$.

Example 4.2. Let $X = \mathbb{R}^2$ with Euclidean norm and usual Euclidean inner product. Let $D := [-1,2] \times [-1,2]$ and $Q : \mathbb{R}^2 \to D$ be defined by

$$Q_D(x,y) := \begin{cases} (-1,-1), & -\infty < x,y < -1, \\ (x,y), & -1 \le x,y < 2, \\ (2,2), & 2 \le x,y < \infty, \\ (-1,y), & -\infty < x < -1, -1 \le y < 2, \\ (-1,2), & -\infty < x < -1, 2 \le y < \infty, \\ (x,2), & -1 \le x < 2, 2 \le y < \infty, \\ (x,-1), & -1 \le x < 2, -\infty < y < -1, \\ (2,-1), & 2 \le x < \infty, -\infty < y < -1, \\ (2,y), & 2 \le x < \infty, -1 \le y < 2, \end{cases}$$

and $B : D \to \mathbb{R}^2$ be defined by

$$Bx := \begin{cases} (0,0), & x,y \in [-1,1], \\ (\frac{x^2}{4}, \frac{y^2}{4}), & otherwise. \end{cases}$$

Clearly, Q_D is a sunny nonexpansive retracttion and $S(D,B) = [-1,1] \times [-1,1]$, and B is a δ-inverse strongly accretive for $\delta \le 1$. If $x,y \in [-1,1]$, then

$$\langle B(x,y) - B(x',y'), (x,y) - (x',y') \rangle = 0 \ge \|B(x,y) - B(x',y')\|^2 = 0.$$

Otherwise, since $x + x' < 4$ for all $x, x' \in [-1,2]$, we have

$$|\frac{x^2}{4} - \frac{x'^2}{4}| = \frac{1}{4}|(x-x')(x+x')| < |x-x'|.$$

Acknowledgment

The authors express their gratitude to the Professor J. J. Nieto and referees for their helpful suggestions which improved final version of this paper.

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