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ON PERIODIC OSCILLATIONS OF SOME POINTS OF A STRING WITH A NONLINEAR BOUNDARY CONDITION

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Abstract. In this paper, we investigate an initial boundary value problem describing oscillation process with a hysteresis type boundary condition. The analogue of the d'Alembert formula is obtained. We study the possibility of periodic oscillations of some points of a string with such condition.

Keywords. Wave equation; String oscillations; Hysteresis; Periodic oscillations.

1. Introduction

There are many papers dedicated to the study of string oscillations with different types of conditions; see, e.g., [4, 6, 7, 8, 9, 10, 13, 14, 15, 18, 19, 20] and the references therein. In this paper, we investigate a boundary value problem describing oscillation process with a hysteresis type boundary condition. This kind of problems arises in a simulation of string oscillations, where the movement is restricted by a sleeve, which is concentrated at a boundary point. A case when time varying on the interval $0 \leq t \leq l$ was considered in [13, 20]. In this paper, we consider a case that $t \in [0, +\infty)$. The analogue of the d'Alembert formula is obtained. The possibility of periodic oscillations of some points of a string with such condition is also investigated.

2. Preliminaries

In this section, we recall some notions and definitions which will need in the sequel (details can be found in [17]).

Let H be a Hilbert space. The inner product in H is denoted by $\langle \cdot, \cdot \rangle$. For a closed convex set $C \subset H$ and $x \in C$, the set

$$N_C(x) = \{\xi \in H : \langle \xi, c - x \rangle \leq 0 \quad \forall c \in C\}$$

denotes the outward normal cone to C at x.

Notice that we always have $0 \in N_C(x)$, $N_{\{x\}}(x) = H$, and $N_C(x) = \{0\}$ for $x \in intC$, where $intC$ is the interior of C. The last relation shows that the outward normal cone is non-trivial only for $x \in \partial C$, where ∂C is the boundary of C.

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Recall that the Hausdorff distance $d_H(C_1,C_2)$ between closed sets C_1 and C_2 is given by the formula

$$d_H(C_1,C_2) = \max\{\sup_{x\in C_2} \operatorname{dist}(x,C_1), \sup_{x\in C_1} \operatorname{dist}(x,C_2)\}.$$

Consider so called "sweeping process"

$$-u'(t) \in N_{C(t)}(u(t)) \quad for \quad a.e. \quad t \in [0,T], \tag{2.1}$$

$$u(0) = u_0 \in C(0). \tag{2.2}$$

A function $u : [0,T] \to H$ is called a solution of sweeping process (2.1)-(2.2) if
(a) $u(0) = u_0$;
(b) $u(t) \in C(t)$ for all $t \in [0,T]$;
(c) u is differentiable at a.e. $t \in [0,T]$;
(d) $-u'(t) \in N_{C(t)}(u(t))$ at a.e. $t \in [0,T]$.

There are many papers dedicated to sweeping processes; see, e.g., [1, 2, 3, 5, 11, 12, 16, 17] and the references therein. The following two theorems are useful for our main results.

Theorem 2.1. [17, Theorem 2] *(Existence) Assume that the map $t \to C(t)$ satisfies*

$$d_H(C(t),C(s)) \le L|t-s|$$

and $C(t) \subset H$ is nonempty, closed and convex for every $t \in [0,T]$. Let $u_0 \in C(0)$. Then there exists a solution $u : [0,T] \to H$ of (2.1),(2.2) which is Lipschitz continuous with constant L. In particular, $|u'(t)| \le L$ for almost every $t \in [0,T]$.

Theorem 2.2. [17, Theorem 3] *(Uniqueness) The solution of (2.1),(2.2) is unique in the class of absolutely continuous functions.*

3. A PROBLEM WITH A HYSTERESIS TYPE BOUNDARY CONDITION

Suppose that a string is located along the segment $[0,l]$. Let $u(x,t)$ be a deviation from the equilibrium position at the time t. Assume that the left end of the string is rigidly fixed, i.e., $u(0,t) = 0$. The right end of the string moves along a vertical needle (without friction) inside a sleeve, represented by $[-h,h]$, where $h > 0$. While $|u(l,t)| < h$, the right end of the string inside of the sleeve remains free, i.e., $u'_x(l,t) = 0$. If the string reaches the boundary points of the sleeve, then the conditions $u(l,t) = h$, or $u(l,t) = -h$, respectively, are satisfied. Notice that we consider the case where the sleeve can move in perpendicular to the axis Ox direction and its movement is given by

$$C(t) = [-h,h] + \xi(t). \tag{3.1}$$

Suppose that the string velocity is zero at the initial time $t = 0$ and the string form is determined by the function $\varphi(\cdot) \in W_2^1[0,l]$, where $\varphi(0) = 0$, $\varphi(l) \in C(0)$.

The mathematical model of such problem can be described as

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \\ u(x,0) = \varphi(x), \\ \dfrac{\partial u}{\partial t}(x,0) = 0, \\ u(0,t) = 0, \\ u(l,t) \in C(t) \\ -u'_x(l,t) \in N_{C(t)}(u(l,t)), \end{cases} \tag{3.2}$$

where the set $N_{C(t)}(u(l,t))$ is the outward normal cone to $C(t)$ at $u(l,t)$, which is defined as

$$N_{C(t)}(u(l,t)) = \{\xi \in R^1 : \xi \cdot (c - u(l,t)) \le 0 \quad \forall c \in C(t)\}.$$

Notice if $u(l,t)$ is an interior point of $C(t)$, then $N_{C(t)}(u(l,t)) = \{0\}$; if $u(l,t) = -h + \xi(t)$, then

$$N_{C(t)}(u(l,t)) = (-\infty, 0];$$

if $u(l,t) = h + \xi(t)$, then

$$N_{C(t)}(u(l,t)) = [0, +\infty).$$

The condition $-u'_x(l,t) \in N_{C(t)}(u(l,t))$ means that if $u(l,t)$ is an interior point of $C(t)$, then $u'_x(l,t) = 0$, i.e., the oscillation process is such as for a string with a free right end (see [18]); when the right end of the string contacts with the boundary sleeve point, the right end of the string is not free any more: there is the force $f(t)$, which blocks this end so $-u'_x(l,t) = -f(t) \in N_{C(t)}(u(l,t))$. Consider a special class of functions, introduced for the first time by Il'in in [8, 9]. Let Q_T be the rectangle $Q_T = [0 \le x \le l] \times [0 \le t \le T]$. As in [8, 9], we suppose that u belongs to the class $\widehat{W}_2^1(Q_T)$ if $u(x,t)$ is continuous in the closed rectangle Q_T and has in this rectangle both generalized partial derivatives $u'_x(x,t)$ and $u'_t(x,t)$, which belong to the class $L_2(Q_T)$. Moreover, $u'_x(\cdot,t)$ belongs to the class $L_2[0 \le x \le l]$ for every fixed t of the segment $[0,T]$ and $u'_t(x,\cdot)$ belongs to the class $L_2[0 \le t \le T]$, for any fixed x of the segment $[0,l]$.

By a solution of (3.2) we that a function $u(x,t)$ such that conditions $u(l,t) \in C(t)$ and $u(0,t) = 0$ hold for all t; conditions $-u'_x(l,t) \in N_{C(t)}(u(l,t))$ and $\dfrac{\partial u}{\partial t}(x,0) = 0$ hold for almost every t; the condition $u(x,0) = \varphi(x)$ holds for all $x \in [0,l]$; for all $T > 0$ $u \in \widehat{W}_2^1(Q_T)$ and the integral identity

$$\int_0^l \int_0^T u(x,t)[\Psi_{tt}(x,t) - \Psi_{xx}(x,t)]dxdt + \int_0^l \Psi'_t(x,0)\varphi(x)dx - \int_0^T \Psi(l,t)u'_x(l,t)dt \\ + \int_0^T \Psi'_x(l,t)u(l,t)dt = 0 \tag{3.3}$$

holds for any function $\Psi \in C^2(Q_T)$, which satisfies the conditions $\Psi(0,t) = 0$, $\Psi(x,T) = 0$, $\Psi'_t(x,T) = 0$.

Theorem 3.1. *Assume that the function $\xi(t)$ satisfies the Lipschitz condition for all $t \ge 0$, and the function $\varphi(x)$ satisfies the Lipschitz condition for all $x \in [0,l]$. Then a solution of problem (3.2) can be represented as*

$$u(x,t) = \frac{\Phi(x-t) + \Phi(x+t)}{2}, \tag{3.4}$$

where the function $\Phi(x)$ has the following form:
if $x \in [0,l]$, then $\Phi(x) = \varphi(x)$;
if $x \in [(i+1)l,(i+2)l]$ and i is an even number, then

$$\Phi(x) = 2\cdot\sum_{k=0}^{\frac{i}{2}} g_{2k}(x-(i+1-2k)l) - \varphi((i+2)l-x);$$

if $x \in [(i+1)l,(i+2)l]$ and i is an odd number, then

$$\Phi(x) = 2\cdot\sum_{k=1}^{\frac{i+1}{2}} g_{2k-1}(x-(i+2-2k)l) + \varphi(x-(i+1)l);$$

$$\Phi(-x) = -\Phi(x),$$

where functions $g_0(t)$ and $g_1(t)$ are solutions of the problems

$$\begin{cases} -g_0'(t) \in N_{C(t)}(g_0(t)) + \varphi'(l-t), t \in [0,l], \\ g_0(0) = \varphi(l), \end{cases}$$

and

$$\begin{cases} -g_1'(t) \in N_{C(t)}(g_1(t)) + \varphi'(t-l), t \in [l,2l], \\ g_1(l) = g_0(l). \end{cases}$$

Functions $g_i(t)$ for even numbers $i \geq 2$ are solutions of the problems

$$\begin{cases} -g_i'(t) \in N_{C(t)}(g_i(t)) + 2\sum_{k=0}^{\frac{i-2}{2}} g_{2k}'(t-il+2kl) + \varphi'(il+l-t), t \in [il,(i+1)l], \\ g_i(il) = g_{i-1}(il), \end{cases}$$

and for odd numbers $i \geq 3$ are solutions of the problems

$$\begin{cases} -g_i'(t) \in N_{C(t)}(g_i(t)) + 2\sum_{k=1}^{\frac{i-1}{2}} g_{2k-1}'(t-l-il+2kl) + \varphi'(t-il), t \in [il,(i+1)l], \\ g_i(il) = g_{i-1}(il). \end{cases}$$

Proof. Suppose formally that a solution of problem (3.2) has form (3.4). Then $u(x,0) = \Phi(x) = \varphi(x)$, where $x \in [0,l]$. Since $\varphi(x)$ satisfies the Lipschitz condition, we get $\Phi \in W_2^1[0,l]$. Since $u(x,0) = 0$ we have $\Phi(x) = -\Phi(-x)$. Let us consider the condition $-u_x'(l,t) \in N_{C(t)}(u(l,t))$. If the solution of problem (3.2) has form (3.4), then

$$-u_t'(l,t) \in N_{C(t)}(u(l,t)) + \Phi'(l-t). \tag{3.5}$$

Consider the case when $t \in [0,l]$. We denote by $g_0(t) = u(l,t)$. Then $g_0(t)$ is a solution of the problem

$$\begin{cases} -\frac{d}{dt}g_0(t) \in N_{C(t)}(g_0(t)) + \varphi'(l-t), \\ g_0(0) = \varphi(l), t \in [0,l]. \end{cases} \tag{3.6}$$

Let us show that problem (3.6) has a unique solution. Consider the function

$$w(t) = g_0(t) + \int_0^t \varphi'(l-s)ds$$

and the set

$$D(t) = C(t) + \int_0^t \varphi'(l-s)ds.$$

Notice that $N_{C(t)}(g_0(t)) = N_{D(t)}(w(t))$. Thus we have the problem

$$-\frac{d}{dt}w(t) \in N_{D(t)}(w(t)), w(0) = \varphi(l) \in D(0), t \in [0,l].$$

According to Theorems 2.1 and 2.2, this problem has a unique solution $w(t)$, which is defined on the whole interval $[0,l]$. The function $w(t)$ is absolutely continuous and its derivative is bounded almost everywhere. Thus problem (3.6) has a unique solution $g_0(t)$, where $g_0(t) \in C(t)$ and $g_0(t)$ satisfies the Lipschitz condition. Therefore

$$\Phi(l-t) + \Phi(l+t) = 2g_0(t)$$

and then

$$\Phi(x) = 2g_0(x-l) - \varphi(2l-x),$$

where $x \in [l,2l]$. Notice that $\Phi(x)$ satisfies the Lipschitz condition on $[l,2l]$. So, $\Phi \in W_2^1[l,2l]$. Let us show that $\Phi(l-0) = \Phi(l+0)$. We have $\Phi(l-0) = \varphi(l)$ and

$$\Phi(l+0) = 2g_0(0) - \varphi(l) = 2\varphi(l) - \varphi(l) = \varphi(l).$$

Consider a case when $t \in [l,2l]$. Define $g_1(t) = u(l,t)$. Consider a problem

$$\begin{cases} -\frac{d}{dt}g_1(t) \in N_{C(t)}(g_1(t)) + \Phi'(l-t), \\ g_1(l) = g_0(l), t \in [l,2l]. \end{cases}$$

Notice that $\Phi(l-t) = -\varphi(t-l)$. Thus we have a problem

$$\begin{cases} -\frac{d}{dt}g_1(t) \in N_{C(t)}(g_1(t)) + \varphi'(t-l), \\ g_1(l) = g_0(l), t \in [l,2l]. \end{cases}$$

Similarly to problem (3.6) we obtain that the last problem has a unique solution $g_1(t)$, where $g_1(t) \in C(t)$ and $g_1(t)$ satisfies the Lipschitz condition. Thus we define $\Phi(x)$, where $x \in [2l,3l]$ as

$$\Phi(x) = 2g_1(x-l) + \varphi(x-2l).$$

Notice that $\Phi(x)$ satisfies the Lipschitz condition on $[2l,3l]$. Hence, $\Phi \in W_2^1[2l,3l]$. Let us show that $\Phi(2l-0) = \Phi(2l+0)$. We have $\Phi(2l-0) = 2g_0(l) - \varphi(0) = 2g_0(l)$ and $\Phi(2l+0) = 2g_1(l) + \varphi(0) = 2g_0(l)$.

Similarly, we consider the case when $t \in [2l,3l]$. Define $g_2(t) = u(l,t)$. Then $g_2(t)$ is a solution of the problem

$$\begin{cases} -\frac{d}{dt}g_2(t) \in N_{C(t)}(g_2(t)) + 2g_0'(t-2l) + \varphi'(3l-t), \\ g_2(2l) = g_1(2l), t \in [2l,3l]. \end{cases}$$

Hence,

$$\Phi(x) = 2g_2(x-l) + 2g_0(x-3l) - \varphi(4l-x),$$

where $x \in [3l,4l]$.

Consider the case when $t \in [3l, 4l]$. Define $g_3(t) = u(l,t)$. Then $g_3(t)$ is a solution of the problem

$$\begin{cases} -\dfrac{d}{dt} g_3(t) \in N_{C(t)}(g_3(t)) + 2g_1'(t-2l) + \varphi'(t-3l), \\ g_3(3l) = g_2(3l), t \in [3l, 4l] \end{cases}$$

and

$$\Phi(x) = 2g_3(x-l) + 2g_1(x-3l) + \varphi(x-4l),$$

where $x \in [4l, 5l]$. Let us show that if $x \in [(i+1)l, (i+2)l]$ and i is an even number, then

$$\Phi(x) = 2 \cdot \sum_{k=0}^{\frac{i}{2}} g_{2k}(x-(i+1-2k)l) - \varphi((i+2)l - x);$$

if $x \in [(i+1)l, (i+2)l]$ and i is an odd number, then

$$\Phi(x) = 2 \cdot \sum_{k=1}^{\frac{i+1}{2}} g_{2k-1}(x-(i+2-2k)l) + \varphi(x-(i+1)l),$$

where functions $g_i(t)$ for even numbers $i \geq 2$ are solutions of the problems

$$\begin{cases} -g_i'(t) \in N_{C(t)}(g_i(t)) + 2\sum\limits_{k=0}^{\frac{i-2}{2}} g_{2k}'(t-il+2kl) + \varphi'(il+l-t), t \in [il, (i+1)l], \\ g_i(il) = g_{i-1}(il), \end{cases}$$

and for odd numbers $i \geq 3$ are solutions of the problems

$$\begin{cases} -g_i'(t) \in N_{C(t)}(g_i(t)) + 2\sum\limits_{k=1}^{\frac{i-1}{2}} g_{2k-1}'(t-l-il+2kl) + \varphi'(t-il), t \in [il, (i+1)l], \\ g_i(il) = g_{i-1}(il). \end{cases}$$

For $i = 2, 3$ the assertion is proved. Let it hold for $i \leq m$ and let us show it for $i = m+1$. Consider the case when m is an even number. Let us show that

$$\Phi(x) = 2\sum_{k=1}^{\frac{m+2}{2}} g_{2k-1}(x-(m+3-2k)l) + \varphi(x-(m+2)l),$$

where $x \in [(m+2)l, (m+3)l]$. Define $u(l,t) = g_{m+1}(t)$, where $t \in [(m+1)l, (m+2)l]$. Then

$$-g_{m+1}'(t) \in N_{C(t)}(g_{m+1}(t)) + \Phi'(l-t).$$

Since

$$\Phi'(l-t) = 2 \cdot \sum_{k=1}^{\frac{m}{2}} g_{2k-1}'(t-l-(m+1-2k)l) + \varphi'(t-l-ml),$$

we have

$$-g_{m+1}'(t) \in N_{C(t)}(g_{m+1}(t)) + 2 \cdot \sum_{k=1}^{\frac{m}{2}} g_{2k-1}'(t-2l-m+2kl) + \varphi'(t-l-ml).$$

Notice that

$$g_{m+1}((m+1)l) = \frac{\Phi((2+m)l) - \Phi(ml)}{2}.$$

Since $\Phi((m+2)l) = 2\sum_{k=0}^{\frac{m}{2}} g_{2k}(l+2kl)$ and $\Phi(ml) = 2\sum_{k=0}^{\frac{m-2}{2}} g_{2k}(l+2kl)$, we have $g_{m+1}((m+1)l) = g_m((m+1)l)$. The problem

$$\begin{cases} -g'_{m+1}(t) \in N_{C(t)}(g_{m+1}(t)) + 2\cdot\sum_{k=1}^{\frac{m}{2}} g'_{2k-1}(t-2l-m+2kl) + \varphi'(t-l-ml), \\ g_{m+1}((m+1)l) = g_m((m+1)l) \end{cases}$$

has a unique solution $g_{m+1}(t)$, which is defined on the whole interval $[(m+1)l,(m+2)l]$. Then

$$g_{m+1}(t) = \frac{\Phi(l-t)+\Phi(l+t)}{2}$$

and we have

$$\Phi(x) = 2g_{m+1}(x-l) - \Phi(2l-x),$$

where $x \in [(m+2)l,(m+3)l]$. Since $\Phi(2l-x) = -\Phi(x-2l)$ and

$$\Phi(x-2l) = 2\sum_{k=1}^{\frac{m}{2}} g_{2k-1}(x-3l-ml+2kl) + \varphi(x-2l-ml),$$

we have

$$\begin{aligned}\Phi(x) &= 2g_{m+1}(x-l) + 2\sum_{k=1}^{\frac{m}{2}} g_{2k-1}(x-3l-ml+2kl) + \varphi(x-2l-ml) \\ &= 2\sum_{k=1}^{\frac{m+2}{2}} g_{2k-1}(x-3l-ml+2kl) + \varphi(x-2l-ml).\end{aligned}$$

Other cases can be analyzed in the same way. Thus we define $\Phi(x)$ for all $x \in R$. Let us show that the function $u(x,t)$, which is defined by equality (3.4), is the solution of problem (3.2), where the function $\Phi(x)$ is defined above. Notice that the function $u \in \widehat{W}_2^1(Q_T)$ for all $T > 0$, because $\Phi(x)$ is continuous on $x \in R$ and $\Phi \in W_2^1[il,(i+1)l]$ for all $i = 0,1,2,\ldots$ Since $u(l,t) = g(t)$, where $g(t) = g_i(t)$, $t \in [il,(i+1)l]$, $g_i(il) = g_{i-1}(il)$ and $g_i(t) \in C(t)$, we conclude that $u(l,t) \in C(t)$ for all $t \geq 0$. Notice that the condition $u(0,t) = 0$ holds for all t; $\frac{\partial u}{\partial t}(x,0) = 0$ holds for almost every t; $u(x,0) = \varphi(x)$ holds for all $x \in [0,l]$. Notice that, for all $t \geq 0$,

$$\Phi(l-t)+\Phi(l+t) = 2g(t).$$

Thus, for almost every t,

$$-\Phi'(l-t)+\Phi'(l+t) = 2g'(t).$$

Since $-g'(t) \in N_{C(t)}(g(t)) + \Phi'(l-t)$, we have

$$-u'_x(l,t) = \frac{-\Phi'(l-t)-\Phi'(l+t)}{2} = -g'(t) - \Phi'(l-t) \in N_{C(t)}(g(t)) = N_{C(t)}(u(l,t)).$$

Let us show that integral equality (3.3) holds for any function $\Psi \in C^2(Q_T)$, which satisfies the conditions $\Psi(0,t)=0$, $\Psi(x,T)=0$, $\Psi'_t(x,T)=0$. We have

$$\begin{aligned}
&\int_0^l \left(\int_0^T u(x,t)\Psi_{tt}(x,t)dt\right)dx - \int_0^T \left(\int_0^l u(x,t)\Psi_{xx}(x,t)dx\right)dt \\
&\quad + \int_0^l \Psi'_t(x,0)\varphi(x)dx - \int_0^T \Psi(l,t)u'_x(l,t)dt + \int_0^T \Psi'_x(l,t)u(l,t)dt \\
&= \int_0^l \big(u(x,T)\Psi'_t(x,T) - u(x,0)\Psi'_t(x,0)\big)\,dx - \int_0^l \int_0^T u'_t(x,t)\Psi'_t(x,t)dtdx \\
&\quad - \int_0^T \big(\Psi'_x(l,t)u(l,t) - \Psi'_x(0,t)u(0,t)\big)\,dt \\
&\quad + \int_0^T \int_0^l u'_x(x,t)\Psi'_x(x,t)dxdt + \int_0^l \Psi'_t(x,0)\varphi(x)dx \\
&\quad - \int_0^T \Psi(l,t)u'_x(l,t)dt + \int_0^T \Psi'_x(l,t)u(l,t)dt.
\end{aligned}$$

We need to prove that

$$\int_0^T \int_0^l u'_x(x,t)\Psi'_x(x,t)dxdt - \int_0^T \int_0^l u'_t(x,t)\Psi'_t(x,t)dxdt = \int_0^T \Psi(l,t)u'_x(l,t)dt.$$

According to (3.4), we have

$$\begin{aligned}
&\frac{1}{2}\int_0^T \int_0^l (\Phi'(x-t)-\Phi'(x+t))\Psi'_x(x,t)dxdt - \frac{1}{2}\int_0^l \int_0^T (\Phi'(x+t)-\Phi'(x-t))\Psi'_t(x,t)dtdx \\
&= \frac{1}{2}\int_0^l \big(\Psi'_x(x,T)(\Phi(x+T)-\Phi(x-T)) - \Psi'_x(x,0)(\Phi(x)-\Phi(x))\big)\,dx \\
&\quad - \frac{1}{2}\int_0^l \int_0^T (\Phi(x+t)-\Phi(x-t))\Psi_{xt}(x,t)dtdx \\
&\quad - \frac{1}{2}\int_0^T \big(\Psi'_t(l,t)(\Phi(l+t)-\Phi(l-t)) - \Psi'_t(0,t)(\Phi(t)-\Phi(-t))\big)\,dt \\
&\quad + \frac{1}{2}\int_0^l \int_0^T (\Phi(x+t)-\Phi(x-t))\Psi_{xt}(x,t)dtdx \\
&= -\frac{1}{2}\int_0^T \Psi'_t(l,t)(\Phi(l+t)-\Phi(l-t))dt.
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
\int_0^T \Psi(l,t)u'_x(l,t)dt &= \frac{1}{2}\int_0^T \Psi(l,t)(\Phi'(l-t)+\Phi'(l+t))dt \\
&= \frac{1}{2}(\Psi(l,T)(\Phi(l+T)-\Phi(l-T)) - \Psi(l,0)(\Phi(l)-\Phi(l))) \\
&\quad - \frac{1}{2}\int_0^T (\Phi(l+t)-\Phi(l-t))\Psi'_t(l,t)dt \\
&= -\frac{1}{2}\int_0^T (\Phi(l+t)-\Phi(l-t))\Psi'_t(l,t)dt.
\end{aligned}$$

The theorem is proved. □

Remark 3.1. Notice that problem (3.2) has a unique solution. Assume that $\varphi(l) \in (-h+\xi(0), h+\xi(0))$. Then the oscillation process occurs as for the string with a free end for all $t \in [0,t_1]$, and the string form is the solution of the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < l, 0 < t < t_1 \\ u(x,0) = \varphi(x), \\ \dfrac{\partial u}{\partial t}(x,0) = 0, \\ u(0,t) = 0, \\ u'_x(l,t) = 0. \end{cases}$$

According to [10], the last problem has a unique solution $u(x,t)$. The relation $u(l,t_1) = \pm h + \xi(t)$ holds at the moment t_1 , and for all $t \in [t_1,t_2]$ a string form is a solution of a problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial^2 v}{\partial t^2}, \quad 0 < x < l, t_1 < t < t_2, \\ v(x,t_1) = u(x,t_1), \\ \dfrac{\partial v}{\partial t}(x,t_1) = u'_t(x,t_1), \\ v(0,t) = 0, \\ v(l,t) = -h+\xi(t) \end{cases}$$

or

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial^2 v}{\partial t^2}, \quad 0 < x < l, t_1 < t < t_2, \\ v(x,t_1) = u(x,t_1), \\ \dfrac{\partial v}{\partial t}(x,t_1) = u'_t(x,t_1), \\ v(0,t) = 0, \\ v(l,t) = h+\xi(t). \end{cases}$$

Each of the above problems has a unique solution for every $t \in [t_1,t_2]$. By a similar reasoning, we get that the original problem has a unique solution.

4. Periodic oscillations of some points of a string

Let us study the possibility of periodic oscillations of some points of a string which is described by the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \\ u(x,0) = 0, \\ \dfrac{\partial u}{\partial t}(x,0) = 0, \\ u(0,t) = 0, \\ u(l,t) \in C(t), \\ -u'_x(l,t) \in N_{C(t)}(u(l,t)), \end{cases} \tag{4.1}$$

where $C(t) = [-h,h] + \xi(t)$ and $\xi(t)$ is a l-periodic function defined by the formula

$$\xi(t) = \begin{cases} \frac{8h}{l}t, & t \in [0, \frac{l}{4}], \\ \frac{-8h}{l}(t - \frac{l}{2}), & t \in [\frac{l}{4}, \frac{3l}{4}], \\ \frac{8h}{l}(t-l), & t \in [\frac{3l}{4}, l]. \end{cases}$$

Notice that $\xi(t)$ satisfies the Lipschitz condition with constant $L = \frac{8h}{l}$. According to Theorem 3.1, the last problem has a unique solution $u(x,t)$, where

$$u(x,t) = \frac{\Phi(x-t) + \Phi(x+t)}{2}.$$

The function $\Phi(x)$ has the following form:
if $x \in [0,l]$, then $\Phi(x) = 0$;
if $x \in [(i+1)l, (i+2)l]$ and i is an even number then

$$\Phi(x) = 2 \cdot \sum_{k=0}^{\frac{i}{2}} g_{2k}(x - (i+1-2k)l);$$

if $x \in [(i+1)l, (i+2)l]$ and i is an odd number, then

$$\Phi(x) = 2 \cdot \sum_{k=1}^{\frac{i+1}{2}} g_{2k-1}(x - (i+2-2k)l);$$

$$\Phi(-x) = -\Phi(x),$$

where functions $g_0(t)$ and $g_1(t)$ are solutions of the problems

$$\begin{cases} -g_0'(t) \in N_{C(t)}(g_0(t)), t \in [0,l], \\ g_0(0) = \varphi(l), \end{cases} \tag{4.2}$$

and

$$\begin{cases} -g_1'(t) \in N_{C(t)}(g_1(t)), t \in [l,2l], \\ g_1(l) = g_0(l). \end{cases} \tag{4.3}$$

Functions $g_i(t)$ for even numbers $i \geq 2$ are solutions of the problems

$$\begin{cases} -g_i'(t) \in N_{C(t)}(g_i(t)) + 2\sum_{k=0}^{\frac{i-2}{2}} g_{2k}'(t - il + 2kl), t \in [il, (i+1)l], \\ g_i(il) = g_{i-1}(il), \end{cases}$$

and for odd number $i \geq 3$ are solutions of the problems

$$\begin{cases} -g_i'(t) \in N_{C(t)}(g_i(t)) + 2\sum_{k=1}^{\frac{i-1}{2}} g_{2k-1}'(t - l - il + 2kl), t \in [il, (i+1)l], \\ g_i(il) = g_{i-1}(il). \end{cases}$$

Consider problem (4.2). Notice that a solution of the last problem can be represented as

$$g_0(t) = \begin{cases} 0, & t \in [0, \frac{l}{8}], \\ \xi(t) - h, & t \in [\frac{l}{8}, \frac{l}{4}], \\ h, & t \in [\frac{l}{4}, \frac{l}{2}], \\ \xi(t) + h, & t \in [\frac{l}{2}, \frac{3l}{4}], \\ -h, & t \in [\frac{3l}{4}, l]. \end{cases}$$

Consider problem (4.3). Notice that a solution of the last problem can be represented as

$$g_1(t) = \begin{cases} \xi(t) - h, & t \in [l, \frac{5l}{4}], \\ h, & t \in [\frac{5l}{4}, \frac{3l}{2}], \\ \xi(t) + h, & t \in [\frac{3l}{2}, \frac{7l}{4}], \\ -h, & t \in [\frac{7l}{4}, 2l]. \end{cases}$$

Let us show that, for all $i \in N$,

$$g_i(t) = \begin{cases} \xi(t) - h, & t \in [il, \frac{l(4i+1)}{4}], \\ h, & t \in [\frac{l(4i+1)}{4}, \frac{l(2i+1)}{2}], \\ \xi(t) + h, & t \in [\frac{l(2i+1)}{2}, \frac{l(4i+3)}{4}], \\ -h, & t \in [\frac{l(4i+3)}{4}, l(i+1)]. \end{cases}$$

For $i = 1$, the assertion is proved. Let it hold for $i \le n$. Let us show it for $i = n+1$. Consider the case when n is an even number, i.e., n=2m. Let us show that

$$g_{2m+1}(t) = \begin{cases} \xi(t) - h, & t \in [2ml + l, 2ml + \frac{5l}{4}], \\ h, & t \in [2ml + \frac{5l}{4}, 2ml + \frac{3l}{2}], \\ \xi(t) + h, & t \in [2ml + \frac{3l}{2}, 2ml + \frac{7l}{4}], \\ -h, & t \in [2ml + \frac{7l}{4}, 2ml + 2l]. \end{cases} \tag{4.4}$$

We have

$$\begin{cases} -g'_{2m+1}(t) \in N_{C(t)}(g_{2m+1}(t)) + 2\sum_{k=1}^{m} g'_{2k-1}(t - 2l - 2ml + 2kl), \\ g_{2m+1}((2m+1)l) = -h, \quad t \in [(2m+1)l, (2m+2)l]. \end{cases}$$

Setting

$$v(t) = g_{2m+1}(t) + 2\int_{(2m+1)l}^{t} \sum_{k=1}^{m} g'_{2k-1}(s - 2l - 2ml + 2kl)ds,$$

we have

$$\begin{aligned} v(t) &= g_{2m+1}(t) + 2\sum_{k=1}^{m} (g_{2k-1}(t - 2l - 2ml + 2kl) - g_{2k-1}((2k-1)l) \\ &= g_{2m+1}(t) + 2\sum_{k=1}^{m} g_{2k-1}(t - 2l - 2ml + 2kl) + 2mh. \end{aligned}$$

Denote by

$$\tilde{\xi} = \xi(t) + 2mh + 2\sum_{k=1}^{m} g_{2k-1}(t - 2l - 2ml + 2kl),$$

$$D(t) = [-h, h] + \tilde{\xi}(t),$$

where $t\in[(2m+1)l,(2m+2)l]$. Taking into account the inductive assumption and representation of the function $\xi(t)$, we have

$$\tilde{\xi}(t)=\begin{cases}(1+2m)\xi(t), & t\in[2ml+l,2ml+\frac{5l}{4}],\\ \xi(t)+4mh, & t\in[2ml+\frac{5l}{4},2ml+\frac{3l}{2}],\\ \xi(t)+4mh+2m\xi(t), & t\in[2ml+\frac{3l}{2},2ml+\frac{7l}{4}],\\ \xi(t), & t\in[2ml+\frac{7l}{4},2ml+2l].\end{cases}$$

Since $v(t)$ is the solution of the problem

$$\begin{cases}-v'(t)\in N_{D(t)}(v(t)),t\in[(2m+1)l,(2m+2)l],\\ v((2m+1)l)=-h,\end{cases}$$

one has

$$v(t)=\begin{cases}-h+\xi(t)+2m\xi(t), & t\in[2ml+l,2ml+\frac{5l}{4}],\\ h+4mh, & t\in[2ml+\frac{5l}{4},2ml+\frac{3l}{2}],\\ \xi(t)+4mh+2m\xi(t)+h, & t\in[2ml+\frac{3l}{2},2ml+\frac{7l}{4}],\\ -h, & t\in[2ml+\frac{7l}{4},2ml+2l].\end{cases}$$

Hence, $g_{2m+1}(t)=v(t)-\tilde{\xi}(t)+\xi(t)$ and we get expression (4.4) for the function $g_{2m+1}(t)$. Other cases can be analyzed in the same way. Denote by $g(t)=g_i(t)$, where $t\in[il,(i+1)l]$, $i=0,1,2,\ldots$. Since $u(l,t)=g(t)$, we have that the right end of the string periodically moves with the period l, starting at time $t=\frac{l}{4}$. Rewrite problem (4.1) as

$$\begin{cases}\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2},\\ u(x,0)=0,\\ \dfrac{\partial u}{\partial t}(x,0)=0,\\ u(0,t)=0,\\ u(l,t)=g(t).\end{cases}$$

The solution of the last problem can be represented as

$$u(x,t)=\sum_{n=0}^{\infty}\underline{g}(t-(2n+1)l+x)-\sum_{n=0}^{\infty}\underline{g}(t-(2n+1)l-x), \tag{4.5}$$

where

$$\underline{g}(t)=\begin{cases}g(t), & t\geq 0,\\ 0, & t<0.\end{cases} \tag{4.6}$$

In view of equality (4.6), the sums (4.5) have only a finite number of terms. Let us show that the function $u(\frac{l}{2},t)$ is $2l$-periodic. We have

$$u(\frac{l}{2},t)=\sum_{n=0}^{\infty}\underline{g}(t-2nl-\frac{l}{2})-\sum_{n=0}^{\infty}\underline{g}(t-2nl-\frac{3l}{2}).$$

On the other hand,

$$\begin{aligned}u(\frac{l}{2},t+2l)&=\sum_{n=0}^{\infty}\underline{g}(t-2nl+\frac{3l}{2})-\sum_{n=0}^{\infty}\underline{g}(t-2nl+\frac{l}{2})\\ &=\underline{g}(t+\frac{3l}{2})-\underline{g}(t+\frac{l}{2})+u(\frac{l}{2},t).\end{aligned}$$

Since
$$\underline{g}(t+\frac{3l}{2})=g(t+\frac{3l}{2})=g(t+\frac{l}{2})=\underline{g}(t+\frac{l}{2}),$$
we obtain $u(\frac{l}{2},t)=u(\frac{l}{2},t+2l)$. Let us show that $u(\frac{l}{4},t)=kh$ for $t=kl$, where $k\in N$. We have
$$u(\frac{l}{4},t)=\sum_{n=0}^{\infty}\underline{g}(t-2nl-\frac{3l}{4})-\sum_{n=0}^{\infty}\underline{g}(t-2nl-\frac{5l}{4}).$$
Let k be an even number. It follows that
$$\begin{aligned}u(\frac{l}{4},kl)&=\sum_{n=0}^{\frac{k}{2}-1}\underline{g}(kl-2nl-\frac{3l}{4})-\sum_{n=0}^{\frac{k}{2}-1}\underline{g}(kl-2nl-\frac{5l}{4})\\&=(\underline{g}((k-1)l+\frac{l}{4})+\underline{g}((k-3)l+\frac{l}{4})+...+\underline{g}(l+\frac{l}{4}))\\&\quad-(\underline{g}((k-2)l+\frac{3l}{4})+\underline{g}((k-4)l+\frac{3l}{4})+...+\underline{g}(\frac{3l}{4}))\\&=\frac{k}{2}(\underline{g}(\frac{l}{4})-\underline{g}(\frac{3l}{4}))=kh.\end{aligned}$$
Let k be an odd number. Then
$$\begin{aligned}u(\frac{l}{4},kl)&=\sum_{n=0}^{\frac{k-1}{2}}\underline{g}(kl-2nl-\frac{3l}{4})-\sum_{n=0}^{\frac{k-1}{2}-1}\underline{g}(kl-2nl-\frac{5l}{4})\\&=(\underline{g}((k-1)l+\frac{l}{4})+\underline{g}((k-3)l+\frac{l}{4})+...+\underline{g}(\frac{l}{4}))\\&\quad-(\underline{g}((k-2)l+\frac{3l}{4})+\underline{g}((k-4)l+\frac{3l}{4})+...+\underline{g}(\frac{3l}{4}+l)\\&=\frac{k}{2}(\underline{g}(\frac{l}{4})-\underline{g}(\frac{3l}{4}))+\frac{1}{2}\underline{g}(\frac{l}{4})+\frac{1}{2}\underline{g}(\frac{3l}{4})=kh.\end{aligned}$$
Thus the function $u(\frac{l}{4},t)$ is not periodic in t. Similarly, we obtain if $x=\frac{3l}{4}$, then $u(\frac{3l}{4},kl)=-kh$.

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REFERENCES

[1] L. Adam, J. Outrata, On optimal control of a sweeping process coupled with an ordinary differential equation, Discrete Contin. Dyn. Syst. Ser. B 19 (2014), 2709-2738.
[2] S. Adly, B. K. Le, Unbounded second-order state-dependent Moreau's sweeping processes in Hilbert spaces, J. Optim. Theory Appl. 169 (2016), 407-423.
[3] C. Castaing, M. Monteiro Marques, BV periodic solutions of an evolution problem associated with continuous moving convex sets, Set-Valued Anal. 3 (1995), 381-399.
[4] M.V. de Hoop, P. Kepley, L. Oksanen, An exact redatuming procedure for the inverse boundary value problem for the wave equation, SIAM J. Appl. Math. 78 (2018), 171-192.
[5] J.F. Edmond, L. Thibault, Relaxation of an optimal control problem involving a perturbed sweeping process, Math. Program. 104 (2005), 347-373.

[6] A. Fiscella, E. Vitillaro, Blow-up for the wave equation with nonlinear source and boundary damping terms, Proc. R. Soc. Edinb. Sect. A. 145 (2015), 759-778.

[7] G.R. Goldstein, J.A. Goldstein, G. Reyes, Overdamping and energy decay for abstract wave equations with strong damping, Asymptotical Anal. 88 (2014), 217-232.

[8] V.A. Il'in, Two-end point boundary control of ibrations described by a finite energy generalized solution of the wave equation, Differential Equations 36 (2000), 1659-1675.

[9] V.A. Il'in, Boundary control of oscillations at one endpoint with the other endpoint fixed in terms of a finite-energy generalized solution of the wave equation, Differential Equations 36 (2000), 1832-1849.

[10] V. A. Il'in, E. I. Moiseev, Optimization of boundary controls of string vibrations, Russian Math. Surveys 60 (2005), 1093-1119.

[11] M. Kamenskii, O. Makarenkov, On the pesponse of autonomous sweeping processes to periodic perturbations, Set-Valued Var. Anal. 24 (2016), 551-563.

[12] M. Kamenskii, O. Makarenkov, L. N. Wadippuli, P. Raynaud de Fitte, Global stability of almost periodic solutions to monotone sweeping processes and their response to non-monotone perturbations, Nonlinear Anal. 30 (2018), 213-224.

[13] M. Kamenskii, C.F. Wen, M. Zvereva, A string oscillations simulation with boundary conditions of hysteresis type, Optimization 67 (2018), 1321-1332.

[14] M. Kamenskii, Y.C. Liou, C.F. Wen, M. Zvereva, On a hyperbolic equation on a geometric graph with hysteresis type boundary conditions, Optimization (2019), doi: 10.1080/02331934.2018.1561694.

[15] M. Kamenskii, C.F. Wen, Z. Zalukaeva, M. Zvereva, The influence function properties for a problem with discontinuous solutions, Appl. Anal. Optim. 1 (2017), 259-281.

[16] P. Krejci, T. Roche, Lipschitz continuous data dependence of sweeping processes in BV spaces, Discrete Contin. Dyn. Syst. B 15 (2011), 637-650.

[17] M. Kunze, M. Monteiro Marques, An introduction to Moreau's sweeping process, Impacts in Mechanical Systems Lecture Notes in Physics 551 (2000), 1–60.

[18] A.N. Tikhonov, A.A. Samarskii, Equations of Mathematical Physics, Dover Publication, New York, 1990.

[19] E. Vitillaro, On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and source, Arch. Ration. Mech. Anal. 223 (2017), 1183-1237.

[20] M. Zvereva, A string oscillations simulation with nonlinear conditions, Mem. Differ. Equ. Math. Phys. 72 (2017), 141-150.