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STRONG CONVERGENCE OF THE TSENG EXTRAGRADIENT METHOD FOR SOLVING VARIATIONAL INEQUALITIES

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Abstract. In this paper, we propose a new iterative algorithm for finding a common element of the set of solutions of the variational inequality problem involving monotone operators and the set of fixed points problems involving quasi-nonexpansive mappings with a demiclosedness property in a Hilbert space. We combine Tseng extragradient method with the Mann approximation method and Yamada's algorithm. The main advantages of our algorithm are that the construction of solutions and the knowledge of the Lipschitz constant of the operators does not require to be known. We proved that the sequence generated by the new algorithm is strongly convergent. Finally, we provide a numerical example to show the effectiveness of the proposed algorithm.

Keywords. Fixed point problem; Tseng extragradient method; Self-adaptive method; Variational inequality problem; Yamada's algorithm.

1. INTRODUCTION

Consider the following variational inequality problem (VIP) of finding a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

where $A : H \to H$ is a single-valued mapping and C is a nonempty closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The solution set of VIP (1.1) is denoted by $VI(C,A)$.

Variational inequality problems play an important role in many fields, such as, mathematical programming, engineering sciences, optimization, economics and so on. Because of its wide applications, it has received much attentions from many authors and has been extensively studied in both finite dimensional spaces and infinite dimensional spaces. There are two general approaches for solving the variational inequality problems: regularized methods and projection methods. Projection methods, which play an important role, can solve the variational inequality problems efficiently.

It is easy to know that $x^* \in C$, x^* solves VIP (1.1) if and only if x^* satisfies the fixed point equation:

$$x^* = P_C(I - \lambda A)x^*, \tag{1.2}$$

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where I is the identify operator, $P_C : H \to C$ is the metric projection and λ is an arbitrary positive constant. If operator A is η-strongly monotone and L-Lipschitz continuous (η, L are two positive constants), then it is easy to verify that $P_C(I-\lambda A)$ is a strict contraction with the constant $\sqrt{1-\lambda(2\eta-\lambda L^2)}$ for any $\lambda \in (0, 2\eta/L^2)$. From Banach's fixed point theorem, (1.2) has a unique solution. The sequence $\{x_n\}$ generated by

$$x_{n+1} = P_C(I-\lambda A)x_n,$$

converges strongly to the unique solution of VIP. This method was named as the gradient method, which is the simplest projection method for solving the variational inequality problems. However, the convergence of this method requires a slightly strong assumption on operator A, which means that operator A is strongly monotone or inverse strongly monotone.

To avoid the strong assumption, Korpelevich [9] proposed the extragradient method for solving saddle points with the following manner:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n). \end{cases}$$

Recently, this method was extended to the situation that operator A only needs to be monotone and L-Lipschitz continuous in Hilbert spaces. If $VI(C,A) \neq \emptyset$ and $\lambda \in (0, 1/L)$, then $\{x_n\}$ converges weakly to an element of $VI(C,A)$.

Regarding the extragradient method, we know this method requires to calculate two projections from H onto C per iteration. In fact, if set C is not "simple", it is difficult to calculate P_C, which might affect the computation complexity and effort. In order to avoid such situation, many authors had improved it in various ways. The first extension of the extragradient method is known as the Tseng extragradient method by Tseng [17]. For each $n \in \mathbb{N}$, the sequence $\{x_n\}$ is generated in the following form:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(A y_n - A x_n). \end{cases} \tag{1.3}$$

Compared with the extragradient method, Tseng's extragradient method also requires to calculate two steps, but it only keeps one step about the calculation of the projection. And operator A is monotone and L-Lipschitz continuous. If $VI(C,A) \neq \emptyset$ and $\lambda \in (0, 1/L)$, the sequence $\{x_n\}$ generated by (1.3) converges weakly to an element of $VI(C,A)$. Recently, this method has attracted attentions by many authors, see, e.g., [2, 12, 15, 16, 18] and the references therein. The second method for solving the variational inequality problems, the second projection onto the feasible set C replaced by a projection onto a special constructible half-space, is the so-called subgradient extragradient method by Censor, Gibali and Reich[3]. The algorithm is of the form:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{w \in H \mid \langle x_n - \lambda A x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \end{cases} \tag{1.4}$$

Compared with the extragradient method, the second step of the subgradient extragradient method is easier to calculate. The conditions of operator A is monotone and L-Lipschitz continuous. If $VI(C,A) \neq \emptyset$ and $\lambda \in (0, 1/L)$, the sequence $\{x_n\}$ generated by (1.4) converges weakly to an element of $VI(C,A)$.

From the extragradient method, Tseng's extragradient method, subgradient extragradient method, we find that they have something in common, that is, they all need to know the Lipschitz constant of operator A. However, in some cases, the Lipschitz constant of operator A is unknown or difficult to determine. Keeping the conditions that the operator is still monotone and Lipschitz continuous, many authors proposed related algorithms, see, e.g., [1, 7, 8, 14, 19] and the references therein. In these algorithms, an adaptive method is essential. From these methods, one projection is still needed. Is there a way to avoid the calculation of projections and can it also solve the variational inequality problems? Certainly, there is one method. In 2011, Yamada [23] introduced an algorithm which was named the hybrid steepest descent method with the following form:

$$x_{n+1} = (I - \mu\alpha_n F)Tx_n, \quad \forall n \in \mathbb{N}, \tag{1.5}$$

where $F : H \to H$ is k-Lipschitz continuous, η-strongly monotone and T is a nonexpansive mapping. Under the condition that $Fix(T) \neq \emptyset$, the sequence $\{x_n\}$ generated by (1.5) converges strongly to a point x^*, which is a unique solution of the variational inequality $\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in Fix(T)$.

The so-called fixed point problem is to

$$\text{find } x^* \in H \text{ such that } Tx^* = x^*.$$

Now, many authors are interested in finding a common solution of the variational inequality problems and the fixed point problems; see, e.g, [4, 11, 13, 20] for related iterative algorithms.

Motivated by the results above, we are concerned with Tseng's extragradient method and Yamada's algorithm. We propose our new iterative algorithm for finding a common element of the set of solutions of the variational inequality problem involving monotone operators and the set of fixed points of a quasi-nonenpansive mapping with a demiclosedness property in a Hilbert space. Our new iterative algorithm combines Tseng's extragradient method, Mann's method and Yamada's algorithm. Besides, it is worth noting that our algorithm does not require the Lipschitz constant of operator A by using self-adaptive method.

The paper is organized as follows. In Section 2, we list some related definitions and lemmas that will be used in our paper. In Section 3, we introduce our algorithm in detail and study its strong convergence. In Section 4, a numerical example is presented to conclude the effectiveness of our algorithm.

2. Preliminaries

In this section, we list some definitions and lemmas which will be used in our paper. Assume that H is a real Hilbert space and C is a nonempty closed convex subset of H.

For any $x \in H$, there is a unique point $z \in C$ such that $\|z - x\| \leq \|y - x\|$ for any $y \in C$. The vector z can be denoted by $P_C x$, which means $\|P_C x - x\| \leq \|y - x\|$, where $P_C : H \to C$ is the metric projection. On P_C, it has following properties.

Lemma 2.1. [6] *The metric projection P_C has following properties:*
(i) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle, \quad \forall x, y \in H;$
(ii) $\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, y \in C;$
(iii) $\|P_C x - P_C y\| \leq \|x - y\|, \quad \forall x, y \in H;$
(iv) $\langle x - P_C x, y - P_C x \rangle \leq 0, \quad \forall x \in H, y \in C.$

Now, we give the definitions of the operator, which will be used in this paper, respectively.

Definition 2.1. Let $A : H \to H$ be an operator. Then
(i) A is η-strongly monotone with $\eta > 0$ if

$$\langle x-y, Ax-Ay\rangle \geq \eta \|x-y\|^2, \quad \forall x,y \in H;$$

(ii) A is monotone if

$$\langle x-y, Ax-Ay\rangle \geq 0, \quad \forall x,y \in H;$$

(iii) A is μ-inverse monotone (μ-ism) with $\mu > 0$ if

$$\langle x-y, Ax-Ay\rangle \geq \mu \|Ax-Ay\|^2, \quad \forall x,y \in H;$$

(iv) A is L-Lipschitz continuous with $L > 0$ if

$$\|Ax-Ay\| \leq L\|x-y\|, \quad \forall x,y \in H;$$

(v) A is nonexpansive if

$$\|Ax-Ay\| \leq \|x-y\|, \quad \forall x,y \in H;$$

(vi) A is quasi-nonexpansive if

$$\|Ax-p\| \leq \|x-p\|, \quad \forall x \in H, p \in Fix(A),$$

where $Fix(A) \neq \emptyset$.

Remark 2.1. (i) P_C is nonexpansive;
(ii) It is well that every nonexpansive mapping with a nonempty set of fixed points is a quasi-nonexpansive mapping. However, a quasi-nonexpansive mapping may not be a nonexpansive mapping.

We now give an example that a quasi-nonexpansive mapping is not a nonexpansive mapping.

Example 2.1. [5] Let $C := \{x \in l_\infty : \|x\|_\infty \leq 1\}$. Define $T : C \to C$ by $Tx := (0, x_1^2, x_2^2, x_3^2, \cdots)$ for $x = (x_1, x_2, x_3, \cdots)$ in C. It is clear that T is continuous. Moreover, $Tx^* = x^*$ if and only if $x^* = 0$. Furthermore, for all $x \in C$,

$$\begin{aligned}
\|Tx - x^*\|_\infty &= \|Tx\|_\infty \\
&= \|(0, x_1^2, x_2^2, x_3^2, \cdots)\|_\infty \\
&\leq \|(0, x_1, x_2, x_3, \cdots)\|_\infty \\
&= \|x\|_\infty \\
&= \|x - x^*\|_\infty.
\end{aligned}$$

By Definition 2.1 (vi), we know that T is a quasi-nonexpansive mapping. However, for $x = (\frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \cdots)$ and $y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \cdots)$, $x, y \in C$,

$$\|x-y\|_\infty = \|(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots)\|_\infty = \frac{1}{2}$$

and

$$\|Tx - Ty\|_\infty = \|(0, \frac{7}{12}, \frac{7}{12}, \frac{7}{12}, \cdots)\|_\infty = \frac{7}{12} > \frac{1}{2} = \|x-y\|_\infty.$$

By Definition 2.1 (v), we know that T is not a nonexpansive mapping. Therefore, a quasi-nonexpansive mapping may not be a nonexpansive mapping.

Lemma 2.2. [21] *Let H be a real Hilbert space. For all $x,y \in H$ and $\lambda \in \mathbb{R}$, one has*
(i) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle$;
(ii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$,
(iii) $\|x+y\|^2 = \|x\|^2 + 2\langle y, x+y\rangle$.

Lemma 2.3. [6] *Let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Then $I-T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, where $\{x_n\}$ is any sequence in H, imply $x \in Fix(T)$.*

Lemma 2.4. [10] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number of n in the set $\{1,2,\cdots,k\}$ such that $a_n < a_{n+1}$.

Lemma 2.5. [22] *Assume that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n,\ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} b_n \leq 0$.
Then $\lim_{n\to\infty} a_n = 0$.

In our paper, we also use the following notations:

- $\to$ denotes strong convergence.
- $\rightharpoonup$ denotes weak convergence.
- $\omega_w(x_n) := \{x \mid \textit{there exists } \{x_{n_k}\}_{j=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty} \textit{ such that } x_{n_k} \rightharpoonup x\}$ denotes the weak cluster point set of $\{x_n\}_{n=1}^{\infty}$.

3. Main results

In this section, we propose a new iterative algorithm with self-adaptive techniques for finding a common element of the set of solutions of the variational inequality problem involving monotone operators and the set of fixed points of a quasi-nonexpansive mapping with a demiclosedness property in a Hilbert space. In our new iterative algorithm, we combine Tseng's extragradient method, Mann's method and Yamada's method. The information of the Lipschitz constant of operator A does not require to be known.

Throughout this paper, we assume that operator $A : H \to H$ is monotone and Lipschitz continuous, operator $U : H \to H$ is a quasi-nonexpansive mapping such that $I-U$ is demiclosed at zero, operator $F : H \to H$ is η-strongly monotone and k-Lipschitz continuous, where $\eta > 0$ and $k > 0$ and $Fix(U) \bigcap VI(C,A) \neq \emptyset$.

Our new iterative algorithm is described as follows.

Algorithm 3.1.
Step 1: Given $\gamma > 0$, $l \in (0,1)$, $\mu \in (0,1)$. Let $x_0 \in H$ be arbitrary.
Step 2: Compute

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \ldots\}$ satisfying

$$\lambda \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \tag{3.1}$$

Step 3: Compute

$$t_n = (1-\beta_n)z_n + \beta_n U z_n,$$

where

$$z_n = y_n - \lambda_n(Ay_n - Ax_n).$$

Step 4: Compute

$$x_{n+1} = (1-\tau\alpha_n F)t_n.$$

Set $n := n+1$ and return to Step 2.

Lemma 3.1. *The sequence $\{\lambda_n\}$ generated by Algorithm 3.1 is well defined and*

$$\min\{\gamma, \frac{\mu l}{L}\} \leq \lambda_n \leq \gamma.$$

Proof. It is obvious that the sequence $\{\lambda_n\}$ is well defined. In fact, since A is L-Lipschitz continuous, we have

$$\|Ax_n - A(P_C(x_n - \lambda Ax_n))\| \leq L\|x_n - P_C(x_n - \lambda Ax_n)\|,$$

which can be transformed into

$$\frac{\mu}{L}\|Ax_n - A(P_C(x_n - \lambda Ax_n))\| \leq \mu\|x_n - P_C(x_n - \lambda Ax_n)\|.$$

It implies that, for all $\lambda \leq \frac{\mu}{L}$, (3.1) is established. So $\{\lambda_n\}$ is well defined. Because λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \ldots\}$ satisfying

$$\lambda \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|,$$

where $\gamma > 0$, $l \in (0,1)$, we have $\lambda_n \leq \gamma$. If $\lambda_n = \gamma$, then the lemma is proved. If $\lambda_n < \gamma$, for all $\lambda \leq \frac{\mu}{L}$, (3.1) is established. For $\lambda_n = \gamma l^p$ $(p > 1)$, we have $\lambda_n l^{-1} > \frac{\mu}{L}$. Hence we obtain $\lambda_n > \frac{\mu l}{L}$. Clearly,

$$\min\{\gamma, \frac{\mu l}{L}\} \leq \lambda_n \leq \gamma.$$

□

Lemma 3.2. *Let $\{z_n\}$ be a sequence generated by Algorithm 3.1. Then, for all $p \in VI(C,A)$, we have*

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - (1-\mu^2)\|x_n - y_n\|^2. \tag{3.2}$$

Proof. Let $p \in VI(C,A)$. It follows that

$$\begin{aligned}
\|z_n - p\|^2 &= \|y_n - p\|^2 + \lambda_n^2\|Ay_n - Ax_n\|^2 - 2\lambda_n\langle Ay_n - Ax_n, y_n - p\rangle \\
&= \|(y_n - x_n) + (x_n - p)\|^2 + \lambda_n^2\|Ay_n - Ax_n\|^2 - 2\lambda_n\langle Ay_n - Ax_n, y_n - p\rangle \\
&= \|x_n - y_n\|^2 + \|x_n - p\|^2 + 2\langle y_n - x_n, x_n - p\rangle + \lambda_n^2\|Ay_n - Ax_n\|^2 \\
&\quad -2\lambda_n\langle Ay_n - Ax_n, y_n - p\rangle \\
&= \|x_n - y_n\|^2 + \|x_n - p\|^2 + 2\langle y_n - x_n, y_n - p\rangle \\
&\quad -2\langle y_n - x_n, y_n - x_n\rangle + \lambda_n^2\|Ay_n - Ax_n\|^2 - 2\lambda_n\langle Ay_n - Ax_n, y_n - p\rangle \\
&= \|x_n - y_n\|^2 + \|x_n - p\|^2 + 2\langle y_n - x_n, y_n - p\rangle - 2\|x_n - y_n\|^2 \\
&\quad +\lambda_n^2\|Ay_n - Ax_n\|^2 - 2\lambda_n\langle Ay_n - Ax_n, y_n - p\rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\langle y_n - x_n, y_n - p\rangle + \lambda_n^2\|Ay_n - Ax_n\|^2 \\
&\quad -2\lambda_n\langle Ay_n - Ax_n, y_n - p\rangle. \qquad (3.3)
\end{aligned}$$

Since $y_n = P_C(x_n - \lambda_n Ax_n)$, by Lemma 2.1 (iv), we have

$$\langle y_n - x_n + \lambda_n Ax_n, y_n - p\rangle \leq 0,$$

which implies that

$$\langle y_n - x_n, y_n - p\rangle \leq -\lambda_n\langle Ax_n, y_n - p\rangle. \qquad (3.4)$$

From (3.3) and (3.4), we obtain

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - 2\lambda_n\langle Ax_n, y_n - p\rangle + \lambda_n^2\|Ay_n - Ax_n\|^2 \\
&\quad -2\lambda_n\langle Ay_n - Ax_n, y_n - p\rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 + \lambda_n^2\|Ay_n - Ax_n\|^2 - 2\lambda_n\langle Ay_n, y_n - p\rangle \\
&\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 + \mu^2\|x_n - y_n\|^2 - 2\lambda_n\langle Ay_n, y_n - p\rangle \\
&\leq \|x_n - p\|^2 - (1-\mu^2)\|x_n - y_n\|^2.
\end{aligned}$$

Therefore,

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - (1-\mu^2)\|x_n - y_n\|^2, \quad \forall p \in VI(C,A).$$

This completes the proof. □

Theorem 3.1. *Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the following conditions:*
(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(ii) $\{\beta_n\} \subset [a,b] \subset (0,1)$.
Assume $\tau \in (0, \frac{2\eta}{k^2})$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to an element $p = P_{Fix(U)\bigcap VI(C,A)}(I - \tau F)p$.

Proof. We prove the mapping $I - \tau F : H \to H$ is a contractive mapping.

$$\begin{aligned}
\|(I-\tau F)x - (I-\tau F)y\|^2 &= \|x-y\|^2 + \tau^2\|Fx - Fy\|^2 - 2\tau\langle x-y, Fx-Fy\rangle \\
&\leq \|x-y\|^2 + k^2\tau^2\|x-y\|^2 - 2\eta\tau\|x-y\|^2 \\
&= (1-\tau(2\eta - \tau k^2))\|x-y\|^2 \\
&\leq (1-h)^2\|x-y\|^2,
\end{aligned}$$

where $h=\frac{1}{2}\tau(2\eta-\tau k^2)$, which means that $I-\tau F: H\to H$ is a contractive mapping with constant $1-h$. Thus, $P_{Fix(U)\bigcap VI(C,A)}(I-\beta F)$ is a contraction mapping. By Banach contraction principle, there exists a unique point $p\in H$ such that $p=P_{Fix(U)\bigcap VI(C,A)}(I-\beta F)p$. Let $p\in Fix(U)\bigcap VI(C,A)$.

$$\begin{aligned}\|(I-\tau\alpha_nF)x-(I-\tau\alpha_nF)y\|^2 &= \|x-y\|^2+\tau^2\alpha_n^2\|Fx-Fy\|^2-2\tau\alpha_n\langle x-y,Fx-Fy\rangle\\ &\leq \|x-y\|^2+k^2\tau^2\alpha_n^2\|x-y\|^2-2\eta\tau\alpha_n\|x-y\|^2\\ &= (1-\tau\alpha_n(2\eta-\tau\alpha_nk^2))\|x-y\|^2\\ &\leq (1-h\alpha_n)^2\|x-y\|^2, \end{aligned}\tag{3.5}$$

where $h=\frac{1}{2}\tau(2\eta-\tau k^2)$, which means $I-\tau\alpha_nF: H\to H$ also is a contractive mapping with constant $1-h\alpha_n$. By Lemma 3.2, we have $\|z_n-p\|\leq\|x_n-p\|$. Since U is quasi-nonexpansive, we have

$$\begin{aligned}\|t_n-p\| &= \|(1-\beta_n)(z_n-p)+\beta_n(Uz_n-p)\|\\ &\leq (1-\beta_n)\|z_n-p\|+\beta_n\|Uz_n-p\|\\ &\leq \|z_n-p\|. \end{aligned}\tag{3.6}$$

So, by (3.5) and (3.6), we obtain

$$\begin{aligned}\|x_{n+1}-p\| &= \|(I-\tau\alpha_nF)t_n-(I-\tau\alpha_nF)p-\tau\alpha_nFp\|\\ &\leq \|(I-\tau\alpha_nF)t_n-(I-\tau\alpha_nF)p\|+\tau\alpha_n\|Fp\|\\ &\leq (1-h\alpha_n)\|t_n-p\|+\tau\alpha_n\|Fp\|\\ &\leq (1-h\alpha_n)\|z_n-p\|+h\alpha_n\frac{\tau}{h}\|Fp\|\\ &\leq (1-h\alpha_n)\|x_n-p\|+h\alpha_n\frac{\tau}{h}\|Fp\|\\ &\leq \max\{\|x_n-p\|,\frac{\tau}{h}\|Fp\|\}\\ &\leq \cdots\\ &\leq \max\{\|x_0-p\|,\frac{\tau}{h}\|Fp\|\}. \end{aligned}$$

This implies that sequence $\{x_n\}$ is bounded. Consequently, sequences $\{y_n\}$, $\{z_n\}$, $\{t_n\}$ and $\{(I-\tau F)x_n\}$ are bounded.

Next, we prove that

$$\|x_{n+1}-p\|^2\leq\alpha_n\|(I-\tau F)t_n-p\|^2+\|x_n-p\|^2-\beta_n(1-\beta_n)\|Uz_n-z_n\|^2.$$

Indeed, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(I-\tau F)t_n+(1-\alpha_n)t_n-p\|^2\\ &= \|\alpha_n((I-\tau F)t_n-p)+(1-\alpha_n)(t_n-p)\|^2\\ &= \alpha_n\|(I-\tau F)t_n-p\|^2+(1-\alpha_n)\|t_n-p\|^2-\alpha_n(1-\alpha_n)\|\tau Ft_n\|^2. \end{aligned}\tag{3.7}$$

Since U is quasi-nonexpansive, we get

$$\begin{aligned} \|t_n - p\|^2 &= \|(1-\beta_n)(z_n - p) + \beta_n(Uz_n - p)\|^2 \\ &= (1-\beta_n)\|z_n - p\|^2 + \beta_n\|Uz_n - p\|^2 - \beta_n(1-\beta_n)\|Uz_n - z_n\|^2 \\ &\leq (1-\beta_n)\|z_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\|Uz_n - z_n\|^2 \\ &= \|z_n - p\|^2 - \beta_n(1-\beta_n)\|Uz_n - z_n\|^2. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\begin{aligned} & \|x_{n+1} - p\|^2 \\ \leq\ & \alpha_n\|(I-\tau F)t_n - p\|^2 + \|t_n - p\|^2 - \alpha_n(1-\alpha_n)\|\tau F t_n\|^2 \\ \leq\ & \alpha_n\|(I-\tau F)t_n - p\|^2 + \|z_n - p\|^2 - \beta_n(1-\beta_n)\|Uz_n - z_n\|^2 - \alpha_n(1-\alpha_n)\|\tau F t_n\|^2 \\ \leq\ & \alpha_n\|(I-\tau F)t_n - p\|^2 + \|x_n - p\|^2 - \beta_n(1-\beta_n)\|Uz_n - z_n\|^2. \end{aligned}$$

Therefore, we have

$$\|x_{n+1} - p\|^2 \leq \alpha_n\|(I-\tau F)t_n - p\|^2 + \|x_n - p\|^2 - \beta_n(1-\beta_n)\|Uz_n - z_n\|^2. \tag{3.9}$$

Next, we prove that if $\liminf_{k\to\infty}(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|) \geq 0$, then

$$\omega_w\{x_{n_k}\} \subset Fix(U)\bigcap VI(C,A).$$

Because the sequence $\{x_n\}$ is bounded, there exists subsequence $\{x_{n_k}\}$. Next, we use $\omega_w\{x_{n_k}\}$ to denote the set of all weak cluster points of the sequence $\{x_{n_k}\}$.

We consider two aspects:
(i) $\omega_w\{x_{n_k}\} \subset VI(C,A)$.
In fact, we have

$$\begin{aligned} \|x_{n_k+1} - p\| - \|x_{n_k} - p\| &= \|(I-\tau\alpha_{n_k})t_{n_k} - p\| - \|x_{n_k} - p\| \\ &\leq (1-h\alpha_{n_k})\|t_{n_k} - p\| + \tau\alpha_{n_k}\|Fp\| - \|x_{n_k} - p\| \\ &\leq (1-h\alpha_{n_k})\|z_{n_k} - p\| + \tau\alpha_{n_k}\|Fp\| - \|x_{n_k} - p\|. \end{aligned} \tag{3.10}$$

Because of $\|z_n - p\| \leq \|x_n - p\|$, by (3.10), we have

$$\begin{aligned} 0 &\leq \liminf_{k\to\infty}(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|) \\ &\leq \liminf_{k\to\infty}(\|z_{n_k} - p\| - \|x_{n_k} - p\|) \\ &\leq \limsup_{k\to\infty}(\|z_{n_k} - p\| - \|x_{n_k} - p\|) \\ &\leq 0, \end{aligned}$$

which implies that

$$\lim_{k\to\infty}(\|z_{n_k} - p\| - \|x_{n_k} - p\|) = 0. \tag{3.11}$$

By (3.2) and (3.11), we obtain that

$$\lim_{k\to\infty}\|x_{n_k} - y_{n_k}\| = 0. \tag{3.12}$$

Therefore, by (3.12), we get $\omega_w\{x_{n_k}\} \subset VI(C,A)$.

(ii) $\omega_w\{x_{n_k}\} \subset Fix(U)$.
In fact, from (3.9), we have

$$\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2 \le \alpha_{n_k}\|(I-\tau F)t_{n_k}-p\|^2-\beta_{n_k}(1-\beta_{n_k})\|Uz_{n_k}-z_{n_k}\|^2. \quad (3.13)$$

Since $\lim_{n\to\infty}\alpha_n = 0$, we have from (3.13) that

$$\begin{aligned} 0 &\le \liminf_{k\to\infty}(\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2) \\ &\le -\limsup_{k\to\infty}\beta_{n_k}(1-\beta_{n_k})\|Uz_{n_k}-z_{n_k}\|^2 \\ &\le -a(1-b)\limsup_{k\to\infty}\|Uz_{n_k}-z_{n_k}\|^2. \end{aligned}$$

Thus,

$$\lim_{k\to\infty}\|Uz_{n_k}-z_{n_k}\| = 0. \quad (3.14)$$

By (3.1) and (3.12), we have

$$\|z_n-y_n\| = \lambda_n\|Ax_n-Ay_n\| \le \mu\|x_n-y_n\|,$$

which implies

$$\lim_{k\to\infty}\|z_{n_k}-y_{n_k}\| = 0. \quad (3.15)$$

Combining (3.12) and (3.15) yields that

$$\|z_{n_k}-x_{n_k}\| \le \|z_{n_k}-y_{n_k}\|+\|y_{n_k}-x_{n_k}\| \to 0$$

as $k\to\infty$. From (3.14) and the demiclosedness of mapping $I-U$, we have

$$\omega_w\{x_{n_k}\} = \omega_w\{z_{n_k}\} \subset Fix(U)$$

Therefore,

$$\omega_w\{x_{n_k}\} \subset Fix(U)\bigcap VI(C,A).$$

Next, we prove that $\lim_{n\to\infty}\|x_n-p\| = 0$. We consider two possible cases.

Case 1: There exists an $N\in\mathbb{N}$ such that $\|x_{n+1}-p\| \le \|x_n-p\|$ for all $n\ge N$. From this, we have

$$\liminf_{n\to\infty}(\|x_{n+1}-p\|-\|x_n-p\|) = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$, where $z\in Fix(U)\bigcap VI(C,A)$. In addition,

$$\begin{aligned} \limsup_{n\to\infty}\langle(I-\tau F)p-p, x_n-p\rangle &= \lim_{j\to\infty}\langle(I-\tau F)p-p, x_{n_j}-p\rangle \\ &= \langle(I-\tau F)p-p, z-p\rangle. \end{aligned}$$

Since $p = P_{Fix(U)\bigcap VI(C,A)}(I-\tau F)p$, by Lemma 2.1 (iv), we have

$$\langle(I-\tau F)p-p, z-p\rangle \le 0.$$

By $\omega_w\{x_n\} \subset Fix(U)\bigcap VI(C,A)$ and $\|z_n-x_n\| \to 0$, we obtain

$$\limsup_{n\to\infty}\langle(I-\tau F)p-p, z_n-p\rangle = \limsup_{n\to\infty}\langle(I-\tau F)p-p, x_n-p\rangle \le 0.$$

Since

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n((I-\tau F)t_n-p)+(1-\alpha_n)(t_n-p)\|^2 \\
&\le (1-\alpha_n)^2\|t_n-p\|^2+2\alpha_n\langle (I-\tau F)t_n-p, x_{n+1}-p\rangle \\
&= (1-\alpha_n)^2\|t_n-p\|^2+2\alpha_n\langle (I-\tau F)t_n-(I-\tau F)p, x_{n+1}-p\rangle \\
&\quad +2\alpha_n\langle (I-\tau F)p-p, x_{n+1}-p\rangle \\
&\le (1-\alpha_n)^2\|t_n-p\|^2+2\alpha_n(1-h)\|t_n-p\|\|x_{n+1}-p\| \\
&\quad +2\alpha_n\langle (I-\tau F)p-p, x_{n+1}-p\rangle \\
&\le (1-\alpha_n)^2\|x_n-p\|^2+2\alpha_n(1-h)\|x_n-p\|\|x_{n+1}-p\| \\
&\quad +2\alpha_n\langle (I-\tau F)p-p, x_{n+1}-p\rangle \\
&\le (1-2\alpha_n+2\alpha_n(1-h))\|x_n-p\|^2+\alpha_n^2\|x_n-p\|^2 \\
&\quad +2\alpha_n\langle (I-\tau F)p-p, x_{n+1}-p\rangle \\
&= (1-2\alpha_n h)\|x_n-p\|^2+2\alpha_n h[\frac{\alpha_n\|x_n-p\|^2}{2h}+\frac{\langle (I-\tau F)p-p, x_{n+1}-p\rangle}{h}].
\end{aligned}$$

Since $\limsup_{n\to\infty}\langle (I-\tau F)p, x_{n+1}-p\rangle \le 0$ and $\lim_{n\to\infty}\alpha_n=0$, we get

$$\limsup_{n\to\infty}[\frac{\alpha_n\|x_n-p\|^2}{2h}+\frac{\langle (I-\tau F)p-p, x_{n+1}-p\rangle}{h}]\le 0.$$

Therefore, by Lemma 2.5, we get

$$\lim_{n\to\infty}\|x_n-p\|=0.$$

Case 2: There exists a subsequence $\{\|x_{n_j}-p\|\}$ of $\{\|x_n-p\|\}$ such that $\|x_{n_j}-p\|\le\|x_{n_j+1}-p\|$ for all $j\in\mathbb{N}$.

By Lemma 2.4, we know there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty}m_k=0$ and for all $k\in\mathbb{N}$,

$$\|x_{m_k}-p\|\le\|x_{m_k+1}-p\|,$$

and

$$\|x_k-p\|\le\|x_{m_k+1}-p\|.$$

Since $\{x_{m_k}\}$ is bounded, there exists a subsequence $\{x_{m_{k_i}}\}$ of $\{x_{m_k}\}$ such that $x_{m_{k_i}}\rightharpoonup z$, where $z\in Fix(U)\bigcap VI(C,A)$. Thus,

$$\begin{aligned}
\limsup_{k\to\infty}\langle (I-\tau F)p-p, x_{m_k}-p\rangle &= \lim_{i\to\infty}\langle (I-\tau F)p-p, x_{m_{k_i}}-p\rangle \\
&= \langle (I-\tau F)p-p, z-p\rangle \le 0.
\end{aligned}$$

From $\|z_{m_k}-x_{m_k}\|\to 0$, we get

$$\limsup_{k\to\infty}\langle (I-\tau F)p-p, z_{m_k}-p\rangle=\limsup_{k\to\infty}\langle (I-\tau F)p-p, x_{m_k}-p\rangle\le 0.$$

On the other hand, we have

$$\begin{aligned}&\|x_{m_k+1}-p\|^2\\ \leq\ & (1-2\alpha_{m_k}h)\|x_{m_k}-p\|^2+2\alpha_{m_k}h[\frac{\alpha_{m_k}\|x_{m_k}-p\|^2}{2h}+\frac{\langle (I-\tau F)p-p,x_{m_k+1}-q\rangle}{h}\\ \leq\ & (1-\alpha_{m_k}h)\|x_{m_k+1}-p\|^2+2\alpha_{m_k}h[\frac{\alpha_{m_k}\|x_{m_k}-p\|^2}{2h}+\frac{\langle (I-\tau F)p-p,x_{m_k+1}-q\rangle}{h}],\end{aligned}$$

which implies

$$\|x_{m_k+1}-p\|^2\leq 2[\frac{\alpha_{m_k}\|x_{m_k}-p\|^2}{2h}+\frac{\langle (I-\tau F)p-p,x_{m_k+1}-q\rangle}{h}].$$

By $\lim_{k\to\infty}\alpha_{m_k}=0$, we have

$$\limsup_{k\to\infty} 2[\frac{\alpha_{m_k}\|x_{m_k}-p\|^2}{2h}+\frac{\langle (I-\tau F)p-p,x_{m_k+1}-q\rangle}{h}]\leq 0.$$

Therefore, by $\|x_k-p\|^2\leq\|x_{m_k+1}-p\|^2\leq 0$, we get $\limsup_{k\to\infty}\|x_k-p\|\leq 0$, which means

$$\lim_{n\to\infty}\|x_n-p\|=0.$$

So, we obtain the conclusion that $\{x_n\}$ converges strongly to an element of $Fix(U)\bigcap VI(C,A)$. This completes the proof. □

4. THE NUMERICAL EXAMPLE

In this section, we present a numerical example to test our new iterative Algorithm. The computations are performed by Matlab R2016a. For the test, we choose $\gamma=1$, $l=0.5$, $\mu=0.4$, $\alpha_n=\frac{1}{n+1}$ and $\beta_n=\frac{n}{2n+1}$. In the following, we give a numerical example in detail.

Example 4.1. In the infinite dimensional Hilbert space $H=L^2([0,1])$, we consider our algorithm with the inner product

$$\langle x,y\rangle:=\int_0^1 x(t)y(t)dt,\quad \forall x,y\in H,$$

and the induced norm

$$\|x\|:=(\int_0^1|x(t)|^2dt)^{\frac{1}{2}},\quad \forall x\in H,$$

where $t\in[0,1]$.

Consider the operator $A:H\to H$ defined by $(Ax)(t)=\max\{0,x(t)\}$, $\forall x\in H$. It is easy to know that operator $A:H\to H$ is monotone and 1-Lipschitz continuous. In fact,

$$(Ax)(t)=\max\{0,x(t)\}=\frac{x(t)+|x(t)|}{2}$$

and

$$\begin{aligned}
\langle Ax(t)-Ay(t), x(t)-y(t)\rangle &= \int_0^1 (Ax(t)-Ay(t))(x(t)-y(t))dt \\
&= \int_0^1 \Big(\frac{x(t)-y(t)+|x(t)|-|y(t)|}{2}\Big)(x(t)-y(t)) \\
&= \int_0^1 \frac{1}{2}[(x(t)-y(t))^2+(|x(t)|-|y(t)|)(x(t)-y(t))]dt \\
&\geq 0.
\end{aligned}$$

So, A is monotone. Note that

$$\begin{aligned}
\|Ax(t)-Ay(t)\|^2 &= \int_0^1 |Ax(t)-Ay(t)|^2 dt \\
&= \int_0^1 \Big|\frac{x(t)-y(t)+|x(t)|-|y(t)|}{2}\Big|^2 dt \\
&= \frac{1}{4}\int_0^1 |x(t)-y(t)+|x(t)|-|y(t)||^2 dt \\
&\leq \int_0^1 |x(t)-y(t)|^2 dt \\
&= \|x(t)-y(t)\|^2.
\end{aligned}$$

So, A is 1-Lipschitz continuous. Let $C := \{x \in H : \|x\| \leq 1\}$. The operator $U : L^2([0,1]) \to L^2([0,1])$ is of the form

$$(Ux)(t) = \sqrt{3}\int_0^1 tx(s)ds, \quad t \in [0,1].$$

Since $0 \in Fix(U)$, we have $Fix(U) \neq \emptyset$. In fact

$$\begin{aligned}
\|Ux(t)-Uy(t)\|^2 &= \int_0^1 |Ux(t)-Uy(t)|^2 dt \\
&= \int_0^1 3[t^2\int_0^1 |x(s)-y(s)|ds]^2 dt \\
&\leq \int_0^1 |x(s)-y(s)|^2 ds \\
&= \|x(t)-y(t)\|^2.
\end{aligned}$$

Therefore, U is nonexpansive, so is U. Let $F : H \to H$ be an operator defined by

$$(Fx)(t) = \frac{1}{2}x(t), \quad t \in [0,1].$$

It is easy to see that this operator is strongly monotone and Lipschitz continuous. For parameter $\tau = 0.5$, we know the solution of the problem is $x^* = 0$. We use the condition $\|x_{n+1} - x_n\| \leq \varepsilon$ to terminate the Algorithm. With two starting points $x_0(t) = t^2$ and $x_0(t) = t + 0.5\cos t$, the numerical results with iteration step and iteration time are showed in the Table. From the numerical results, we know the algorithm we proposed is effective.

x_0	ε	Iter. Step	Iter. Time
t^2	10^{-1}	5	4.58
$t+0.5\cos t$	$0.5*10^{-1}$	3	7.09

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