PARALLEL COMPUTING PROXIMAL METHOD FOR NONSMOOTH CONVEX OPTIMIZATION WITH FIXED POINT CONSTRAINTS OF QUASI-NONEXPANSIVE MAPPINGS

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Abstract. We present a parallel computing proximal method for solving the problem of minimizing the sum of convex functions over the intersection of fixed point sets of quasi-nonexpansive mappings in a real Hilbert space. We also provide a convergence analysis of the method for constant and diminishing step sizes under certain assumptions as well as a convergence-rate analysis for a diminishing step size. Numerical comparisons show that the performance of the algorithm is comparable with existing subgradient methods.

Keywords. Fixed point; Nonsmooth convex optimization; Parallel computing; Proximal method; Quasi-nonexpansive mapping.

1. INTRODUCTION

In this paper, we consider the following problem [7, Problem 2.1] (see [3, 9, 10] for applications of Problem 1.1):

Problem 1.1. Let \( H \) be a real Hilbert space. Suppose that

\[(A1) \quad Q_i: H \to H \ (i \in \mathcal{I} := \{1, 2, \ldots, I\}) \text{ is quasi-firmly nonexpansive;}
(A2) \quad f_i: H \to \mathbb{R} \ (i \in \mathcal{I}) \text{ is convex and continuous with } \text{dom}(f_i) := \{x \in H: f_i(x) < +\infty\} = H.\]

Then,

\[\text{minimize } f(x) := \sum_{i \in \mathcal{I}} f_i(x) \text{ subject to } x \in X := \bigcap_{i \in \mathcal{I}} \text{Fix}(Q_i),\]

where one assumes that there exists a solution of Problem 1.1 (see Sections 2 and 4 for the details).

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Algorithms for solving this problem were proposed in [7, 9]. Reference [7] proposed parallel and incremental subgradient methods for solving Problem 1.1 and provided convergence as well as convergence-rate analyses. Reference [9, 10] proposed stochastic fixed point optimization algorithms for solving a convex stochastic optimization problem that minimizes the expectation of \( f_s \) over \( \text{Fix}(Q_1) \). The stochastic fixed point optimization algorithms can be applied to the classifier ensemble problem.

There are methods for solving Problem 1.1, where \( Q_i \) is taken to be a nonexpansive mapping, which is a stronger assumption than a quasi-nonexpansive mapping. Subgradient methods were presented in [4, 5, 6, 11], while proximal methods were presented in [8, 16].

In this paper, we present a parallel method for solving Problem 1.1. The method is obtained by combining the parallel method in [7] with the proximal method in [8]. We also present a convergence analysis for a constant step size and a diminishing step size. The analysis shows that the proposed method with a small constant step size may approximate a solution to Problem 1.1 (Theorem 3.1) and that with a diminishing step size it converges to a solution under certain assumptions (Theorem 3.2). We also provide a convergence-rate analysis with a diminishing step size (Theorem 3.3). Finally, we numerically compare the proposed method with the existing subgradient methods.

This paper is organized as follows. Section 2 gives the mathematical preliminaries. Section 3 presents the parallel proximal method for solving Problem 1.1 and analyzes its convergence. Section 4 numerically compares the behaviors of the proposed method and the existing ones. Section 5 concludes the paper with a brief summary.

2. Mathematical Preliminaries

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and its induced norm \( \| \cdot \| \). We use the standard notation \( \mathbb{N} \) for the natural numbers including zero and \( \mathbb{R}^N \) for the \( N \)-dimensional Euclidean space.

2.1. Quasi-nonexpansivity and demiclosedness. The fixed point set of a mapping \( Q : H \to H \) is denoted by

\[
\text{Fix}(Q) := \{ x \in H : Q(x) = x \}.
\]

\( Q \) is said to be quasi-nonexpansive [2, Definition 4.1(iii)] if \( \|Q(x) - y\| \leq \|x - y\| \) for all \( x \in H \) and for all \( y \in \text{Fix}(Q) \). When a quasi-nonexpansive mapping has one fixed point, its fixed point set is closed and convex [2, Proposition 2.6]. \( Q \) is said to be quasi-firmly nonexpansive [1, Section 3] if, for all \( x \in H \) and for all \( y \in \text{Fix}(Q) \),

\[
\|Q(x) - y\|^2 + \|(\text{Id} - Q)(x)\|^2 \leq \|x - y\|^2,
\]

where \( \text{Id}(x) := x \ (x \in H) \). Any quasi-firmly nonexpansive mapping satisfies the quasi nonexpansivity condition. Moreover, \( Q \) is quasi-firmly nonexpansive if and only if \( R := 2Q - \text{Id} \) is quasi-nonexpansive [2, Proposition 4.2], which implies that \( (1/2)(\text{Id} + R) \) is quasi-firmly nonexpansive when \( R \) is quasi-nonexpansive. Let \( x, u \in H \) and \( (x_n)_{n \in \mathbb{N}} \subset H \). \( \text{Id} - Q \) is said to be demiclosed if a weak convergence of \( (x_n) \) to \( x \) and \( \lim_{n \to +\infty} \|x_n - Q(x_n) - u\| = 0 \) imply \( x - Q(x) = u \). \( \text{Id} - Q \) is demiclosed when \( Q \) is nonexpansive, i.e., \( \|Q(x) - Q(y)\| \leq \|x - y\| \) \((x, y \in H) \) [2, Theorem 4.17]. The metric projection \( P_C \) onto a nonempty, closed convex subset \( C \)
of \( H \) is firmly nonexpansive, i.e., \( \|P_C(x) - P_C(y)\|^2 + \|(\text{Id} - P_C)(x) - (\text{Id} - P_C)(y)\|^2 \leq \|x - y\|^2 \) \((x, y \in H)\). Moreover, \( \text{Fix}(P_C) = C \) \([2, \text{Proposition 4.8, (4.8)}]\).

2.2. Convexity, proximal point, and subdifferentiability. A function \( f: H \to \mathbb{R} \) is said to be convex if, for all \( x, y \in H \) and for all \( \alpha \in [0, 1] \), \( f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \). A function \( f \) is said to be strictly convex \([2, \text{Definition 8.6}]\) if, for all \( x, y \in H \) and for all \( \alpha \in (0, 1) \), \( x \neq y \) implies \( f(\alpha x + (1 - \alpha)y) < \alpha f(x) + (1 - \alpha)f(y) \). \( f \) is strongly convex with constant \( \beta \) \([2, \text{Definition 10.5}]\) if there exists \( \beta > 0 \) such that, for all \( x, y \in H \) and for all \( \alpha \in (0, 1) \),

\[
f(\alpha x + (1 - \alpha)y) + \beta \alpha(1 - \alpha)/2 \|x - y\|^2 \leq \alpha f(x) + (1 - \alpha)f(y).
\]

Let \( f: H \to (-\infty, +\infty] \) be proper, lower semicontinuous, and convex. Then, the proximity operator of \( f \) \([2, \text{Definition 12.23}]\), \([14]\), denoted by \( \text{Prox}_f \), maps every \( x \in H \) to the unique minimizer of \( f(\cdot) + (1/2)\|x - \cdot\|^2 \); i.e.,

\[
\{ \text{Prox}_f(x) \} = \arg\min_{y \in H} \left[ f(y) + \frac{1}{2} \|x - y\|^2 \right] \quad (x \in H).
\]

The uniqueness and existence of \( \text{Prox}_f(x) \) are guaranteed for all \( x \in H \) \([2, \text{Definition 12.23}]\), \([13]\). We call \( \text{Prox}_f(x) \) the proximal point of \( f \) at \( x \). Let \( \text{dom}(f) := \{ x \in H : f(x) < +\infty \} \) be the domain of a function \( f: H \to (-\infty, +\infty] \).

The subdifferential \([2, \text{Definition 16.1}]\) of \( f \) is defined by

\[
\partial f(x) := \{ u \in H : f(y) \geq f(x) + \langle y - x, u \rangle \quad (y \in H) \} \quad (x \in H).
\]

We call \( u \in \partial f(x) \) the subgradient of \( f \) at \( x \).

**Proposition 2.1.** \([2, \text{Propositions 12.26, 12.27, 12.28, and 16.14}]\) Let \( f: H \to (-\infty, +\infty] \) be proper, lower semicontinuous, and convex. Then, the following conclusions hold:

(i) Let \( x, p \in H \). \( p = \text{Prox}_f(x) \) if and only if \( x - p \in \partial f(p) \) (i.e., \( \langle y - p, x - p \rangle + f(p) \leq f(y) \) for all \( y \in H \)).

(ii) \( \text{Prox}_f \) is firmly nonexpansive with \( \text{Fix}(\text{Prox}_f) = \arg\min_{x \in H} f(x) \).

(iii) If \( f \) is continuous at \( x \in \text{dom}(f) \), \( \partial f(x) \) is nonempty. Moreover, there exists \( \delta > 0 \) such that \( \partial f(B(x; \delta)) \) is bounded, where \( B(x; \delta) \) stands for a closed ball with center \( x \) and radius \( \delta \).

The following propositions will be used to prove the main theorems in this paper.

**Proposition 2.2.** \([15, \text{Lemma 3.1}]\) Suppose that \( (x_n)_{n \in \mathbb{N}} \subset H \) weakly converges to \( \hat{x} \in H \) and \( \hat{x} \neq \hat{x} \). Then, \( \liminf_{n \to +\infty} \|x_n - \hat{x}\| < \liminf_{n \to +\infty} \|x_n - \hat{x}\| \).

**Proposition 2.3.** \([2, \text{Theorem 9.1}]\) When \( f: H \to \mathbb{R} \) is convex, \( f \) is weakly lower semicontinuous if and only if \( f \) is lower semicontinuous.

**Proposition 2.4.** \([12, \text{Lemma 2.1}]\) Let \( (\Gamma_n)_{n \in \mathbb{N}} \subset \mathbb{R} \) and suppose that \( (\Gamma_n)_{j \in \mathbb{N}} \subset \mathbb{R} \) exists such that \( \Gamma_{n_j} < \Gamma_{n_{j+1}} \) for all \( j \in \mathbb{N} \). Define \( (\tau(n))_{n \geq n_0} \subset \mathbb{N} \) by \( \tau(n) := \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\} \) for some \( n_0 \in \mathbb{N} \). Then, \( (\tau(n))_{n \geq n_0} \) is increasing and \( \lim_{n \to +\infty} \tau(n) = +\infty \). Moreover, \( \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \) and \( \Gamma_n \leq \Gamma_{\tau(n)+1} \) for all \( n \geq n_0 \).
Algorithm 1 Parallel Proximal Method for solving Problem 1.1

Require: \((y_n)_{n \in \mathbb{N}} \subset (0, +\infty)\)
1: \(n \leftarrow 0, x_0 \in H\)
2: loop
3: \(\text{for } i = 1 \text{ to } I \text{ do}\)
4: \(x_{n,i} := Q_i(\text{Prox}_{\gamma_if_i}(x_n))\)
5: \(\text{end for}\)
6: \(x_{n+1} := \frac{1}{I} \sum_{i \in \mathcal{I}} x_{n,i}\)
7: \(n \leftarrow n + 1\)
8: \(\text{end loop}\)

3. THE PARALLEL PROXIMAL METHOD

Algorithm 1 is the proposed algorithm for solving Problem 1.1.

Let us consider a network system with \(I\) users and assume that user \(i\) has its own private objective function \(f_i\) and mapping \(Q_i\) and tries to minimize \(f_i\) over \(\text{Fix}(Q_i)\). Moreover, let us assume that each user can communicate with other users. Then, at iteration \(n\), each user can have \(x_n\) in common. Since user \(i\) has its own objective function \(f_i\), it computes \(y_{n,i} := \text{Prox}_{\gamma_if_i}(x_n)\). Moreover, user \(i\) has its own constraint set \(\text{Fix}(Q_i)\), with which it tries to find a fixed point of \(Q_i\) by using \(x_{n,i} := Q_i(y_{n,i})\). Since the users can communicate with each other, user \(i\) can receive all \(x_{n,i}\), and hence, user \(i\) can compute \(x_{n+1} := (1/I) \sum_{i \in \mathcal{I}} x_{n,i}\).

Let us compare Algorithm 1 with the existing parallel subgradient method [7, Algorithm 3.1] for solving Problem 1.1. The parallel subgradient method [7, Algorithm 3.1] is as follows:

\[
Q_{\alpha,i} := \alpha \text{Id} + (1 - \alpha)Q_i, \\
g_{n,i} \in \partial f_i(Q_{\alpha,i}(x_n)), \\
x_{n,i} := Q_{\alpha,i}(x_n) - \lambda_n g_{n,i}, \\
x_{n+1} := \frac{1}{I} \sum_{i \in \mathcal{I}} x_{n,i}. 
\]  

(3.1)

The difference between Algorithms 1 and (3.1) is the form of \(x_{n,i}\), i.e., Algorithm 1 uses \(x_{n,i} = Q_i(\text{Prox}_{\gamma_if_i}(x_n))\), while algorithm (3.1) uses \(x_{n,i} := Q_{\alpha,i}(x_n) - \lambda_n g_{n,i}\). Section 4 compares the behaviors of Algorithm 1 and algorithm (3.1) for concrete optimization problems.

First, we prove the following lemma.

**Lemma 3.1.** Suppose that (A1) and (A2) hold and define \(y_{n,i} := \text{Prox}_{\gamma_if_i}(x_n)\) for all \(i \in \mathcal{I}\) and for all \(n \in \mathbb{N}\). Then, Algorithm 1 satisfies that, for all \(x \in X\) and for all \(n \in \mathbb{N}\),

\[
\|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 - \frac{1}{I} \sum_{i \in \mathcal{I}} \left\{ \|x_n - y_{n,i}\|^2 + \|x_{n,i} - y_{n,i}\|^2 \right\} + \frac{2}{I} \gamma_n \sum_{i \in \mathcal{I}} (f_i(x) - f_i(y_{n,i})).
\]

**Proof.** Let \(x \in X\) and \(n \in \mathbb{N}\) be fixed arbitrarily. The definition of \(y_{n,i} := \text{Prox}_{\gamma_if_i}(x_n)\) and Proposition 2.1(i) ensure that, for all \(i \in \mathcal{I}\),

\[
\langle x - y_{n,i}, x_n - y_{n,i} \rangle \leq \gamma_n (f_i(x) - f_i(y_{n,i})),
\]
which, together with $2 \langle x, y \rangle = \| x \|^2 + \| y \|^2 - \| x - y \|^2 \ (x, y \in H)$, implies that
\[
2 \gamma_n (f_i(x) - f_i(y_{n,i})) \geq \| x - y_{n,i} \|^2 + \| x_n - y_{n,i} \|^2 - \| x - x_n \|^2.
\]
Accordingly, for all $i \in \mathcal{I}$,
\[
\| y_{n,i} - x \|^2 \leq \| x_n - x \|^2 - \| x_n - y_{n,i} \|^2 + 2 \gamma_n (f_i(x) - f_i(y_{n,i})).
\]
(3.2)
The definition of $x_{n,i} := Q_i(y_{n,i})$ and (A1) guarantee that, for all $i \in \mathcal{I}$,
\[
\| x_{n,i} - x \|^2 \leq \| y_{n,i} - x \|^2 - \| x_{n,i} - y_{n,i} \|^2.
\]
Hence, (3.2) and (3.3) imply that
\[
\| x_{n,i} - x \|^2 \leq \| x_n - x \|^2 - \| x_n - y_{n,i} \|^2 - \| x_{n,i} - y_{n,i} \|^2 + 2 \gamma_n (f_i(x) - f_i(y_{n,i})).
\]
Summing the above inequality from $i = 1$ to $i = I$ and the convexity of $\| \cdot \|^2$ ensure that
\[
\| x_{n+1} - x \|^2 \leq \sum_{i \in \mathcal{I}} \| x_{n,i} - x \|^2 \\
\leq 1 \| x_n - x \|^2 - \sum_{i \in \mathcal{I}} \left\{ \| x_n - y_{n,i} \|^2 + \| x_{n,i} - y_{n,i} \|^2 \right\} + 2 \gamma_n \sum_{i \in \mathcal{I}} (f_i(x) - f_i(y_{n,i})),
\]
which completes the proof. \( \square \)

The convergence analysis of Algorithm 1 depends on the following.

**Assumption 3.1.** The sequence $(y_{n,i})_{n \in \mathbb{N}} (i \in \mathcal{I})$ is bounded.

Assume that, for all $i \in \mathcal{I}$, $\arg\min_{y \in H} f_i(x) (= \text{Fix(Prox}_{f_i}) \neq \emptyset$ and $\text{Fix}(Q_i)$ is bounded. Then, we can choose in advance of running the algorithm a bounded, closed convex set $C_i$ (e.g., $C_i$ is a closed ball with a large enough radius) satisfying $C_i \supset \text{Fix}(Q_i)$. Accordingly, we can compute
\[
x_{n,i} := P_{C_i} \left[ Q_i(y_{n,i}) \right] \in C_i
\]
(3.4)instead of $x_{n,i}$ in Algorithm 1. The boundedness of $C_i \ (i \in \mathcal{I})$ implies that $(x_{n,i})_{n \in \mathbb{N}} (i \in \mathcal{I})$ is bounded. Accordingly, $(x_n)_{n \in \mathbb{N}}$ is also bounded. Moreover, Proposition 2.1(ii) ensures that, for all $i \in \mathcal{I}$, for all $n \in \mathbb{N}$, and for all $x \in \text{Fix(Prox}_{f_i})$, $\| y_{n,i} - x \| \leq \| x_n - x \|$. Hence, the boundedness of $(x_n)_{n \in \mathbb{N}}$ guarantees that $(y_{n,i})_{n \in \mathbb{N}} (i \in \mathcal{I})$ is bounded. Hence, it can be assumed that $(x_{n,i})_{n \in \mathbb{N}} (i \in \mathcal{I})$ in Algorithm 1 is as in (3.4) in place of Assumption 3.1.

We also have the following lemma.

**Lemma 3.2.** Suppose that (A1), (A2), and Assumption 3.1 hold. Then, $(x_{n,i})_{n \in \mathbb{N}} (i \in \mathcal{I})$ and $(x_n)_{n \in \mathbb{N}}$ are bounded.

**Proof.** Assumption (A1) ensures that, for all $x \in X$, for all $i \in \mathcal{I}$, and for all $n \in \mathbb{N}$,
\[
\| x_{n,i} - x \| \leq \| y_{n,i} - x \|,
\]
which, together with Assumption 3.1, implies that $(x_{n,i})_{n \in \mathbb{N}} (i \in \mathcal{I})$ is bounded. Hence, the definition of $x_n$ implies that $(x_n)_{n \in \mathbb{N}}$ is also bounded. \( \square \)
3.1. **Constant step-size rule.** The following is a convergence analysis of Algorithm 1 with a constant step size, which indicates that Algorithm 1 with a small constant step size may approximate a solution of Problem 1.1.

**Theorem 3.1.** Suppose that (A1), (A2), and Assumption 3.1 hold. Then, Algorithm 1 with $\gamma_n := \gamma > 0$ satisfies that

$$
\liminf_{n \to +\infty} \sum_{i \in I} \|y_{n,i} - Q_i(y_{n,i})\|^2 \leq IM_1 \gamma \quad \text{and} \quad \liminf_{n \to +\infty} \sum_{i \in I} f_i(y_{n,i}) \leq f^*,
$$

where $M_1 := \sup\{(2/1)\sum_{i \in I}(f_i(x) - f_i(y_{n,i})): n \in \mathbb{N}\} < +\infty$ for some $x \in X$ and $f^*$ is the optimal value of Problem 1.1.

**Proof.** Let $x \in X$ be fixed arbitrarily. The definition of $\partial f_i(x)$ and the Cauchy-Schwarz inequality imply that, for all $i \in I$, for all $n \in \mathbb{N}$, and for all $u_i \in \partial f_i(x)$,

$$
f_i(x) - f_i(y_{n,i}) \leq \langle x - y_{n,i}, u_i \rangle \leq \|y_{n,i} - x\| \|u_i\|,
$$

which, together with $\tilde{B} := \max_{i \in I} \sup\{\|y_{n,i} - x\|: n \in \mathbb{N}\} < +\infty$ (by Assumption 3.1), implies that

$$
M_1 := \sup \left\{ \frac{2}{I} \sum_{i \in I} (f_i(x) - f_i(y_{n,i})): n \in \mathbb{N} \right\} \leq 2\tilde{B} \max_{i \in I} \|u_i\| < +\infty. \quad (3.5)
$$

We first show that

$$
\liminf_{n \to +\infty} \sum_{i \in I} \left\{ \left\|x_n - y_{n,i}\right\|^2 + \left\|x_{n,i} - y_{n,i}\right\|^2 \right\} \leq IM_1 \gamma. \quad (3.6)
$$

If (3.6) does not hold, there exists $\delta > 0$ such that

$$
\liminf_{n \to +\infty} \sum_{i \in I} X_{n,i} > IM_1 \gamma + 2\delta.
$$

Accordingly, the property of the limit inferior of $(\sum_{i \in I} \{\|x_n - y_{n,i}\|^2 + \|x_{n,i} - y_{n,i}\|^2\})_{n \in \mathbb{N}}$ ensures that $n_0 \in \mathbb{N}$ exists such that, for all $n \geq n_0$,

$$
\sum_{i \in I} X_{n,i} > IM_1 \gamma + \delta. \quad (3.7)
$$

Accordingly, Lemma 3.1 with $\gamma_n := \gamma (n \in \mathbb{N})$ guarantees that, for all $n \geq n_0$,

$$
\|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 - \frac{1}{I} \sum_{i \in I} X_{n,i} + \frac{2}{I} \sum_{i \in I} (f_i(x) - f_i(y_{n,i}))
$$

$$
< \|x_n - x\|^2 - \frac{1}{I} (IM_1 \gamma + \delta) + M_1 \gamma
$$

$$
= \|x_n - x\|^2 - \frac{\delta}{I}
$$

$$
< \|x_{n_0} - x\|^2 - \frac{\delta}{I} (n + 1 - n_0).
$$

The right side of the above inequality approaches minus infinity as $n$ diverges. Hence, we have a contradiction. This implies that (3.6) holds. Therefore,

$$
\liminf_{n \to +\infty} \sum_{i \in I} \|y_{n,i} - x_{n,i}\|^2 = \liminf_{n \to +\infty} \sum_{i \in I} \|Q_i(y_{n,i})\|^2 \leq IM_1 \gamma.
$$
Next, we show that

$$\liminf_{n \to +\infty} \sum_{i \in \mathcal{I}} f_i(y_{n,i}) \leq f^*.$$  \hfill (3.8)

Assume that (3.8) does not hold. An argument similar to the one for obtaining (3.7) implies that there exist $\zeta > 0$ and $m_0 \in \mathbb{N}$ such that, for all $n \geq m_0$,

$$\sum_{i \in \mathcal{I}} f_i(y_{n,i}) - f^* > \zeta.$$

Lemma 3.1 thus ensures that, for all $n \geq m_0$ and for all $x^* \in X^* := \{x^* \in X : f(x^*) = f^* = \inf_{x \in X} f(x)\}$,

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \frac{2}{i} \gamma \left( f^* - \sum_{i \in \mathcal{I}} f_i(y_{n,i}) \right)$$

$$< \|x_n - x^*\|^2 - \frac{2}{i} \gamma \zeta$$

$$< \|x_{m_0} - x^*\|^2 - \frac{2}{i} \gamma \zeta (n + 1 - m_0),$$

which is a contradiction. Accordingly, (3.8) holds. This completes the proof. \hfill \square

3.2. **Diminishing step-size rule.** The following is a convergence analysis of Algorithm 1 with a diminishing step size.

**Theorem 3.2.** Suppose that (A1), (A2), and Assumption 3.1 hold and $\text{Id} - Q_i$ ($i \in \mathcal{I}$) is demiclosed.* Let $(x_n)_{n \in \mathbb{N}}$ be the sequence generated by Algorithm 1 with $(y_n)_{n \in \mathbb{N}}$ satisfying $\lim_{n \to +\infty} y_n = 0$ and $\sum_{n=0}^{+\infty} y_n = +\infty$. Then, there exists a subsequence of each of $(x_n)_{n \in \mathbb{N}}$, $(x_{n,i})_{n \in \mathbb{N}}$, and $(y_{n,i})_{n \in \mathbb{N}}$ ($i \in \mathcal{I}$) that weakly converges to a solution of Problem 1.1. Moreover, $(x_n)_{n \in \mathbb{N}}$, $(x_{n,i})_{n \in \mathbb{N}}$, and $(y_{n,i})_{n \in \mathbb{N}}$ ($i \in \mathcal{I}$) strongly converge to a unique solution of Problem 1.1 if one of the following holds:

(i) One $f_i$ is strongly convex;

(ii) $H$ is finite-dimensional, and one $f_i$ is strictly convex.

**Proof.** We consider two cases.

Case 1: Suppose that there exists $m_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ and for all $x^* \in X^*$, $n \geq m_0$ implies $\|x_{n+1} - x^*\| \leq \|x_n - x^*\|$, where $X^* := \{x^* \in X : f(x^*) = f^* = \inf_{x \in X} f(x)\}$. Then, there exists $c := \lim_{n \to +\infty} \|x_n - x^*\|$. Let $x^* \in X^*$ be fixed arbitrarily. Lemma 3.1, together with a discussion similar to that of (3.5), guarantees that there exists

$$M_2 := \sup \left\{ \frac{2}{i} \sum_{i \in \mathcal{I}} (f_i(x^*) - f_i(y_{n,i})) : n \in \mathbb{N} \right\} < +\infty$$

such that, for all $n \geq m_0$,

$$\frac{1}{i} \sum_{i \in \mathcal{I}} \left\{ \|x_n - y_{n,i}\|^2 + \|x_{n,i} - y_{n,i}\|^2 \right\} \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M_2 y_n. \hfill (3.9)$$

* See Section 4 for an example in which $Q_i$ is quasi-firmly nonexpansive and $\text{Id} - Q_i$ is demiclosed.
Accordingly, the conditions \( \lim_{n \to +\infty} y_n = 0 \) and \( c := \lim_{n \to +\infty} \|x_n - x^*\| \) mean that
\[
\lim_{n \to +\infty} \|x_n - y_{n,i}\| = 0 \quad \text{and} \quad \lim_{n \to +\infty} \|x_{n,i} - y_{n,i}\| = 0 \quad (i \in \mathcal{I}). \tag{3.10}
\]
From Lemma 3.1, for all \( x \in X \) and for all \( k \in \mathbb{N} \), we have
\[
\frac{2}{T} \sum_{i \in \mathcal{I}} \left( f_i(y_{k,i}) - f_i(x) \right) \leq \|x_k - x\|^2 - \|x_{k+1} - x\|^2, \tag{3.11}
\]
which implies that, for all \( n \in \mathbb{N} \) and for all \( x \in X \),
\[
\frac{2}{T} \sum_{k=0}^{n} \gamma_k N_k(x) \leq \|x_0 - x\|^2 - \|x_{n+1} - x\|^2 \leq \|x_0 - x\|^2.
\]
Accordingly, for all \( x \in X \),
\[
\frac{2}{T} \sum_{k=0}^{+\infty} \gamma_k N_k(x) < +\infty. \tag{3.12}
\]
Here, we show that, for all \( x \in X \),
\[
\liminf_{n \to +\infty} N_n(x) \leq 0. \tag{3.13}
\]
Assume that (3.13) does not hold; i.e., there exists \( x_0 \in X \) such that \( \liminf_{n \to +\infty} N_n(x_0) > 0 \). Then, \( m_1 \in \mathbb{N} \) and \( \theta > 0 \) exist such that, for all \( n \geq m_1 \), \( N_n(x_0) \geq \theta \). From (3.12) and \( \sum_{n=0}^{+\infty} \gamma_n = +\infty \), we have
\[
+\infty = \frac{2\theta}{T} \sum_{k=m_1}^{+\infty} \gamma_k \leq \frac{2}{T} \sum_{k=m_1}^{+\infty} \gamma_k N_k(x_0) < +\infty,
\]
which is a contradiction. Hence, (3.13) holds, i.e., for all \( x \in X \),
\[
\liminf_{n \to +\infty} \sum_{i \in \mathcal{I}} f_i(y_{n,i}) \leq \sum_{i \in \mathcal{I}} f_i(x) := f(x). \tag{3.14}
\]
The definition of \( u_{n,i} \in \partial f_i(x_n) \) and the Cauchy-Schwarz inequality ensure that, for all \( i \in \mathcal{I} \) and for all \( n \in \mathbb{N} \),
\[
f_i(x_n) - f_i(y_{n,i}) \leq \langle x_n - y_{n,i}, u_{n,i} \rangle \leq \|x_n - y_{n,i}\| \|u_{n,i}\|.
\]
Proposition 2.1(iii) and the boundedness of \( (x_n)_{n \in \mathbb{N}} \) (see also Lemma 3.2) guarantee that there exists \( B_1 := \max_{i \in \mathcal{I}} \sup \|u_{n,i}\| : n \in \mathbb{N} \| < +\infty \) such that, for all \( n \in \mathbb{N} \),
\[
f(x_n) = \sum_{i \in \mathcal{I}} f_i(x_n) \leq B_1 \sum_{i \in \mathcal{I}} \|x_n - y_{n,i}\| + \sum_{i \in \mathcal{I}} f_i(y_{n,i}).
\]
Therefore, (3.10) and (3.14) lead to the finding that, for all \( x \in X \),
\[
\liminf_{n \to +\infty} f(x_n) \leq B_1 \liminf_{n \to +\infty} \sum_{i \in \mathcal{I}} \|x_n - y_{n,i}\| + \liminf_{n \to +\infty} \sum_{i \in \mathcal{I}} f_i(y_{n,i}) \leq f(x). \tag{3.15}
\]
Accordingly, a subsequence \((x_{n_l})_{l \in \mathbb{N}} \) of \((x_n)_{n \in \mathbb{N}} \) exists such that, for all \( x \in X \),
\[
\lim_{l \to +\infty} f(x_{n_l}) = \liminf_{n \to +\infty} f(x_n) \leq f(x). \tag{3.16}
\]
Since \((x_{n_l})_{l \in \mathbb{N}} \) is bounded (see also Lemma 3.2), there exists \((x_{n_{l,m}})_{m \in \mathbb{N}} \) \( \subset (x_{n_l})_{l \in \mathbb{N}} \) such that \((x_{n_{l,m}})_{m \in \mathbb{N}} \) weakly converges to \( x_* \in H \). From (3.10), \((y_{n_{l,m},i})_{i \in \mathcal{I}} \) weakly converges to \( x_* \).
Hence, (3.10) and the demiclosedness of $\text{Id} - Q_i$ ensure that $x_\ast \in \text{Fix}(Q_i)$ ($i \in \mathcal{I}$), i.e., $x_\ast \in X$. Proposition 2.3 ensures that the continuity and convexity of $f$ (by (A2)) imply that $f$ is weakly lower semicontinuous, which means that

$$f(x_\ast) \leq \liminf_{m \to +\infty} f(x_{n_{lm}}).$$

Therefore, (3.16) leads to the finding that, for all $x \in X$,

$$f(x_\ast) \leq \liminf_{m \to +\infty} f(x_{n_{lm}}) = \lim_{m \to +\infty} f(x_{n_{lm}}) \leq f(x),$$

that is, $x_\ast \in X^\ast$. Let us take another subsequence $(x_{n_{lk}})_{k \in \mathbb{N}}$ (⊂ $(x_{n_i})_{i \in \mathbb{N}}$) such that $(x_{n_{lk}})_{k \in \mathbb{N}}$ weakly converges to $x_{**} \in H$. A discussion similar to the one for obtaining $x_\ast \in X^\ast$ guarantees that $x_{**} \in X^\ast$. Here, it is proven that $x_\ast = x_{**}$. Now, let us assume that $x_\ast \neq x_{**}$. Then, the existence of $c := \lim_{n \to +\infty} \|x_n - x^\ast\|$ ($x^\ast \in X^\ast$) and Proposition 2.2 imply that

$$c = \lim_{m \to +\infty} \|x_{n_{lm}} - x_\ast\| \leq \lim_{m \to +\infty} \|x_{n_{lm}} - x_{**}\| = \lim_{n \to +\infty} \|x_n - x_{**}\| = \lim_{k \to +\infty} \|x_{n_{lk}} - x_\ast\| = c,$$

which is a contradiction. Hence, $x_\ast = x_{**}$. Accordingly, any subsequence of $(x_{n_i})_{i \in \mathbb{N}}$ converges weakly to $x_\ast \in X^\ast$; i.e., $(x_{n_i})_{i \in \mathbb{N}}$ converges weakly to $x_\ast \in X^\ast$. This means that $x_\ast$ is a weak cluster point of $(x_n)_{n \in \mathbb{N}}$ and belongs to $X^\ast$. A discussion similar to the one for obtaining $x_\ast = x_{**}$ guarantees that there is only one weak cluster point of $(x_n)_{n \in \mathbb{N}}$, so we can conclude that, in Case 1, $(x_n)_{n \in \mathbb{N}}$ weakly converges to a point in $X^\ast$.

Case 2: Suppose that, for all $m \in \mathbb{N}$, there exist $n \in \mathbb{N}$ and $x_0^\ast \in X^\ast$ such that $n \geq m$ and

$$\|x_{n+1} - x_0^\ast\| > \|x_n - x_0^\ast\|.$$  

This implies that $(x_{n_j})_{j \in \mathbb{N}}$ (⊂ $(x_n)_{n \in \mathbb{N}}$) exists such that, for all $j \in \mathbb{N}$,

$$\|x_{n_j+1} - x_0^\ast\| > \|x_{n_j} - x_0^\ast\| =: \Gamma_{n_j}.$$  

Proposition 2.4 thus guarantees that $m_1 \in \mathbb{N}$ exists such that, for all $n \geq m_1$, $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, where $\tau(n)$ is defined as in Proposition 2.4. From Lemma 3.1 (see also (3.9)), for all $n \geq m_1$, we have

$$\frac{1}{l} \sum_{i \in \mathcal{I}} \left( \|x_{\tau(n)} - y_{\tau(n),i}\|^2 + \|x_{\tau(n),i} - y_{\tau(n),i}\|^2 \right) \leq \Gamma_{\tau(n)}^2 - \Gamma_{\tau(n)+1}^2 + \tilde{M}_2 \gamma_{\tau(n)}$$

$$\leq \tilde{M}_2 \gamma_{\tau(n)},$$

where

$$\tilde{M}_2 := \sup \left\{ \frac{2}{l} \sum_{i \in \mathcal{I}} (f_i(x^\ast) - f_i(y_{\tau(n),i})) : n \in \mathbb{N} \right\}$$

is finite by Assumption 3.1 (see also (3.5)). Hence, the condition $\lim_{n \to +\infty} \gamma_{\tau(n)} = 0$ implies that

$$\lim_{n \to +\infty} \|x_{\tau(n)} - y_{\tau(n),i}\| = 0 \quad \text{and} \quad \lim_{n \to +\infty} \|x_{\tau(n),i} - y_{\tau(n),i}\| = 0 \quad (i \in \mathcal{I}).$$  

(3.17)

From (3.11), for all $n \geq m_1$,

$$\frac{2}{l} \gamma_{\tau(n)} N_{\tau(n)}(x_0^\ast) \leq \Gamma_{\tau(n)}^2 - \Gamma_{\tau(n)+1}^2 \leq 0,$$
which, together with \( \gamma_{\tau(n)} \geq 0 \ (n \geq m_1) \), implies that \( N_{\tau(n)}(x_0^*) \leq 0 \). Accordingly,

\[
\limsup_{n \to +\infty} \sum_{i \in \mathcal{I}} f_i (y_{\tau(n),i}) \leq f^*.
\]

An argument, which is similar to the one for obtaining (3.15), together with (3.17), implies that

\[
\limsup_{n \to +\infty} f (x_{\tau(n)}) \leq f^*.
\]

Choose a subsequence \( (x_{\tau(n)})_{k \in \mathbb{N}} \) of \( (x_{\tau(n)})_{n \geq m_1} \) arbitrarily. Then,

\[
\limsup_{k \to +\infty} f (x_{\tau(n_k)}) \leq \limsup_{n \to +\infty} f (x_{\tau(n)}) \leq f^*.
\]  \hfill (3.18)

The boundedness of \( (x_{\tau(n_k)})_{k \in \mathbb{N}} \) ensures that \( (x_{\tau(n_k)})_{l \in \mathbb{N}} (\subset (x_{\tau(n_k)})_{k \in \mathbb{N}}) \) exists such that \( (x_{\tau(n_k)})_{l \in \mathbb{N}} \) weakly converges to \( x^* \in H \). Then, (3.17) and the demiclosedness of \( \text{Id} - Q_i \) ensure that \( x^* \in X \). Moreover, Proposition 2.3 and (3.18) guarantee that

\[
f(x^*) \leq \liminf_{l \to +\infty} f (x_{\tau(n_k)}) \leq \limsup_{l \to +\infty} f (x_{\tau(n_k)}) \leq f^*,
\]

that is, \( x^* \in X^* \). Therefore, \( (x_{\tau(n_k)})_{l \in \mathbb{N}} \) weakly converges to \( x^* \in X^* \). From Cases 1 and 2, there exists a subsequence of \( (x_n)_{n \in \mathbb{N}} \) that weakly converges to a point in \( X^* \).

Suppose that assumption (i) in Theorem 3.2 holds. The strong convexity of \( f := \sum_{i \in \mathcal{I}} f_i \) implies that \( X^* \) consists of one point, denoted by \( x^* \). In Case 1, the strong convexity of \( f \) guarantees that there exists \( \beta > 0 \) such that, for all \( \alpha \in (0, 1) \) and for all \( l \in \mathbb{N} \),

\[
(\beta/2) \alpha (1 - \alpha) \|x_{n_l} - x^*\|^2 \leq \alpha f(x_{n_l}) + (1 - \alpha) f^* - f(\alpha x_{n_l} + (1 - \alpha)x^*).
\]

Accordingly, from the existence of \( c := \lim_{n \to +\infty} \|x_n - x^*\| \) and (3.16), we have

\[
\frac{\beta}{2} \alpha (1 - \alpha) \lim_{l \to +\infty} \|x_{n_l} - x^*\|^2 \leq \lim_{l \to +\infty} (\alpha f(x_{n_l}) + (1 - \alpha) f^*) + \limsup_{l \to +\infty} (-f(\alpha x_{n_l} + (1 - \alpha)x^*))
\]

\[
\leq f^* - \liminf_{l \to +\infty} f(\alpha x_{n_l} + (1 - \alpha)x^*),
\]

which, together with the weak convergence of \( (x_{n_l})_{l \in \mathbb{N}} \) to \( x^* \) and Proposition 2.3, implies that

\[
\frac{\beta}{2} \alpha (1 - \alpha) \lim_{l \to +\infty} \|x_{n_l} - x^*\|^2 \leq f^* - f(\alpha x^* + (1 - \alpha)x^*) = 0.
\]

Hence, \( (x_{n_l})_{l \in \mathbb{N}} \) strongly converges to \( x^* \). Therefore, from [2, Theorem 5.11], the whole sequence \( (x_n)_{n \in \mathbb{N}} \) strongly converges to \( x^* \). From (3.10), \( (x_{n,l})_{n \in \mathbb{N}} \) and \( (y_{n,l})_{n \in \mathbb{N}} \) \( (i \in \mathcal{I}) \) strongly converge to \( x^* \). In Case 2, the strong convexity of \( f \) leads to the deduction that, for all \( \alpha \in (0, 1) \) and for all \( l \in \mathbb{N} \),

\[
\frac{\beta}{2} \alpha (1 - \alpha) \limsup_{l \to +\infty} \|x_{\tau(n_{l,k})} - x^*\|^2 \leq \alpha \limsup_{l \to +\infty} f(x_{\tau(n_{l,k})}) + (1 - \alpha) f^*
\]

\[
- \liminf_{l \to +\infty} f(\alpha x_{\tau(n_{l,k})} + (1 - \alpha)x^*).
\]
The weak convergence of \((x_{\tau(n_k)})_{i \in \mathbb{N}}\) to \(x^*\), the weakly lower semicontinuity of \(f\) (by Proposition 2.3), and (3.18) imply that
\[
\frac{\beta}{2} \alpha (1 - \alpha) \limsup_{i \to +\infty} \left\| x_{\tau(n_k)} - x^* \right\|^2 \leq f^* - f(\alpha x^* + (1 - \alpha)x^*) = 0,
\]
which implies that \((x_{\tau(n_k)})_{i \in \mathbb{N}}\) strongly converges to \(x^*\). When another subsequence \((x_{\tau(n_m)})_{m \in \mathbb{N}}\) \((\subset (x_{\tau(n_k)})_{k \in \mathbb{N}}\) can be chosen, a discussion similar to the one for showing the weak convergence of \((x_{\tau(n_k)})_{i \in \mathbb{N}}\) to a point in \(X^*\) guarantees that \((x_{\tau(n_m)})_{m \in \mathbb{N}}\) also weakly converges to a point in \(X^*\). Accordingly, a discussion similar to the one for showing the strong convergence of \((x_{\tau(n_k)})_{i \in \mathbb{N}}\) to \(x^*\) ensures that \((x_{\tau(n_m)})_{m \in \mathbb{N}}\) strongly converges to the same \(x^*\). Hence, it is guaranteed that \((x_{\tau(n_k)})_{k \in \mathbb{N}}\) strongly converges to \(x^*\). Since \((x_{\tau(n_k)})_{k \in \mathbb{N}}\) is an arbitrary subsequence of \((x_{\tau(n)})_{n \geq m_1}\), \((x_{\tau(n)})_{n \geq m_1}\) strongly converges to \(x^*\); i.e.,
\[
\lim_{n \to +\infty} \Gamma_{\tau(n)} = \lim_{n \to +\infty} \left\| x_{\tau(n)} - x^* \right\| = 0.
\]
Accordingly, Proposition 2.4 ensures that
\[
\limsup_{n \to +\infty} \left\| x_n - x^* \right\| \leq \limsup_{n \to +\infty} \Gamma_{\tau(n)+1} = 0,
\]
which implies that, in Case 2, the whole sequence \((x_n)_{n \in \mathbb{N}}\) converges to \(x^*\). Moreover, Lemma 3.1 and \(\lim_{n \to +\infty} \gamma_n = 0\) imply that
\[
\lim_{n \to +\infty} \left\| x_n - y_{n,i} \right\| = \lim_{n \to +\infty} \left\| x_{n,i} - y_{n,i} \right\| = 0, \quad (i \in \mathcal{I}).
\]
Therefore, \((x_{n,i})_{n \in \mathbb{N}}\) and \((y_{n,i})_{n \in \mathbb{N}}\) \((i \in \mathcal{I})\) converge to \(x^*\).

Suppose that assumption (ii) in Theorem 3.2 holds. Let \(x^* \in X^*\) be the unique solution to Problem 1.1. In Case 1, it is guaranteed that \((x_n)_{n \in \mathbb{N}}\) converges to \(x^* \in X^*\). From (3.10), \((x_{n,i})_{n \in \mathbb{N}}\) and \((y_{n,i})_{n \in \mathbb{N}}\) \((i \in \mathcal{I})\) strongly converge to \(x^*\). Moreover, in Case 2, the convergence of \((x_{\tau(n_k)})_{i \in \mathcal{I}}\) to \(x^*\) is guaranteed. A discussion similar to the one for showing the strong convergence of \((x_{\tau(n)})_{n \geq m_1}\) to \(x^*\) ensures that \((x_{\tau(n)})_{n \geq m_1}\) converges to \(x^* \in X^*\). Proposition 2.4 thus guarantees that the whole sequence \((x_n)_{n \in \mathbb{N}}\) converges to \(x^*\). Lemma 3.1 and \(\lim_{n \to +\infty} \gamma_n = 0\) imply that
\[
\lim_{n \to +\infty} \left\| x_n - y_{n,i} \right\| = \lim_{n \to +\infty} \left\| x_{n,i} - y_{n,i} \right\| = 0, \quad (i \in \mathcal{I}).
\]
Therefore, \((x_{n,i})_{n \in \mathbb{N}}\) and \((y_{n,i})_{n \in \mathbb{N}}\) \((i \in \mathcal{I})\) converge to \(x^*\). This completes the proof.

The following is a convergence-rate analysis of Algorithm 1 with a diminishing step size.

**Theorem 3.3.** Suppose that the assumptions in Theorem 3.1 hold and a monotone decreasing sequence \((\gamma_n)_{n \in \mathbb{N}}\) satisfies \(\lim_{n \to +\infty} \gamma_n = 0\), \(\lim_{n \to +\infty} (n \gamma_n)^{-1} = 0\), \(\sum_{n=0}^{+\infty} \gamma_n = +\infty\), and \(\lim_{n \to +\infty} n^{-1} \sum_{k=0}^{n-1} \gamma_k = 0\). Then, Algorithm 1 satisfies that, for all \(n \geq 1\),
\[
\sum_{i \in \mathcal{I}} \left( \frac{1}{n} \sum_{k=0}^{n-1} \left\| y_{k,i} - Q_i(y_{k,i}) \right\| \right)^2 \leq \frac{1}{n} \left\| x_0 - x \right\|^2 + \frac{\tilde{M}_1}{n} \sum_{k=0}^{n-1} \gamma_k,
\]
and
\[
\sum_{i \in \mathcal{I}} f_i \left( \frac{1}{n} \sum_{k=0}^{n-1} y_{k,i} \right) \leq f^* + \frac{IB}{2n \gamma_n},
\]
where \( x^* \) is a solution of Problem 1.1,

\[
\tilde{M}_1 := \sup \left\{ 2 \sum_{i \in \mathcal{I}} (f_i(x^*) - f_i(y_{n,i})) : n \in \mathbb{N} \right\} < +\infty,
\]

and

\[
B := \sup \left\{ \|x_n - x^*\|^2 : n \in \mathbb{N} \right\} < +\infty.
\]

**Proof.** Let \( x^* \in X^* \). Lemma 3.1 implies that, for all \( n \geq 1 \),

\[
\frac{1}{n} \sum_{i \in \mathcal{I}} \sum_{k=0}^{n-1} \left\{ \|x_k - y_{k,i}\|^2 + \|x_{k,i} - y_{k,i}\|^2 \right\} \leq \|x_0 - x\|^2 + \frac{\tilde{M}_1}{n} \sum_{k=0}^{n-1} \gamma_k,
\]

which in turn implies that

\[
\sum_{i \in \mathcal{I}} \left( \frac{n}{n} \sum_{k=0}^{n-1} \|x_{k,i} - y_{k,i}\|^2 \right) \leq \frac{1}{n} \sum_{i \in \mathcal{I}} \sum_{k=0}^{n-1} \left\{ \|x_k - y_{k,i}\|^2 + \|x_{k,i} - y_{k,i}\|^2 \right\} \leq \frac{1}{n} \left( \frac{n}{n} \sum_{k=0}^{n-1} \gamma_k \right) \leq \frac{\tilde{M}_1}{n} \sum_{k=0}^{n-1} \gamma_k.
\]

Lemma 3.1 indicates that, for all \( k \in \mathbb{N} \),

\[
\sum_{i \in \mathcal{I}} f_i(y_{k,i}) - f^* \leq \frac{1}{2\gamma_k} \left\{ \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 \right\}.
\]

Summing the above inequality from \( k = 0 \) to \( k = n - 1 \) implies that, for all \( n \geq 1 \),

\[
\frac{1}{n} \sum_{k=0}^{n-1} \sum_{i \in \mathcal{I}} f_i(y_{k,i}) - f^* \leq \frac{1}{2n} \sum_{k=0}^{n-1} \sum_{i \in \mathcal{I}} \frac{1}{\gamma_k} \left\{ \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 \right\}.
\]

The definition of \( X_n \) means that

\[
X_n = \frac{\|x_0 - x^*\|}{\gamma_0} + \sum_{k=1}^{n-1} \left( \frac{\|x_k - x^*\|^2}{\gamma_k} - \frac{\|x_k - x^*\|^2}{\gamma_{k-1}} \right) - \frac{\|x_n - x^*\|^2}{\gamma_{n-1}},
\]

which, together with \( \gamma_n \leq \gamma_{n-1} \) \((n \geq 1)\) and \( B := \sup \{\|x_n - x^*\|^2 : n \in \mathbb{N} \} < +\infty \) (by Lemma 3.2), implies that

\[
X_n \leq \frac{B}{\gamma_0} + B \sum_{k=1}^{n-1} \left( \frac{1}{\gamma_k} - \frac{1}{\gamma_{k-1}} \right) = \frac{B}{\gamma_{n-1}} \leq \frac{B}{\gamma_n}.
\]

The convexity of \( f_i \) thus ensures that, for all \( n \geq 1 \),

\[
\sum_{i \in \mathcal{I}} f_i \left( \frac{1}{n} \sum_{k=0}^{n-1} y_{k,i} \right) - f^* \leq \frac{IB}{2n\gamma_n},
\]

which completes the proof. \( \square \)
Let us consider the rate of convergence of Algorithm 1 with $\gamma_n := n^{-1/2}$ ($n \geq 1$). The step size $(\gamma_n)_{n \geq 1}$ is monotone decreasing and satisfies $\lim_{n \to +\infty} \gamma_n = 0$, $\lim_{n \to +\infty} (n\gamma_n)^{-1} = 0$, and $\sum_{n=0}^{+\infty} \gamma_n = +\infty$. Moreover, the Cauchy-Schwarz inequality and $\sum_{k=0}^{n-1} k^{-1} \leq 1 + \ln n$ mean that

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\sqrt{k}} \leq \frac{\sqrt{n}}{n} \sum_{k=0}^{n-1} \frac{1}{\sqrt{k}} \leq \sqrt{\frac{1 + \ln n}{n}},$$

which implies that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \gamma_k = 0.$$ 

Theorem 3.3 indicates that Algorithm 1 with $\gamma_n := n^{-1/2}$ satisfies that, for all $n \geq 1$,

$$\sum_{i \in \mathcal{I}} \left( \frac{1}{n} \sum_{k=0}^{n-1} \|y_{k,i} - Q_i(y_{k,i})\|^2 \right) = O \left( \sqrt{\frac{1 + \ln n}{n}} \right)$$

and

$$\sum_{i \in \mathcal{I}} f_i \left( \frac{1}{n} \sum_{k=0}^{n-1} y_{k,i} \right) \leq f^* + \frac{IB}{2\sqrt{n}},$$

where $O$ stands for the Landau notation (see [10] for a convergence rate analysis of stochastic approximation methods).

4. Numerical Comparisons

Let us compare the performance of Algorithm 1 with the one of the existing parallel subgradient method (PSM) [7, Algorithm 3.1] (see (3.1)) and incremental subgradient method (ISM) [7, Algorithm 4.1] for the following problem (see also [7, Problem 5.1]): Let $a_{i,j} > 0$, $b_{i,j}, d_i \in \mathbb{R}$ ($i \in \mathcal{I}, j = 1, 2, \ldots, N$), and $c_i := (c_{i,j})_{j=1}^N \in \mathbb{R}^N$ ($i \in \mathcal{I}$) with $c_{i,j} > 0$. Then,

minimize $f(x) := \sum_{i \in \mathcal{I}} f_i(x)$ subject to $x \in X := \bigcap_{i \in \mathcal{I}} \text{Fix}(Q_i) = \bigcap_{i \in \mathcal{I}} \text{lev}_{\leq 0} g_i$, \hspace{1cm} (4.1)

where $f_i : \mathbb{R}^N \to \mathbb{R}$ and $Q_i : \mathbb{R}^N \to \mathbb{R}^N$ are defined for all $x := (x_j)_{j=1}^N \in \mathbb{R}^N$ by

$$f_i(x) := \sum_{j=1}^N a_{i,j} |x_j - b_{i,j}|$$

and

$$Q_i(x) := \begin{cases} x - \frac{g_i(x)}{\|z_i(x)\|^2} z_i(x), & \text{if } g_i(x) > 0, \\ x, & \text{if } x \in \text{lev}_{\leq 0} g_i := \{ x \in \mathbb{R}^N : g_i(x) \leq 0 \}, \end{cases}$$

$g_i : \mathbb{R}^N \to \mathbb{R}$ is defined for all $x \in \mathbb{R}^N$ by

$$g_i(x) := \begin{cases} (c_i, x) + d_i, & \text{if } (c_i, x) > -d_i, \\ 0, & \text{otherwise}, \end{cases}$$
and $z_i(x)$ is any vector in $\partial g_i(x)$. The above mapping $Q_i$ is called the subgradient projection related to $g_i$. $Q_i$ satisfies quasi-firm nonexpansivity, and $I - Q_i$ satisfies the demiclosedness condition [1, Lemma 3.1].

The experiment was conducted on a MacBook Air (13-inch, 2017) with a 1.8 GHz Intel (R) Core (TM) i5 CPU processor, 8 GB, 1600 MHz DDR3 memory, and Mac OS Catalina (Version 10.15) operating system. PSM, ISM, and Algorithm 1 were written in Python 3.7.4 with the NumPy 1.17.2 package. We set $I = 256$ and $N = 1000$ and randomly chose $a_{i,j} \in (0,100]$, $b_{i,j} \in [-100,100)$, $d_i \in [-1,0)$, and $c_{i,j} \in [-0.5,0.5)$. The stopping condition was $n = 10000$. The step sizes were as follows:

Constant step sizes: $\gamma_n := 10^{-1}, 10^{-3}$, 

Diminishing step sizes: $\gamma_n := \frac{10^{-1}}{n+1}, \frac{10^{-3}}{n+1}$.

The performance measures were as follows: for $n \in \mathbb{N}$,

$$F_n := \frac{1}{10} \sum_{s=1}^{10} \sum_{i \in \mathcal{I}} f_i(x_n(s))$$

and

$$D_n := \frac{1}{10} \sum_{s=1}^{10} \sum_{i \in \mathcal{I}} \|x_n(s) - Q_i(x_n(s))\|,$$

where $(x_n(s))_{n \in \mathbb{N}}$ is the sequence generated by each of the three algorithms with the randomly chosen initial point $x_0(s) \in [0,1)^N$ ($s = 1, 2, \ldots, 10$). If $(D_n)_{n \in \mathbb{N}}$ converges to 0, the algorithms converge to a point in $X$.

Figure 1 shows that the algorithms with $\gamma_n = \lambda_n = 10^{-1}$ did not converge to a point in $X$. Figure 2 indicates that, although the values of $D_{10000}$ generated by the algorithms with $\gamma_n = \lambda_n = 10^{-3}$ were less than those generated by the algorithms with $\gamma_n = \lambda_n = 10^{-1}$, the algorithms with $\gamma_n = \lambda_n = 10^{-3}$ did not converge to a point in $X$. These results imply that it would be difficult to set an appropriate constant step size in advance.
Meanwhile, Figures 3 and 4 show that Algorithm 1 with diminishing step sizes $\gamma_n = 10^{-1}/(n + 1), 10^{-3}/(n + 1)$ converged to a point in $X$, as guaranteed by Theorem 3.2. These figures also show that $F_n$ remains stable. Accordingly, from Theorem 3.2, Algorithm 1 converged to a solution of problem (4.1). Figures 3 and 4 also indicate that Algorithm 1 performs comparably to PSM and ISM.
5. The Conclusion

This paper presented a parallel proximal method for solving the minimization problem of the sum of convex functions over the intersection of fixed point sets of quasi-nonexpansive mappings in a real Hilbert space. It also provided convergence and convergence-rate analyses. Numerical comparisons showed that the performance of the algorithm is almost the same as those of the existing methods.

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References


