

LOCAL CONVERGENCE ANALYSIS OF JARRATT-TYPE SCHEMES FOR SOLVING EQUATIONS

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Abstract. The aim of this paper is to present a local convergence analysis for Jarratt-type schemes to solve equations with Banach space valued operators. The convergence conditions are weaker than the corresponding ones in earlier results and some numerical examples are provided.

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1. INTRODUCTION

Let $\mathcal{E}_1, \mathcal{E}_2$ stand for Banach spaces and $\Omega \subseteq \mathcal{E}_1$ be a nonempty and open set. Let $\mathcal{E}_1 : \Omega \rightarrow \mathcal{E}_2$ be a continuously differentiable operator in the sense of Fréchet. The task of solving equation of the form

$$F(x) = 0 \tag{1.1}$$

is very important since problems from many disciplines can be reduced to (1.1); see, e.g., [1, 2, 3, 4, 5, 8, 13, 21] and the references therein. A locally unique solution x_* is desirable in a closed form but this can be achieved only in special cases. That explains why researchers and practitioners utilize some iterative schemes to approximate x_* . The most widely used schemes are the so called Newton-type.

Recently, there are many results, based on the Lipschitz-type conditions on the higher order derivatives, on local and semi-local convergence analysis of Newton-type schemes; see [1, 2, 3, 4, 5, 10, 11, 12, 14, 15, 16, 17, 18, 19, 22] and the references therein. The convergence domain in these studies is small in general. These conditions limit the applicability of these method. As a motivational example, let us define a function F on $\mathcal{E}_1 = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

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Letting $x^* = 1$, we have

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \quad F'(1) = 3, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

It is obvious that function F does not have bounded third derivative in \mathcal{E}_1 .

In this paper, we study the following Jarratt-type scheme defined by

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ z_k &= x_k + \frac{2}{3}(y_k - x_k), \\ u_k &= x_k - A_k F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= u_k - \left[\frac{3}{2}F'(z_k)^{-1}A_k + F'(x_k)^{-1}(I - \frac{3}{2}A_k) \right] F(u_k), \\ A_k &= (6F'(z_k) - 2F'(x_k))^{-1}(3F'(z_k) + F'(x_k)) \end{aligned} \tag{1.2}$$

for $n = 0, 1, 2, \dots$. Our local convergence analysis uses only hypotheses on the first Fréchet derivative, whereas the order of convergence is established using (COC) and (ACOC) (see Remark 2.1 (b)). Hence, we expand the applicability of method (1.2). The organization is as follows. In Section 2, we give the local convergence of method (1.2). In last section, Section 3, numerical examples are provided.

2. LOCAL CONVERGENCE

Let us consider some scalar functions and parameters which will appear in the local convergence analysis. Consider $w_0 : [0, +\infty) \rightarrow [0, +\infty)$ to be a continuous and increasing function with $w_0(0) = 0$. Suppose that the equation

$$w_0(t) = 1 \tag{2.1}$$

has at least one positive solution. Denote by ρ_0 the smallest such solution. Let us also consider function $w : [0, \rho_0) \rightarrow [0, +\infty)$ to be continuous and increasing with $w(0) = 0$. Define functions g_1, g_2, h_1 and h_2 on the interval $[0, \rho_0)$ by

$$\begin{aligned} g_1(t) &= \frac{\int_0^1 w((1-\theta)t) d\theta}{1 - w_0(t)}, \\ h_1(t) &= g_1(t) - 1, \\ g_2(t) &= \frac{1}{3}(1 + 2g_1(t)) \end{aligned}$$

and

$$h_2(t) = g_2(t) - 1.$$

We have $h_1(0) = -1$ and $h_1(t) \rightarrow +\infty$ as $t \rightarrow \rho_0^-$. The intermediate value theorem guarantees that equation $h_1(t) = 0$ has at least one solution in $(0, \rho_0)$. Denote by r_1 the smallest such solution. We also have that $h_2(0) = -\frac{2}{3}$ and $h_2(t) \rightarrow +\infty$ as $t \rightarrow \rho_0^-$. Denote by r_2 the smallest solution of equation $h_2(t) = 0$ in $(0, \rho_0)$. Suppose that equation

$$p(t) = 1 \tag{2.2}$$

has at least one solution in $(0, \rho_0)$, where $p(t) = \frac{1}{2}(3w_0(g_2(t)t) + w_0(t))$. Denote by ρ_1 the smallest such solution. Set $\rho_2 = \min\{\rho_0, \rho_1\}$. Define functions g_3 and h_3 on the interval $[0, \rho_2)$ by

$$g_3(t) = g_1(t) + \frac{3}{4} \frac{(w_0(g_2(t)t) + w_0(t)) \int_0^1 v(\theta t) d\theta}{(1-p(t))(1-w_0(t))}$$

and

$$h_3(t) = g_3(t) - 1,$$

where $v : [0, \rho_2) \rightarrow [0, +\infty)$ is a continuous and increasing function. We get $h_3(0) = -1$ and $h_3(t) \rightarrow +\infty$ as $t \rightarrow \rho_2^-$. Denote by r_3 the smallest solution of equation $h_3(t) = 0$ in $(0, \rho_2)$. Suppose that equation

$$w_0(g_2(t)t) = 1 \tag{2.3}$$

and

$$w_0(g_3(t)t) = 1 \tag{2.4}$$

have at least one solution in $(0, \rho_2)$. Denote by ρ_3 and ρ_4 , the smallest such solutions, respectively. Set $\rho = \min\{\rho_3, \rho_4\}$. Define functions g_4 and h_4 on $(0, \rho)$ by

$$g_4(t) = \left[\begin{aligned} & \frac{\int_0^1 w((1-\theta)g_3(t)t) d\theta}{1-w_0(g_3(t)t)} \\ & + \frac{(w_0(g_3(t)t) + w_0(t)) \int_0^1 v(\theta g_3(t)t) d\theta}{(1-w_0(g_3(t)t))(1-w_0(t))} \\ & + \frac{3}{8} \frac{(3v(g_2(t)t) + v(t))(w_0(g_2(t)t) + w_0(t)) \int_0^1 v(\theta g_3(t)t) d\theta}{(1-w_0(g_2(t)t))(1-w_0(t))(1-p(t))} \end{aligned} \right] g_3(t)$$

and

$$h_4(t) = g_4(t) - 1.$$

We obtain $h_4(0) = -1$ and $h_4(t) \rightarrow +\infty$ as $t \rightarrow \rho^-$. Denote by r_4 the smallest solution of equation $h_4(t) = 0$ in $(0, \rho)$. Moreover, define the radius of convergence r by

$$r = \min\{r_i\}, \quad i = 1, 2, 3, 4. \tag{2.5}$$

Then, we have that, for each $t \in [0, r)$,

$$0 \leq w_0(t) < 1, \tag{2.6}$$

$$0 \leq p(t) < 1, \tag{2.7}$$

$$0 \leq w_0(g_2(t)t) < 1, \tag{2.8}$$

$$0 \leq w_0(g_3(t)t) < 1 \tag{2.9}$$

and

$$0 \leq g_i(t) < 1, \quad i = 1, 2, 3, 4. \tag{2.10}$$

Let $B(q, a), \bar{B}(q, a)$ denote the open and closed balls in \mathcal{E}_1 , respectively, with center $q \in \mathcal{E}_1$ and radius $a > 0$. We utilize in the local convergence analysis the proceeding notation and the conditions (C):

- (c1) $F : \Omega \subset \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is a continuously differentiable operator in the sense of Fréchet and there exists $x_* \in \Omega$ such that $F(x_*) = 0$ and $F'(x_*)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$.

(c2) There exists function $w_0 : [0, +\infty) \rightarrow [0, +\infty)$ continuous and increasing with $w_0(0) = 0$ such that, for each $x \in \Omega$,

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq w_0(\|x - x_*\|).$$

Set $\Omega_0 = \Omega \cap B(x_*, \rho_0)$, where ρ_0 is defined in (2.1).

(c3) There exist functions $w : [0, \rho_0) \rightarrow [0, +\infty)$, $v : [0, \rho_0) \rightarrow [0, +\infty)$ with $w(0) = 0$ such that, for each $x, y \in \Omega_0$,

$$\|F'(x_*)^{-1}(F(y) - F(x))\| w(\|y - x\|)$$

and

$$\|F'(x_*)^{-1}F'(x)\| \leq v(\|x - x_*\|).$$

(c4) $\bar{B}(x_*, r) \subseteq \Omega$, where r is defined in (2.5).

(c5) There exists $r_* \geq r$ such that

$$\int_0^1 w_0(\theta r_*) d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \bar{U}(x_*, r_*)$.

Next, we present the local convergence analysis of method (1.2).

Theorem 2.1. *Suppose that conditions (C) hold. Then, the sequence $\{x_k\}$ starting at $x_0 \in B(x_*, r) - \{x_*\}$ and generated by method (1.2) is well defined, remains in $B(x_*, r)$ for each $k = 0, 1, 2, \dots$ and converges to x_* so that*

$$\|y_k - x_*\| \leq g_1(\|x_k - x_*\|)\|x_k - x_*\| \leq \|x_k - x_*\| < r, \quad (2.11)$$

$$\|z_k - x_*\| \leq g_2(\|x_k - x_*\|)\|x_k - x_*\| \leq \|x_k - x_*\|, \quad (2.12)$$

$$\|u_k - x_*\| \leq g_3(\|x_k - x_*\|)\|x_k - x_*\| \leq \|x_k - x_*\| \quad (2.13)$$

and

$$\|x_{k+1} - x_*\| \leq g_4(\|x_k - x_*\|)\|x_k - x_*\| \leq \|x_k - x_*\|, \quad (2.14)$$

where the functions g_i are defined as previous. Moreover, the limit point x_* is the only solution of the equation $F(x) = 0$ in Ω_1 defined in (c5).

Proof. Mathematical induction is used to prove estimates (2.11)-(2.14). Let $x \in B(x_*, r) - \{x_*\}$. By (2.1), (2.5) and (c2), it follows that

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq w_0(\|x - x_*\|) \leq w_0(r) < 1,$$

which along the Banach perturbation Lemma [3] shows that $F'(x)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$ and

$$\|F'(x)^{-1}F'(x_*)\| \leq \frac{1}{1 - w_0(\|x - x_*\|)}. \quad (2.15)$$

We also have that y_0 and z_0 are well defined by the first and second substep of method (1.2) for $n = 0$. By (2.5), (2.10) (for $i = 1, 2$), (c1), the first condition in (c3) and (2.15), we have that

$$\begin{aligned}
 \|y_0 - x_*\| &= \|x_0 - x_* - F'(x_0)^{-1}F(x_0)\| \\
 &\leq \|F'(x_0)^{-1}F'(x_*)\| \left\| \int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*)) - F'(x_0))(x_0 - x_*)d\theta \right\| \\
 &\leq \frac{\int_0^1 w((1 - \theta)\|x_0 - x_*\|)d\theta}{1 - w_0(\|x_0 - x_*\|)} \|x_0 - x_*\| \\
 &\leq g_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r.
 \end{aligned} \tag{2.16}$$

Hence, (2.11) holds for $n = 0$, and $y_0 \in B(x_*, r)$. Moreover, using (2.16), we have

$$\begin{aligned}
 \|z_0 - z_*\| &= \|x_0 - x_* + \frac{2}{3}[(y_0 - x_*) + (x_* - x_0)]\| \\
 &= \frac{1}{3}\|x_0 - x_* + 2(y_0 - x_*)\| \\
 &\leq \frac{1}{3}(\|x_0 - x_*\| + 2\|y_0 - x_*\|) \\
 &\leq \frac{1}{3}(1 + 2g_1(\|x_0 - x_*\|))\|x_0 - x_*\| \\
 &= g_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r.
 \end{aligned} \tag{2.17}$$

So, (2.1) holds for $n = 0$ and $z_0 \in B(x_*, r)$. We show that A_0 is well defined. Using (2.5), (2.9), (c2), (2.16) and (2.17), we get that

$$\begin{aligned}
 &\|(2F'(x_*))^{-1}[3F'(z_0) - F'(x_0) - 2F'(x_*)]\| \\
 &\leq \frac{1}{2}[3\|F'(x_*)^{-1}(F'(z_0) - F'(x_*))\| + \|F'(x_*)^{-1}(F'(x_0) - F'(x_*))\|] \\
 &\leq \frac{1}{2}(3w_0(\|z_0 - x_*\|) + w_0(\|x_0 - x_*\|)) \\
 &\leq \frac{1}{2}(3w_0(g_2(\|x_0 - x_*\|)\|x_0 - x_*\|) + w_0(\|x_0 - x_*\|)) \\
 &= p(\|x_0 - x_*\|) \leq p(r) < 1.
 \end{aligned}$$

It follows that $(3F'(z_0) - F'(x_0))^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$, A_0 is well defined and

$$\|(3F'(z_0) - F'(x_0))^{-1}F'(x_*)\| \leq \frac{1}{2(1 - p(\|x_0 - x_*\|))}. \tag{2.18}$$

We can write

$$\begin{aligned}
 I - A_0 &= (6F'(z_0) - 2F'(x_0))^{-1}[6F'(z_0) - 2F'(x_0) - (3F'(x_0) + F'(x_0))] \\
 &= \frac{3}{2}(3F'(z_0) - F'(x_0))^{-1}[(F'(z_0) - F'(x_*)) + (F'(x_*) - F'(x_0))],
 \end{aligned} \tag{2.19}$$

which shows that

$$\|I - A_0\| \leq \frac{3w_0(\|z_0 - x_*\|) + w_0(\|x_0 - x_*\|)}{1 - p(\|x_0 - x_*\|)}. \tag{2.20}$$

Then, u_0 is well defined by the third substep of method (1.2) for $n = 0$ and

$$u_0 - x_* = (x_0 - x_* - F'(x_0)^{-1}F(x_0)) + (I - A_0)F'(x_0)^{-1}F'(x_*)F'(x_*)^{-1}F(x_0). \tag{2.21}$$

We can write by (c1) that

$$F(x) = F(x) - F(x_*) = \int_0^1 F'(x_* + \theta(x - x_*)) d\theta(x - x_*).$$

From the second condition in (c3), we have

$$\|F'(x_*)^{-1}F(x)\| \leq \int_0^1 v(\theta\|x - x_*\|) d\theta\|x - x_*\|, \quad (2.22)$$

since $\|x_* + \theta(x - x_*) - x_*\| = \theta\|x - x_*\| < r$ (i.e., $x_* + \theta(x - x_*) \in B(x_*, r)$ for each $\theta \in [0, 1]$).

To continue, use (2.5), (2.10) (for $i = 3$), (2.15)-(2.18) and (2.20)-(2.22) to get that

$$\begin{aligned} \|u_0 - x_*\| &\leq \|x_0 - x_* - F'(x_0)^{-1}F(x_0)\| \\ &\quad + \|I - A_0\| \|F'(x_0)^{-1}F(x_*)\| \|F'(x_*)^{-1}F(x_0)\| \\ &\leq g_1(\|x_0 - x_*\|) \|x_0 - x_*\| \\ &\quad + \frac{3}{4} \frac{(w_0(\|z_0 - x_*\|) + w_0(\|x_0 - x_*\|)) \int_0^1 v(\theta\|x_0 - x_*\|) d\theta}{(1 - p(\|x_0 - x_*\|))(1 - w_0(\|x_0 - x_*\|))} \|x_0 - x_*\| \\ &= g_3(\|x_0 - x_*\|) \|x_0 - x_*\| \leq \|x_0 - x_*\| < r. \end{aligned} \quad (2.23)$$

Hence, (2.13) holds for $n = 0$ and $u_0 \in B(x_*, r)$. Since $z_0, u_0 \in B(x_*, r)$, we have as in (2.15) for $x = z_0, u_0$ that $F'(z_0)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1), F'(u_0)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$,

$$\|F'(z_0)^{-1}F'(x_*)\| \leq \frac{1}{1 - w_0(\|z_0 - x_*\|)}, \quad (2.24)$$

$$\|F'(u_0)^{-1}F'(x_*)\| \leq \frac{1}{1 - w_0(\|u_0 - x_*\|)} \quad (2.25)$$

and x_1 is well defined by the last substep of method (1.2). Moreover, we can write

$$\begin{aligned} x_1 - x_* &= u_0 - x_* - F'(x_0)^{-1}F(u_0) \\ &\quad - \frac{3}{2}(F'(z_0)^{-1} - F'(x_0)^{-1})A_0F(u_0) \\ &= (u_0 - x_* - F'(u_0)^{-1}F(u_0)) + F'(u_0)^{-1}[(F'(x_0) - F'(x_*)) \\ &\quad + (F'(x_*) - F'(u_0))]F'(x_0)^{-1}F(u_0) \\ &\quad - \frac{3}{2}F'(z_0)^{-1}[(F'(x_0) - F'(x_*)) \\ &\quad + (F'(x_*) - F'(z_0))]F'(x_0)^{-1}A_0F(u_0). \end{aligned} \quad (2.26)$$

Using (2.5), (2.10) (for $i = 4$), (2.15), (2.22) (for $x = u_0$) and (2.23)-(2.27), and the triangle inequality, we obtain that

$$\begin{aligned}
\|x_1 - x_*\| &\leq \|u_0 - x_* - F'(x_0)^{-1}F(u_0)\| \\
&\quad + \|F'(u_0)^{-1}F'(x_*)\| \|F'(x_*)^{-1}(F'(x_0) - F'(x_*))\| \\
&\quad + \|F'(x_*)^{-1}(F'(x_*) - F'(u_0))\| \|F'(x_0)^{-1}F'(x_*)\| \|F'(x_*)^{-1}F(u_0)\| \\
&\quad + \frac{3}{2} \|F'(z_0)^{-1}F'(x_*)\| \|F'(x_*)^{-1}(F'(x_0) - F'(x_*))\| \\
&\quad + \|F'(x_*)^{-1}(F'(x_*) - F'(z_0))\| \\
&\quad \times \|F'(x_0)^{-1}F'(x_*)\| \|F'(x_*)^{-1}A_0\| \|F'(x_*)^{-1}F(u_0)\| \\
&\leq \frac{\int_0^1 w((1-\theta)\|u_0 - x_*\|)d\theta \|u_0 - x_*\|}{1 - w_0(\|u_0 - x_*\|)} \\
&\quad + \frac{(w_0(\|u_0 - x_*\|) + w_0(\|x_0 - x_*\|)) \int_0^1 v(\theta\|u_0 - x_*\|)d\theta \|u_0 - x_*\|}{(1 - w_0(\|u_0 - x_*\|))(1 - w_0(\|x_0 - x_*\|))} \\
&\quad + \frac{3(3v(\|z_0 - x_*\|) + v(\|x_0 - x_*\|))(w_0(\|z_0 - x_*\|) + w_0(\|x_0 - x_*\|))}{8(1 - w_0(\|z_0 - x_*\|))(1 - w_0(\|x_0 - x_*\|))(1 - p(\|x_0 - x_*\|))} \\
&\quad \times \int_0^1 v(\theta\|u_0 - x_*\|)d\theta \|u_0 - x_*\| \\
&\leq g_4(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r.
\end{aligned} \tag{2.28}$$

So, (2.14) holds for $n = 0$ and $x_1 \in B(x_*, r)$. If $x_i, y_i, z_i, u_i, x_{i+1}$ replace x_0, y_0, z_0, u_0, x_1 in the preceding estimates, we complete the induction for (2.11)-(2.14). Then, it follows from the estimate

$$\|x_{i+1} - x_*\| \leq c\|x_i - x_*\| < r, \tag{2.29}$$

where $c = g_4(\|x_0 - x_*\|) \in [0, 1)$, that $\lim_{i \rightarrow +\infty} x_i = x_*$ and $x_{i+1} \in B(x_*, r)$. Let

$$Q = \int_0^1 F'(x_* + \theta(y_* - x_*))d\theta$$

for some $y_* \in \Omega_1$ with $F(y_*) = 0$. In view of (c2) and (c5), we get that

$$\begin{aligned}
\|F'(x_*)^{-1}(Q - F'(x_*))\| &\leq \int_0^1 w_0(\theta\|y_* - x_*\|)d\theta \\
&\leq \int_0^1 w_0(\theta r_*)d\theta < 1,
\end{aligned} \tag{2.30}$$

so $Q^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$. Finally, in view of

$$0 = F(y_*) - F(x_*) = Q(y_* - x_*),$$

we deduce that $x_* = y_*$. This completes the proof. \square

Remark 2.1. (a) Let $w_0(t) = L_0t$, $w(t) = Lt$. The radius $\tilde{\rho}_1 = \frac{2}{2L_0+L}$ was obtained by Argyros as the convergence radius for Newton's method under condition (c1)-(c3) [1, 2, 3]. Notice that the convergence radius for Newton's method given independently by Rheinboldt [17] and Traub [21] was given by

$$\tilde{\rho} = \frac{2}{3L} < \tilde{\rho}_1.$$

Let $f(x) = e^x - 1$. Then $x^* = 0$. Setting $\Omega = B(0, 1)$, we have that $L_0 = e - 1 < L = e^{\frac{1}{L_0}}$. So $\tilde{\rho} = 0.24252961 < \tilde{\rho}_1 = 0.3827$.

Moreover, the new error bounds [17, 21] are:

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0 \|x_n - x^*\|} \|x_n - x^*\|^2,$$

whereas the old ones [20, 21]

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L \|x_n - x^*\|} \|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise if $L_0 < L$. Clearly, the radius of convergence of method (1.2) given by r is smaller than $\tilde{\rho}_1$.

(b) We can use the computational order of convergence (COC) [23]

$$\xi = \frac{\ln \frac{\|x_{n+2} - x_{n+1}\|}{\|x_{n+1} - x_n\|}}{\ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}}, \quad \text{for each } n = 1, 2, \dots$$

or the approximate computational order of convergence (ACOC) [23]

$$\xi^* = \frac{\ln \frac{\|x_{n+2} - x^*\|}{\|x_{n+1} - x^*\|}}{\ln \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}}, \quad \text{for each } n = 0, 1, 2, \dots$$

(c) Using (h2) and

$$\begin{aligned} \|G'(x^*)^{-1}G'(x)\| &= \|G'(x^*)^{-1}(G'(x) - G'(x^*)) + I\| \\ &\leq 1 + \|G'(x^*)^{-1}(G'(x) - G'(x^*))\| \leq 1 + q_0(\|x - x^*\|), \end{aligned}$$

the second condition in (h3) can be replaced by

$$w(t) = 1 + w_0(t)$$

or

$$w(t) = 1 + w_0(\rho_0).$$

Notice that if, in particular, $w_0(t) = L_0 t$, then, we can choose $w(t) = 2$, since $t \in [0, \frac{1}{L_0})$.

3. NUMERICAL EXAMPLES

Example 3.1. Let $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}^3$, $\Omega = \bar{U}(0, 1)$, $x^* = (0, 0, 0)^T$. Define function F on Ω for $w = (x, y, z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $w_0(t) = (e-1)t$, $w(t) = e^{\frac{1}{e-1}t}$ and $v(t) = e^{\frac{1}{e-1}}$.

Therefore, we have

$$\begin{aligned} r_1 = r_2 &= 0.38269191223238574472986783803208, \\ r_3 &= 0.36509279210603506893306757774553 \end{aligned}$$

and

$$r_4 = r = 0.22889602845498427385884099294344.$$

Example 3.2. Returning back to the motivational example at the introduction of this paper, we have $w_0(t) = w(t) = L_0t$ and $v(t) = 2$ where $L_0 = 96.662907$. Then, the radius are:

$$r_2 = r_2 = 0.0068968199628702108600064590859802,$$

$$r_3 = 0.0068968199628702099926447210975766$$

and

$$r_4 = r = 0.002743265590198637297880690510965.$$

Example 3.3. Let $\mathcal{E}_1 = \mathcal{E}_2 = C[0, 1], \Omega = \bar{U}(x^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [1, 2, 6, 7, 8, 9, 12] defined by

$$x(s) = \int_0^1 G(s, t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt,$$

where the kernel G is the Green's function defined on the interval $[0, 1] \times [0, 1]$ by

$$G(s, t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.1), where $F : C[0, 1] \longrightarrow C[0, 1]$ is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s, t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt.$$

Since

$$\| \int_0^1 G(s, t)dt \| \leq \frac{1}{8},$$

we have

$$F'(x)y(s) = y(s) - \int_0^1 G(s, t)(\frac{3}{2}x(t)^{1/2} + x(t))dt.$$

From $F'(x^*(s)) = I$, we have

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}(\frac{3}{2}\|x - y\|^{1/2} + \|x - y\|).$$

It follows that $w_0(t) = w(t) = \frac{1}{8}(\frac{3}{2}t^{1/2} + t), v(t) = 1 + w_0(t)$. Hence

$$r_1 = r_2 = r_3 = 2.13333333333333333333333333333333$$

and

$$r_4 = 1.342649884740344745637230516877.$$

So we choose $r = 1$.

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