Appl. Set-Valued Anal. Optim. 1 (2019), No. 3, pp. 221-229
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.1.2019.3.03

MINIMALITY CONDITIONS FOR CONVEX COMPOSITE FUNCTIONS AND EFFICIENCY CONDITIONS IN VECTOR OPTIMIZATION

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Abstract. Starting from a sufficient minimality condition for convex composite functions using the notion of sharp minima, we formulate equivalent conditions and have a look at the continuous case. By using these results in scalar optimization, we obtain a sufficient condition for strictly efficient points as well as for superefficient points in vector optimization. Moreover, a necessary condition for Henig properly efficient points is given.

Keywords. Convex composite function; Sharp minimum; Strict efficiency; Superefficiency; Henig properly efficient point.

2010 Mathematics Subject Classification. 49J52, 90C26.

1. INTRODUCTION

If one wants to treat vector optimization problems using scalar functions, it is important to put together matching notions of efficiency and restrictive definitions of minimality. For the latter, the concept of sharp minima appears to be appropriate since convex composite functions, i.e. functions $g \circ f$ with convex g and continuous / differentiable f, are often to minimize, especially when g is the distance function with respect to the negative of some ordering cone.

Starting from a sufficient minimality condition for convex composite functions under rather general assumptions and the formulation of equivalent assertions, the notion of strict efficiency [1] (resp. strong efficiency [12]) will prove suitable for dealing with a vector optimization problem of the form

$$\min_K f(w),\ w \in A \subseteq X, \tag{1.1}$$

where $f : X \longrightarrow Y$ is a differentiable function with X and Y normed spaces and $K \subseteq Y$ a (closed) convex cone, which consists in finding points of $f(A)$ that are efficient with respect to the ordering cone K. Strict / strong efficiency is stronger than efficiency in the usual sense, but implied by superefficiency, which was introduced in [2]; see also [12]. In order to obtain a necessary condition for efficiency, Henig's notion of efficient points (see [10]) turns out to be helpful. Throughout this paper, X and Y are (not necessarily complete) normed spaces with their topological duals X^* and Y^* as well as

$$g\colon Y \longrightarrow \mathbb{R} \cup \{+\infty\} \text{ a convex function and } f\colon X \longrightarrow Y.$$

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Received October 11, 2019; Accepted December 18, 2019

$B(x, r)$ stands for the open ball with center x and radius r, and the closed unit balls of Y, X^* and Y^* are denoted by $B(Y)$, $B(X^*)$ and $B(Y^*)$, respectively. Moreover, the notations cor(S), int(S) and cl(S) refer to the algebraic interior, the topological interior and the closure of a set S. A nonempty subset K of Y is said to be a cone if $tK \subseteq K$ for any $t \geq 0$ and a convex cone if, in addition, $K + K \subseteq K$. A cone is called pointed if $K \cap (-K) = \{0\}$. We say that a convex subset $\mathbf{B}$ of a convex cone K is a base of K if $K = \mathbb{R}_+\mathbf{B}$ and $0 \notin \mathrm{cl}(\mathbf{B})$. As usual, we define

$$\mathrm{dom}\, g := \{z \in Y \mid g(z) \neq +\infty\},$$

$$\partial g(y) := \{y^* \in Y^* \mid y^*(z - y) \leq g(z) - g(y) \ \ \forall\, z \in Y\},$$

$$\mathrm{dom}\, \partial g := \{y \in \mathrm{dom}\, g \mid \partial g(y) \neq \emptyset\},$$

$$\sigma_M(z) := \sup\, \{l^*(z) \mid l^* \in M\} \text{ for } M \subseteq X^*, z \in X \text{ or } M \subseteq Y^*, z \in Y,$$

$$N(A, x) := \{v^* \in X^* \mid v^*(w - x) \leq 0 \ \ \forall\, w \in A\} \text{ for } x \in A \subseteq X, \text{ the normal cone to } A \text{ at } x,$$

$$K^* := \{y^* \in Y \mid y^*(k) \geq 0\ \forall\, k \in K\} \text{ for a convex cone } K \subseteq Y, \text{ the dual cone of } K,$$

$$d_S(y) := \inf\, \{\|s - y\| \mid s \in S\} \text{ for a subset } S \subset Y, y \in Y,$$

$$g'_+(y, z) := \inf\, \{\frac{1}{t}(g(y + tz) - g(y)) \mid t > 0\} \text{ for } y \in \mathrm{dom}\, g, z \in Y,$$

$$M \circ l := \{y^* \circ l \mid y^* \in M\} \text{ for } l : X \longrightarrow Y,\ M \subseteq Y^*.$$

2. Minimality conditions for convex composite functions

First, let us take a glance at the finite-dimensional case with g being a norm to find a suitable notion of minimality for convex composite functions $g \circ f$: Let $m \geq n$, $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, f Fréchet-differentiable at x with rank $f'(x) = n$.

Set $\varepsilon := \min\ \{\|f'(x)(h)\| \mid \|h\| = 1\}$ and choose $\delta > 0$ such that

$$\|f(w) - f(x) - f'(x)(w - x)\| \leq \frac{\varepsilon}{2}\, \|w - x\| \quad \text{for } w \in B(x, \delta).$$

One has for such w

$$\begin{aligned} \varepsilon\|w - x\| &\leq \|f'(x)(w - x)\| \\ &= \|f(x) - f(w) + f(w) - f(x) - f'(x)(w - x)\| \\ &\leq \|f(w) - f(x)\| + \frac{\varepsilon}{2}\|w - x\|. \end{aligned}$$

Hence $\|f_x(w)\| \geq \|f_x(x)\| + \frac{\varepsilon}{2}\|w - x\|$, where $f_x(\cdot) = f(\cdot) - f(x)$.

By this observation in the finite-dimensional case, it seems appropriate to look for (especially sufficient) minimality conditions for convex composite functions in more general settings that use the notion of sharp minima.

Definition 2.1. Let A be a subset of a normed space X, $h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $x \in A$ such that $h(x) \neq +\infty$. We say that h has a sharp minimum on A at x if there exist $r > 0$, $a > 0$ satisfying

$$h(v) \geq h(x) + a\|v - x\| \quad \text{for all } v \in B(x, r) \cap A.$$

Taking advantage of the fact that the subdifferential of a convex function g is always weak*-compact (and hence norm bounded since it is convex) at points of cor(dom g) ([9, Theorem 2.1]), a sufficient minimality condition can be given without any continuity assumption on g:

Theorem 2.1. ([9, Theorem 2.3]) *Let A be a subset of X, f Fréchet-differentiable at $x \in A$ and $f(x) \in \text{cor}(\text{dom } g) \cap \text{dom } \partial g$. If there exists $b > 0$ such that*

$$\sigma_{\partial g(f(x))}(f'(x)(w - x)) \ \geq \ b\|w - x\| \quad \text{for all} \ \ w \ \in \ A, \tag{2.1}$$

then $g \circ f$ has a sharp minimum on A at x.

Under certain conditions, statements equivalent to (2.1) can be found. Suppose that

$$bB(X^*) \subseteq \partial g(f(x)) \circ f'(x) + N(A, x)$$

holds for some $b > 0$ and let $u^* \in bB(X^*)$, $w \in A$. Then,

$$u^*(w - x) \leq u^*(w - x) - v^*(w - x) = y^*(f'(x)(w - x))$$

for some $y^* \in \partial g(f(x))$, $v^* \in N(A, x)$, and thus (2.1) is satisfied:

$$\sigma_{\partial g(f(x))}(f'(x)(w - x)) \geq \sigma_{bB(X^*)}(w - x) = b\|w - x\|.$$

However, if A is convex, the converse is also true, as we can see by applying the lemma below to

$$L = bB(X^*),\ M = \partial g(f(x)) \circ f'(x) \text{ and } C = A.$$

Lemma 2.1. *Suppose that $L, M \subseteq X^*$ (L and M are nonempty, M weak*-compact and convex), $C \subseteq X$ convex and $x \in C$ such that*

$$\sigma_M(v - x) \ \geq \ \sigma_L(v - x) \quad \text{for all} \ \ v \in C. \tag{2.2}$$

Then, $L \subseteq M + N(C, x)$.

Proof. Assume to the contrary that there exists $l^* \in L \setminus (M + N(C, x))$. Because of the convexity and weak*-closedness of $M + N(C, x)$, we can find some $u \in X$, $\varepsilon > 0$ such that

$$l^*(u) - \varepsilon \geq \sigma_M(u) + \sigma_{N(C,\,x)}(u) = \sigma_M(u)$$

(see [5, Theorem 3.18] and notice that $\sigma_{N(C,\,x)}(u) = 0$ since $N(C, x)$ is a cone). We have that $-N(C, x) = K^*$ for $K := \{k \in X \mid \exists\, t > 0\ (x + tk \in C)\}$, and thus u belongs to the (norm) closure of K. Hence, there exist $h \in K$, $s > 0$ satisfying

$$\sigma_M(h - u) < \frac{\varepsilon}{4},\ l^*(u - h) < \frac{\varepsilon}{4},\ x + sh \in C.$$

One gets

$$\begin{aligned} \sigma_M(h) &\leq \ \sigma_M(u) + \sigma_M(h - u) \\ &< \ l^*(u) - \varepsilon + \frac{\varepsilon}{4} \\ &< \ l^*(h) + \frac{\varepsilon}{4} - \varepsilon + \frac{\varepsilon}{4} \\ &\leq \ \sigma_L(h) - \frac{\varepsilon}{2} \end{aligned}$$

and

$$\sigma_M(x + sh - x) \ \leq \ \sigma_L(x + sh - x) - \frac{s\varepsilon}{2},$$

which contradicts (2.2). ☐

For a further statement, equivalent to (2.1), consider the Gerstewitz scalarization function (see [6])

$$\varphi_{D,\,p} : V \ni v \longmapsto \inf\{t \in \mathbb{R} \mid v \in tp - D\} \in \mathbb{R} \cup \{\pm\infty\},$$

where V is a topological vector space, $\emptyset \neq D \subset V$, $p \in V \setminus \{0\}$. It was generalized to

$$\psi_{S,\,e} : 2^Z \ni T \longmapsto \inf\{s \in \mathbb{R} \mid se + S \subseteq T + K\} \in \mathbb{R} \cup \{\pm\infty\},$$

where Z is a Hausdorff locally convex space, K is a solid convex cone, $e \in \operatorname{int} K$, and $\emptyset \neq S \subseteq Z$ ($S + K \neq Z$, i.e., S is K-proper), by Gutiérrez, Miglierina, Molho and Novo in [7]. Furthermore, Gutiérrez, Jiménez, Miglierina and Molho [8] proved that

$$\psi_{S,\,e}(T) \leq c \;\Leftrightarrow\; ce + S \subseteq \operatorname{cl}(T + K) \text{ for } T \subseteq Z,\ c \in \mathbb{R}.$$

Applying this result to $Z = X^*$, endowed with the weak*-topology, $K = N(A, x)$ and $S = bB(X^*)$, one derives if $e \in \operatorname{int} N(A, x)$,

$$\psi_{bB(X^*),\,e}(\partial g(f(x)) \circ f'(x)) \leq 0 \;\Leftrightarrow\; bB(X^*) \subseteq \partial g(f(x)) \circ f'(x) + N(A, x)$$

(since $\partial g(f(x)) \circ f'(x) + N(A, x)$ is weak*-closed) and thus a minimality condition, expressed by means of a scalarization function.

If, in addition, g is continuous at $f(x) \in \operatorname{dom} g$, a chain rule can be applied to $g \circ f$ ([3, Propositions 2.47 and 2.49]), and one gets altogether the new theorem below, which follows from our previous observations.

Theorem 2.2. *Let A be a convex subset of X, f Fréchet-differentiable at $x \in A$, $f(x) \in \operatorname{dom} g$ and g continuous at $f(x)$. Consider the following statements*:
1.) *$g \circ f$ has a sharp minimum on A at x.*
2.) *There exists $b > 0$ such that*

$$\sigma_{\partial g(f(x))}(f'(x)(w - x)) \geq b\|w - x\| \quad \text{for all } w \in A.$$

3.) *There exists $a > 0$ such that*

$$g'_+(f(x),\, f'(x)(v - x)) \geq a\|v - x\| \quad \text{for all } v \in A.$$

4.) $0 \in \operatorname{int}(\partial g(f(x)) \circ f'(x) + N(A, x))$
5.) *There exists $c > 0$ such that*

$$\psi_{cB(X^*),\,e}(\partial g(f(x)) \circ f'(x)) \leq 0 \quad (e \in N(A, x)).$$

Statements 1.), 2.), 3.) and 4.) *are equivalent. If $N(A, x)$ is assumed to be nonempty* and $e \in N(A, x)$, *then all five statements are equivalent.*

3. Efficiency Conditions in Vector Optimization

We start this section with a list of the three types of solution concepts for vector optimization problems, which will be used in the sequel:

Definition 3.1. Let Z be a locally convex Hausdorff space, $K \subseteq Z$ be a closed, convex and pointed cone, and $z \in M \subseteq Z$.
1.) (see [1, 12]) z is a **strictly (strong) efficient point** of M if for each neighborhood V of 0 there is a neighborhood U of 0 such that

$$(M - z) \cap (U - K) \subseteq V.$$

2.) (see [10]) Let **B** be a base of K. Then z is called a **Henig properly efficient point** of M with respect to **B** if for each convex neighborhood V of 0 such that $0 \notin \mathbf{B} + V$ there exists a convex neighborhood U of 0 such that $U \subseteq V$ and

$$\mathbb{R}_+(M - z) \cap -(\mathbf{B} + U) = \emptyset.$$

3.) (see [2, 12]) z is called a **superefficient point** of M if for each neighborhood V of 0 there is a neighborhood U of 0 such that

$$\mathbb{R}_+(M - z) \cap (U - K) \subseteq V.$$

Note that each strictly efficient point z is efficient in the usual sense, i.e.,

$$(M - z) \cap -K = \{0\}.$$

In general, each superefficient point is Henig properly (with respect to some base **B**), but the converse is not true.

Now, a sufficient efficiency condition for problem (1.1) can be given by using Theorem 2.1.

Theorem 3.1. *Let $K \subseteq Y$ be a closed, convex and pointed cone, and let f be Fréchet-differentiable at $x \in A \subseteq X$ and suppose that $\mathbb{R}_+(A - x)$ has a weakly compact base. If for each $w \in A \setminus \{x\}$ there exists some $l^* \in K^*$ such that*

$$l^*(f'(x)(w - x)) > 0,$$

then there are some $r > 0$, and $a > 0$ satisfying

$$d_{-K}(f(w) - f(x)) \geq a\|w - x\| \quad \text{for all } w \in A \cap B(x, r)$$

and $f(x)$ is a strictly efficient point of $f(A \cap B(x, r))$.

Proof. It follows from $\partial d_{-K}(0) = K^* \cap B(Y^*)$ ([4, Theorem 1]) that

$$d_{-K}(f'(x)(v - x)) > 0 \quad \text{for all } v \in A \setminus \{x\}. \tag{3.1}$$

Let **C** be a weakly compact base of $\mathbb{R}_+(A - x)$ and choose $c > 0$ such that $\|u\| \leq c$ for all $u \in \mathbf{C}$. By (3.1) and the compactness of **C**, there is some $b > 0$ satisfying

$$d_{-K}(f'(x)(u)) \geq b \quad \text{for all } u \in \mathbf{C}.$$

Let $w \in A \setminus \{x\}$ and $tu = w - x$ for some $t > 0$, $u \in \mathbf{C}$. Then,

$$d_{-K}(f'(x)(w - x)) = t d_{-K}(f'(x)(u)) \geq tb \geq \frac{tb}{c}\|u\| = \frac{b}{c}\|w - x\|.$$

Applying Theorem 2.1 to $g = d_{-K}(\cdot - f(x))$ and f, there exist $r > 0$, and $a > 0$ such that

$$d_{-K}(f(w) - f(x)) \geq a\|w - x\| \quad \text{for all } w \in A \cap B(x, r).$$

Now pick an arbitrary $\varepsilon > 0$ and choose $\gamma > 0$ satisfying

$$\|f(v) - f(x)\| < \varepsilon \quad \text{for } \|v - x\| < \frac{\gamma}{a} \tag{3.2}$$

and $\delta < 0$ satisfying

$$|d_{-K}(z)| < \gamma \quad \text{for } \|z\| < \delta.$$

Let $p \in (f(A \cap B(x, r)) - f(x)) \cap (B(0, \delta) - K) : p = f(w) - f(x) = z - k$, where $w \in A \cap B(x, r), z \in B(0, \delta), k \in K$. We can find $y^* \in K^* \cap B(Y^*)$ such that

$$y^*(z - k) = d_{-K}(z - k)$$

and get

$$\gamma > d_{-K}(z) \geq y^*(z) \geq y^*(z-k) = d_{-K}(z-k) = d_{-K}(f(w)-f(x)) \geq a\|w-x\|.$$

It follows from (3.2) that $p \in B(0, \varepsilon)$, i.e.,

$$(f(A \cap B(x, r)) - f(x)) \cap (B(0, \delta) - K) \subseteq B(0, \varepsilon).$$

□

Corollary 3.1. *If, in addition to the conditions of Theorem 3.1, f is continuously differentiable, then exists some $s > 0$ such that $f(x)$ is a superefficient point of $f(A \cap B(x,s))$.*

Proof. There are some $r \geq s > 0$, and $c > 0$ satisfying

$$\|f(v) - f(x)\| \leq c\|v - x\| \text{ for all } v \in B(x, s).$$

Therefore

$$d_{-K}(f(w) - f(x)) \geq a\|w - x\| \geq \frac{a}{c}\|f(w) - f(x)\| \text{ for all } w \in A \cap B(x, s).$$

From [11, Theorem 4.6], we can complete the proof. □

In the setting of Theorem 3.1, we can add a further equivalent statement to the ones of Theorem 2.2, provided that A is convex (compare [13, Theorem 3.3]).

Proposition 3.1. *Let K and f be as in Theorem 3.1 and $x \in A \subseteq X$.*
1.) $X^* = K^* \circ f'(x) + N(A, x) \Rightarrow d_{-K}(f(\cdot) - f(x))$ has a sharp minimum on A at x.
2.) If A is convex, the converse is also true.

Proof. 1.) We have that

$$X^* = \bigcup_{m \geq 1} (K^* \cap mB(Y^*)) \circ f'(x) + N(A, x).$$

Since $\partial d_{-K}(0) = K^* \cap B(Y^*)$, we have that $K^* \cap B(Y^*)$ is weak*-compact, so is $K^* \cap mB(Y^*)$ for $m \geq 1$.
Therefore, $(K^* \cap mB(Y^*)) \circ f'(x) + N(A, x)$ is (weak*-)closed. By the completeness of X^* as well as the Baire Category Theorem, one can find $n \geq 1, l^*, r > 0$ satisfying

$$B(l^*, r) \subseteq (K^* \cap nB(Y^*)) \circ f'(x) + N(A, x).$$

Furthermore, choose $p \geq 1$ such that

$$-l^* \in (K^* \cap pB(Y^*)) \circ f'(x) + N(A, x).$$

It follows that

$$B(0, r) \subseteq (K^* \cap nB(Y^*)) \circ f'(x) + (K^* \cap pB(Y^*)) \circ f'(x) + N(A, x).$$

Thus

$$B(0, \frac{r}{n+p}) \subseteq (K^* \cap B(Y^*)) \circ f'(x) + N(A, x),$$

which shows that

$$0 \in \operatorname{int}((K^* \cap B(Y^*)) \circ f'(x) + N(A, x)).$$

Now apply Theorem 2.1.

2.) We know from Theorem 2.2 that if $d_{-K}(f(\cdot) - f(x))$ has a sharp minimum on A at x, then

$$0 \in \operatorname{int}((K^* \cap B(Y^*)) \circ f'(x) + N(A, x)) \subseteq \operatorname{int}(K^* \circ f'(x) + N(A, x)).$$

Thus,

$$X^* = K^* \circ f'(x) + N(A, x),$$

since $K^* \circ f'(x)$ and $N(A, x)$ are cones.

□

Finally, we state a necessary efficiency condition.

Theorem 3.2. *Let $K \subseteq Y$ be a closed, convex and pointed cone, f Fréchet-differentiable at $x \in A \subseteq X$ such that $f'(x)$ has a bounded inverse. If $f(x)$ is a Henig properly efficient point of $f(A)$ with respect to a bounded base* **B**, *then there are some $r > 0$, $a > 0$ satisfying*

$$d_{-K}(f(v) - f(x)) \ \geq \ a\|v - x\| \quad \text{for all } v \in A \cap B(x, r).$$

resp.

$$f(v) - f(x) \ \notin \ a\|v - x\|B(Y) - K \quad \textit{for all} \ \ v \in \ A \cap B(x, r) \setminus \{x\}.$$

In particular, for any $w \in A \setminus \{x\}$ exists $k^* \in K^*$ such that

$$k^*(f'(x)(w - x)) > 0,$$

provided A is convex.

Proof. Choose $c > 0$ such that $\|u\| \leq c$ for all $u \in \mathbf{B}$ and set $q := \|f'(x)^{-1}\|$. Assume to the contrary that

$$\forall\, n \geq 1 \ \exists\, w_n \in \ A \cap \ B(x, \frac{1}{n}) \setminus \{x\} \ \text{ s.t. } f(w_n) - f(x) \in \ \frac{\|w_n - x\|}{nq} B(Y) - K,$$

i.e.

$$\forall\, n \geq 1 \ \exists\, w_n \in \ A \cap \ B(x, \frac{1}{n}) \setminus \{x\}, \ b_n \in B(Y), \ t_n > 0, \ k_n \in \mathbf{B}$$

$$\text{s.t. } f(w_n) - f(x) = \frac{\|w_n - x\|}{nq} b_n - t_n k_n. \tag{3.3}$$

Let $r(w_n - x) := f(w_n) - f(x) - f'(x)(w_n - x)$. Then

$$\begin{aligned} \frac{q}{\|w_n - x\|}(f'(x)(w_n - x) + r(w_n - x)) &= \frac{q}{\|w_n - x\|}(f(w_n) - f(x)) \\ &= \frac{1}{n} b_n - \frac{q t_n}{\|w_n - x\|} k_n. \end{aligned}$$

It follows that

$$\frac{q\|r(w_n - x)\|}{\|w_n - x\|} + \frac{1}{n} + \frac{qct_n}{\|w_n - x\|} \geq \frac{q\|r(w_n - x)\|}{\|w_n - x\|} + \frac{1}{n} + \frac{qt_n\|k_n\|}{\|w_n - x\|}$$
$$\geq \frac{q\|f'(x)(w_n - x)\|}{\|w_n - x\|}$$
$$= \frac{\|f'(x)^{-1}\|\|f'(x)(w_n - x)\|}{\|w_n - x\|}$$
$$\geq \frac{\|w_n - x\|}{\|w_n - x\|} = 1.$$

One gets for sufficiently large n

$$\frac{1}{2} \leq \frac{qct_n}{\|w_n - x\|}$$

resp.

$$2c \geq \frac{\|w_n - x\|}{qt_n} \geq 0.$$

Hence, there exists a converging subsequence

$$\left(\frac{\|w_{n_j} - x\|}{qt_{n_j}}\right)_{j \in \mathbb{N}}.$$

Thus,

$$\frac{\|w_{n_j} - x\|}{n_j qt_{n_j}} \longrightarrow 0 \quad \text{as} \quad j \longrightarrow +\infty. \tag{3.4}$$

It follows from (3.3) that

$$\frac{1}{t_{n_j}}(f(w_{n_j}) - f(x)) = \frac{\|w_{n_j} - x\|}{n_j qt_{n_j}} b_{n_j} - k_{n_j},$$

but this contradicts the Henig efficiency of $f(x)$ because of (3.4). Thus, there are some $r > 0$, and $a > 0$ satisfying

$$f(v) - f(x) \notin a\|v - x\|B(Y) - K \quad \text{for all} \quad v \in A \cap B(x, r) \setminus \{x\}$$

resp.

$$d_{-K}(f(v) - f(x)) \geq a\|v - x\| \quad \text{for all} \quad v \in A \cap B(x, r) \setminus \{x\}. \tag{3.5}$$

Now suppose that A is convex and $w \in A \setminus \{x\}$. From (3.5), one has

$$d_{-K}\left(\frac{1}{t}(f(x + t(w - x)) - f(x))\right) \geq a\|w - x\|$$

for sufficiently small $t > 0$. Therefore,

$$\max\{k^*(f'(x)(w - x)) \mid k^* \in K^* \cap B(Y^*)\} = d_{-K}(f'(x)(w - x)) \geq a\|w - x\|.$$

□

4. The conclusion

We considered the scalar optimization problem of minimizing convex composite functions and formulated equivalent minimality conditions, especially in the continuous case. By these results, we are able to find sufficient conditions for strictly efficient resp. superefficient points in vector optimization with respect to closed and convex ordering cones. Moreover, a necessary efficiency condition for Henig properly efficient points could be given.

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