BOUNDED PERTURBATION RESILIENCE OF GENERALIZED VISCOSITY ITERATIVE ALGORITHMS FOR SPLIT VARIATIONAL INCLUSION PROBLEMS

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Abstract. In this paper, we propose a generalized viscosity approximation method combing a sequence of contractive mappings to solve a split variational inclusion problem. The bounded perturbation resilience of the method is investigated in Hilbert spaces. Under mild conditions, we prove that our algorithms strongly converge to a solution of the split variational inclusion problem, which is also the unique solution of some variational inequality problem. Furthermore, we show the convergence and effectiveness of the algorithms via two numerical examples. Our results extend and improve the related results in the literature.

Keywords. Split variational inclusion problem; Maximal monotone operator; Contractive mapping; Viscosity iteration.

1. INTRODUCTION

Let H_1 and H_2 be two real Hilbert spaces with inner product$\langle , \rangle$ and induced norm $\| \cdot \|$. In this paper, we deal with the split variational inclusion problem (SVIP, for short) of the following type: Find a point $x^* \in H_1$ such that

$$0 \in B_1(x^*), \tag{1.1}$$

and $y^* = Ax^* \in H_2$ solves

$$0 \in B_2(y^*), \tag{1.2}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings. We denote the solution set of SVIP (1.1)-(1.2) by

$$S = \{x^* \in H_1 : 0 \in B_1(x^*) \text{ and } Ax^* \in H_2 : 0 \in B_2(Ax^*)\}.$$

The SVIP has been applied to solving many real life problems, such as, modelling intensity-modulated radiation therapy treatment planning [7, 8], and modelling of inverse problems arising from phase retrieval [4]. Recently, a lot of projection algorithms were proposed to solve the SVIP (1.1)-(1.2) in infinite-dimensional spaces; see, e.g., [2, 5, 6, 11, 12, 13, 23, 25] and the references therein.

In 2012, Byrne *et al.* [5] gave the following two algorithms for the split variational inclusion problem: For any $x_0 \in H_1$, $\{x_n\}$ is a sequence defined as follows

$$x_{n+1} = J_\lambda^{B_1}(x_n - \tau A^*(I - J_\lambda^{B_2})Ax_n)$$

and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_\lambda^{B_1}(x_n - \tau A^*(I - J_\lambda^{B_2})Ax_n).$$

Under appropriate conditions, they obtained weak and strong convergence theorems, respectively, in the framework of Hilbert spaces.

The following lemma is very useful for constructing out iterative algorithms.

Lemma 1.1. [26] *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}Fix(T) \neq \emptyset$. Then the following equality holds:*

$$Fix(J_\lambda^B(I - \tau A^*(I - T)A)) = B^{-1}0 \cap A^{-1}Fix(T), \tag{1.3}$$

where λ and τ are two real numbers.

In 2014, Kazmi and Rizvi [21] proposed the following iteration algorithm

$$\begin{cases} y_n = J_\lambda^{B_1}(x_n - \tau A^*(I - J_\lambda^{B_2})Ax_n), \\ x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n)Ty_n. \end{cases}$$

They proved that the sequence $\{x_n\}$ generated by the above algorithm converges strongly a common element both in the fixed-point set of nonexpansive mapping T and the solution set of SVIP (1.1)-(1.2).

On the other hand, the bounded perturbation resilience and superiorization of iterative methods are extensively studied in many references, for example, [3, 9, 16, 18, 20, 29]. The problem has received much attention due to its applications in convex feasibility problems [10], image reconstruction [15] and inverse problems of radiation therapy [14], and so on.

An algorithm operator **P** is said to be bounded perturbation resilient [9] if the iteration $x_{n+1} = \mathbf{P}x_n$ is replaced by $x_{n+1} = \mathbf{P}(x_n + \beta_n v_n)$, where $\beta_n \geq 0$ for all $n \geq 0$ and $v_n \in H$ with

$$\sum_{n=0}^{\infty} \beta_n < \infty \quad and \quad \|v_n\| \leq M. \tag{1.4}$$

is still convergent.

Consider the following minimization problem:

$$\min_{x \in C} f(x), \tag{1.5}$$

where f is a continuous differentiable, convex function. Xu [29] proposed the superiorization techniques for the relaxed projected scaled gradient method. Xu's algorithm is as follows

$$x_{n+1} = (1 - \tau_n)x_n + \tau_n P_C(x_n - \gamma_n D(x_n)\nabla f(x_n) + e(x_n)), \quad n \geq 0, \tag{1.6}$$

where τ_n is a sequence in $[0,1]$, and $D(x_n)$ is a diagonal scaling matrix. He established weak convergence of the above algorithm in [29] under appropriate conditions imposed on $\{\tau_n\}$, and $\{\gamma_n\}$.

Very recently, Guo and Cui [18] introduced a modified proximal gradient algorithm with bounded perturbations. Indeed, their iterative sequence $\{x_n\}$ is generated as follows

$$x_{n+1} = t_n h(x_n) + (1 - t_n)prox_{\gamma_n}(x_n - \gamma_n \nabla f(x_n) + e(x_n)), \quad n \geq 0. \tag{1.7}$$

Under suitable conditions, they obtained solution theorems in Hilbert spaces.

In this paper, based on the works [9, 26, 29], we propose a generalized viscosity approximation method for solving SVIP (1.1)-(1.2). We show that our iterative algorithm is bounded perturbation resilience and show the convergence point of the iterative algorithm which is also the unique solution of some variational inequality problem. Finally, two numerical examples are given to demonstrate the effectiveness of our iterative algorithms:

$$x_{n+1} = \alpha_n h_n(x_n) + (1-\alpha_n)J_\lambda^{B_1}(x_n - \tau_n A^*(I - J_\lambda^{B_2})Ax_n + e(x_n)), \tag{1.8}$$

where $\{h_n\}$ is a sequence of contractive mappings. We will show that the superiorization of (1.8) yields a sequence $\{x_n\}$ generated by the following iterative process:

$$\begin{cases} y_n = x_n + \beta_n v_n, \\ x_{n+1} = \alpha_n h_n(y_n) + (1-\alpha_n)J_\lambda^{B_1}((I - \tau_n A^*(I - J_\lambda^{B_2})A(y_n) + e(y_n)), \end{cases} \tag{1.9}$$

In addition, we also discuss the convergence of the generalized viscosity algorithm and prove that it is bounded perturbation resilient.

2. PRELIMINARIES

We adopt the following notations:

(1) $x_n \to x$ indicates that $\{x_n\}$ is strongly convergent to x; $x_n \rightharpoonup x$ indicates that $\{x_n\}$ is weakly convergent to x.

(2) $Fix(T)$ denotes the set of fixed points of T, that is, $Fix(T) = \{x \in H : Tx = x\}$.

(3) $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

We list the following lemmas, definitions and propositions, which are needed in the proof of main results.

Lemma 2.1. *Let H be a real Hilbert space. There holds the following inequality:*

$$\|x-y\|^2 \le \|x\|^2 + 2\langle x+y, y\rangle, \ \forall x, y \in H.$$

Definition 2.1. A mapping $F : H \to H$ is said to be

(i) Lipschizian if there exists a positive constant L such that

$$\|Fx - Fy\| \le L\|x-y\|, \quad \forall x, y \in H.$$

In particular, if $L = 1$, we say that F is nonexpansive, that is,

$$\|Fx - Fy\| \le \|x-y\|, \ \forall x, y \in H.$$

If $L \in [0, 1)$, we say that F is contractive.

(ii) α-averaged mapping (α-*av*, for short) if

$$F = (1-\alpha)I + \alpha T,$$

where $\alpha \in (0, 1)$ and $T : H \to H$ is nonexpansive.

Definition 2.2. A mapping $B : H \to H$ is said to be

(i) monotone if

$$\langle Bx - By, x - y\rangle \ge 0, \ \ \forall x, y \in H.$$

(ii) η-*strongly* monotone if there exists a positive constant η such that

$$\langle Bx - By, x - y\rangle \ge \eta\|x-y\|^2, \ \ \forall x, y \in H.$$

(iii) α-*inverse* strongly monotone (α-ism, for short) if there exists a positive constant α such that

$$\langle Bx - By, x - y \rangle \geq \alpha \|Bx - By\|^2, \quad \forall x, y \in H.$$

In particular, if $\alpha = 1$, we say that B is firmly nonexpansive, namely,

$$\langle Bx - By, x - y \rangle \geq \|Bx - By\|^2, \quad \forall x, y \in H.$$

Definition 2.3. Let a mapping $B : H \to H$ be monotone. Then B is maximal monotone if there exists no monotone operator $A : H \to 2^H$ such that *graA* properly contains *graB*, i.e., for every $(x, u) \in H \times H$,

$$(x, u) \in graB \iff \forall (y, v) \in graB, \ \langle x - y, u - v \rangle \geq 0.$$

Lemma 2.2. [28] *Let $h : H \to H$ be a ρ-contraction with $\rho \in (0, 1)$ and $T : H \to H$ be a nonexpansive mapping. Then*

(i) $I - h$ *is $(1 - \rho)$-strongly monotone:*

$$\langle (I - h)x - (I - h)y, x - y \rangle \geq (1 - \rho)\|x - y\|^2, \quad \forall x, y \in H.$$

(ii) $I - T$ *is monotone:*

$$\langle (I - T)x - (I - T)y, x - y \rangle \geq 0, \quad \forall x, y \in H.$$

Proposition 2.1. [27]

(i) *If $T_1, T_2, \cdots, T_n$ are averaged mappings, we have that $T_n T_{n-1} \cdots T_1$ is averaged. In particular, if T_i is α_i-av, i=1,2, where $\alpha_i \in (0, 1)$, then $T_2 T_1$ is $(\alpha_2 + \alpha_1 - \alpha_2 \alpha_1)$-av.*

(ii) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then,*

$$\bigcap_{i=1}^N Fix(T_i) = Fix(T_1 \ldots T_N).$$

(iii) *A mapping T is nonexpansive if and only if $I - T$ is $\frac{1}{2}$-ism.*

(iv) *If T is ν-ism, then, for $\tau > 0$, τT is $\frac{\nu}{\tau}$-ism.*

(v) *T is averaged if and only if $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $0 < \alpha < 1$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Lemma 2.3. [17] *Let H be a real Hilbert space, and let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in H weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particulary, if $y = 0$, then $x \in Fix(T)$.*

Lemma 2.4. *(see [19, 30]) Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n \delta_n, \ n \geq 0,$$

$$s_{n+1} \leq s_n - \eta_n + \varphi_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$, (η_n) is a sequence of nonnegative real numbers and (δ_n) and (φ_n) are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\lim_{n \to \infty} \varphi_n = 0$,

(iii) $\lim_{k \to \infty} \eta_{n_k} = 0$ *implies* $\limsup_{k \to \infty} \delta_{n_k} \leq 0$ *for any subsequence $(n_k) \subset (n)$.*

Then $\lim_{n \to \infty} s_n = 0$.

3. MAIN RESULTS

In 2000, Moudafi [22] proposed the viscosity approximation method for a nonexpansive mapping T with a nonempty fixed-point set $Fix(T)$ in a Hilbert space H:

$$x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n)Tx_n,$$

where $h : H \to H$ is ρ-contractive mapping and $\{\alpha_n\} \subset (0,1)$ is a real sequence, and proved that the sequence $\{x_n\}$ converges strongly to a fixed point of the mapping T if the sequence $\{\alpha_n\}$ satisfies appropriate conditions. It was also proved that the fixed point also uniquely solves the following variational inequality:

$$\langle (I-h)x^*, \tilde{x} - x^* \rangle \geq 0, \ \forall \tilde{x} \in Fix(T). \tag{3.1}$$

Recently, the viscosity approximation method have been extended and studied in the framework of Banach spaces; see [24, 28] and the references therein.

Re-writing algorithm (1.8), one has

$$\begin{aligned} x_{n+1} &= \alpha_n h_n(x_n) + (1-\alpha_n)J_\lambda^{B_1}(x_n - \tau_n A^*(I - J_\lambda^{B_2})Ax_n + e(x_n)) \\ &= \alpha_n h_n(x_n) + (1-\alpha_n)(J_\lambda^{B_1}(x_n - \tau_n A^*(I - J_\lambda^{B_2})Ax_n) + \tilde{e}_n). \end{aligned} \tag{3.2}$$

This algorithm is a outer perturbed version. Since $J_{\lambda_n}^{B_1}$ is nonexpansive, we have

$$\begin{aligned} \|\tilde{e}_n\| &= \|J_\lambda^{B_1}(x_n - \tau_n A^*(I - J_\lambda^{B_2})Ax_n + e(x_n)) - J_\lambda^{B_1}(x_n - \tau_n A^*(I - J_\lambda^{B_2})Ax_n)\| \\ &\leq \|e(x_n)\|. \end{aligned} \tag{3.3}$$

Next, we use S to denote the solution set of the SVIP (1.1)-(1.2) and use the generalized viscosity iterative method to find solutions of the SVIP (1.1)-(1.2).

Theorem 3.1. *Let H_1, H_2 be two real Hilbert spaces and A be a bounded linear operator with $L = \|A^*A\|$, where A^* is the adjoint of A. Suppose that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone operators and $S \neq \emptyset$. Let h_n be a ρ_n-contractive self-mapping of H_1 with $0 \leq \rho_l = \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n = \rho_u < 1$. Suppose that the contractive mapping sequence $\{h_n(x)\}$ is uniformly convergent for any $x \in E$, where E is any bounded subset of H_1. Given $x_0 \in H_1$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n h_n(x_n) + (1-\alpha_n)J_\lambda^{B_1}(x_n - \tau_n A^*(I - J_\lambda^{B_2})Ax_n + e(x_n)), \tag{3.4}$$

where $\tau_n \in (0, \frac{2}{L})$. Assume the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \tau_n \leq \limsup_{n\to\infty} \tau_n < \frac{2}{L}$;
(iv) $\sum_{n=0}^{\infty} \|e(x_n)\| < \infty$.

Then $\{x_n\}$ converges strongly to x^*, where x^* is a solution of the SVIP (1.1)-(1.2), which is also the unique solution of variational inequality problem (3.1).

Proof. Setting

$$V_{\tau_n} := J_\lambda^{B_1}(I - \tau_n A^*(I - J_\lambda^{B_2})A),$$

we have that V_{τ_n} is $\frac{2 + \tau_n L}{4}$-av. Note that $I - \tau_n A^*(I - J_\lambda^{B_2})A$ is $\frac{\tau_n L}{2}$-av as $0 < \tau_n < 2/L$ and $J_\lambda^{B_1}$ is $\frac{1}{2}$-av [1]. It follows from Proposition 2.1 that the composite V_{τ_n} is $\frac{2 + \tau_n L}{4}$-av.

Next, we split the proof into several steps.

Step 1. Show that $\{x_n\}$ is bounded.

For any $z \in S$, we get

$$
\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n h_n(x_n) + (1-\alpha_n)(V_{\tau_n} x_n + \tilde{e}_n) - z\| \\
&= \|\alpha_n(h_n(x_n) - z) + (1-\alpha_n)(V_{\tau_n} x_n - z) + (1-\alpha_n)\tilde{e}_n\| \\
&\leq \alpha_n \|h_n(x_n) - h_n(z)\| + \alpha_n \|h_n(z) - z\| + (1-\alpha_n)\|V_{\tau_n} x_n - z\| + \|\tilde{e}_n\| \\
&\leq \alpha_n \rho_n \|x_n - z\| + \alpha_n \|h_n(z) - z\| + (1-\alpha_n)\|x_n - z\| + \|\tilde{e}_n\| \\
&= (1 - \alpha_n(1-\rho_n))\|x_n - z\| + \alpha_n(1-\rho_n)\frac{\|h_n(z) - z\| + \|\tilde{e}_n\|/\alpha_n}{1 - \rho_n}.
\end{aligned}
\tag{3.5}
$$

Since $\{h_n\}$ is uniformly convergent on bounded sets, it is easy to get that $\{h_n(z)\}$ is bounded. From the conditions (ii), (iv) and $\alpha_n > 0$, we find that $\{\|\tilde{e}_n\|/\alpha_n\}$ is also bounded. Thus, there exists some $M_1 > 0$ such that $\sup\{\|h_n(z) - z\| + \|\tilde{e}_n\|/\alpha_n\} \leq M_1$ for all $n \geq 0$. It follows that

$$
\|x_n - z\| \leq \max\{\|x_0 - z\|, \ \frac{M_1}{1 - \rho_u}\},
$$

which implies that sequence $\{x_n\}$ is bounded, so are $\{h_n(x_n)\}$, $\{V_{\tau_n} x_n\}$ and $\{A^*(I - J_\lambda^{B_2})Ax_n\}$.

Step 2. Show that, for any sequence $(n_k) \subset (n)$, if $\eta_{n_k} \to 0$, then $\lim_{k \to \infty} \|x_{n_k} - V_{\tau_{n_k}} x_{n_k}\| = 0$.

Fixing $z \in S$, we have

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\leq \|\alpha_n h_n(x_n) + (1-\alpha_n)V_{\tau_n} x_n - z\|^2 + 2\langle \alpha_n h_n(x_n) + (1-\alpha_n)V_{\tau_n} x_n - z, (1-\alpha_n)\tilde{e}_n\rangle + \|\tilde{e}_n\|^2 \\
&\leq \alpha_n^2 \|h_n(x_n) - z\|^2 + (1-\alpha_n)^2 \|V_{\tau_n} x_n - z\|^2 + 2\alpha_n(1-\alpha_n)\langle h_n(x_n) - z, V_{\tau_n} x_n - z\rangle \\
&\quad + (2\alpha_n \|h_n(x_n) - z\| + 2(1-\alpha_n)\|x_n - z\| + \|\tilde{e}_n\|)\|\tilde{e}_n\| \\
&\leq 2\alpha_n^2(\|h_n(x_n) - h_n(z)\|^2 + \|h_n(z) - z\|^2) + (1-\alpha_n)^2 \|V_{\tau_n} x_n - z\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle h_n(x_n) - z, V_{\tau_n} x_n - z\rangle + M_2\|\tilde{e}_n\| \\
&\leq 2\alpha_n^2(\|h_n(x_n) - h_n(z)\|^2 + \|h_n(z) - z\|^2) + (1-\alpha_n)^2 \|V_{\tau_n} x_n - z\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)(\|h_n(x_n) - h_n(z)\|\|x_n - z\| + \langle h_n(z) - z, V_{\tau_n} x_n - z\rangle) + M_2\|\tilde{e}_n\| \\
&\leq (1 - \alpha_n(2 - \alpha_n(1 + 2\rho_n^2) - 2(1-\alpha_n)\rho_n))\|x_n - z\|^2 + 2\alpha_n(1-\alpha_n)\langle h_n(z) - z, V_{\tau_n} x_n - z\rangle \\
&\quad + 2\alpha_n^2 \|h_n(z) - z\|^2 + M_2\|\tilde{e}_n\| \\
&\leq (1 - \alpha_n(2 - \alpha_n(1 + 2\rho_u^2) - 2(1-\alpha_n)\rho_u))\|x_n - z\|^2 + 2\alpha_n(1-\alpha_n)\langle h_n(z) - z, V_{\tau_n} x_n - z\rangle \\
&\quad + 2\alpha_n^2 \|h_n(z) - z\|^2 + M_2\|\tilde{e}_n\|,
\end{aligned}
\tag{3.6}
$$

where

$$
M_2 = \sup\{2\alpha_n \|h_n(x_n) - z\| + 2(1-\alpha_n)\|x_n - z\| + \|\tilde{e}_n\|\}.
$$

Since V_{τ_n} is $\frac{2 + \tau_n L}{4}$-av, we can rewrite

$$
V_{\tau_n} = J_\lambda^{B_1}(I - \tau_n A^*(I - J_\lambda^{B_2})A) = (1 - w_n)I + w_n T_n,
\tag{3.7}
$$

where $w_n = \frac{2+\tau_n L}{4}$, T_n is a nonexpansive mapping. From condition (iii), we have

$$\frac{1}{2} < \liminf_{n\to\infty} w_n \leq \limsup_{n\to\infty} w_n < 1.$$

Thus, it follows from (3.2) and (3.6) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n h_n(x_n) + (1-\alpha_n)(V_{\tau_n} x_n + \tilde{e}_n) - z\|^2 \\
&\leq \|\alpha_n h_n(x_n) + (1-\alpha_n)V_{\tau_n} x_n - z\|^2 + M_2\|\tilde{e}_n\| \\
&= \|V_{\tau_n} x_n - z + \alpha_n(h_n(x_n) - V_{\tau_n} x_n)\|^2 + M_2\|\tilde{e}_n\| \\
&= \|V_{\tau_n} x_n - z\|^2 + \alpha_n^2\|h_n(x_n) - V_{\tau_n} x_n\|^2 + 2\alpha_n\langle V_{\tau_n} x_n - z, h_n(x_n) - V_{\tau_n} x_n\rangle + M_2\|\tilde{e}_n\| \\
&= \|(1-w_n)x_n + w_n T_n x_n - z\|^2 + \alpha_n^2\|h_n(x_n) - V_{\tau_n} x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\tau_n} x_n - z, h_n(x_n) - V_{\tau_n} x_n\rangle + M_2\|\tilde{e}_n\| \\
&= (1-w_n)\|x_n - z\|^2 + w_n\|T_n x_n - T_n z\|^2 - w_n(1-w_n)\|T_n x_n - x_n\|^2 \\
&\quad + \alpha_n^2\|h_n(x_n) - V_{\tau_n} x_n\|^2 + 2\alpha_n\langle V_{\tau_n} x_n - z, h_n(x_n) - V_{\tau_n} x_n\rangle + M_2\|\tilde{e}_n\| \\
&\leq \|x_n - z\|^2 - w_n(1-w_n)\|T_n x_n - x_n\|^2 + \alpha_n^2\|h_n(x_n) - V_{\tau_n} x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\tau_n} x_n - z, h_n(x_n) - V_{\tau_n} x_n\rangle + M_2\|\tilde{e}_n\|.
\end{aligned} \tag{3.8}$$

Set

$$s_n = \|x_n - z\|^2, \quad \gamma_n = \alpha_n(2 - \alpha_n(1+2\rho_u^2) - 2(1-\alpha_n)\rho_u),$$

$$\delta_n = \frac{2(1-\alpha_n)\langle h_n(z) - z, V_{\tau_n} x_n - z\rangle + 2\alpha_n\|h_n(z) - z\|^2 + M_2\frac{\|\tilde{e}_n\|}{\alpha_n}}{2 - \alpha_n(1+2\rho_u^2) - 2(1-\alpha_n)\rho_u},$$

$$\eta_n = w_n(1-w_n)\|T_n x_n - x_n\|^2,$$

and

$$\varphi_n = \alpha_n^2\|h_n(x_n) - V_{\tau_n} x_n\|^2 + 2\alpha_n\langle V_{\tau_n} x_n - z, h_n(x_n) - V_{\tau_n} x_n\rangle + M_2\|\tilde{e}_n\|.$$

Note that $\gamma_n \to 0$, $\Sigma_{n=0}^\infty \gamma_n = \infty$ and $\varphi_n \to 0$ as $n \to \infty$. In order to complete the proof by using Lemma 2.4, it suffices to verify that $\eta_{n_k} \to 0$ $(k \to \infty)$ implies that $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ for any subsequence $(n_k) \subset (n)$. Indeed, as $k \to \infty$, $\eta_{n_k} \to 0$ implies that $\|T_{n_k} x_{n_k} - x_{n_k}\| \to 0$ due to condition (iii). Thus, from (3.7), we have

$$\|x_{n_k} - V_{\tau_{n_k}} x_{n_k}\| = w_{n_k}\|x_{n_k} - T_{n_k} x_{n_k}\| \to 0. \tag{3.9}$$

Step 3. Show that

$$\omega_w(x_{n_k}) \subset S, \tag{3.10}$$

where $\omega_w(x_{n_k})$ is the set of all weak cluster points of $\{x_{n_k}\}$. To see (3.10), we take $\tilde{x} \in \omega_w\{x_{n_k}\}$ and assume that $\{x_{n_{k_j}}\}$ is a subsequence of $\{x_{n_k}\}$ weakly converging to $\tilde{x}$. Without loss of generality, we still use $\{x_{n_k}\}$ to denote $\{x_{n_{k_j}}\}$. Since τ_{n_k} is bounded, we can assume $\tau_{n_k} \to \tau$. Then $0 < \tau < \frac{2}{L}$. Setting

$$V_\tau = J_\lambda^{B_1}(I - \tau A^*(I - J_\lambda^{B_2})A),$$

we have that V_τ is nonexpansive. It follows that

$$\|V_{\tau_{n_k}}x_{n_k} - V_\tau x_{n_k}\| = \|J_\lambda^{B_1}(x_{n_k} - \tau_{n_k}A^*(I - J_\lambda^{B_2})Ax_{n_k}) - J_\lambda^{B_1}(x_{n_k} - \tau A^*(I - J_\lambda^{B_2})Ax_{n_k})\|$$

$$\le \|\tau_{n_k}A^*(I - J_\lambda^{B_2})Ax_{n_k} - \tau A^*(I - J_\lambda^{B_2})Ax_{n_k}\|$$

$$\le |\tau_{n_k} - \tau|\|A^*(I - J_\lambda^{B_2})Ax_{n_k}\|. \tag{3.11}$$

Since $\tau_{n_k} \to \tau$ as $k \to \infty$, we immediately derive from the last relation that $\|V_{\tau_{n_k}}x_{n_k} - V_\tau x_{n_k}\| \to 0$. As a result, we find

$$\|x_{n_k} - V_\tau x_{n_k}\| \le \|x_{n_k} - V_{\tau_{n_k}}x_{n_k}\| + \|V_{\tau_{n_k}}x_{n_k} - V_\tau x_{n_k}\| \to 0. \tag{3.12}$$

Using Lemma 2.3, we get $\omega_w(x_{n_k}) \subset Fix(V_\tau)$. By Lemma 1.1, we obtain $\omega_w(x_{n_k}) \subset S$ immediately. Since $\{h_n(x)\}$ is uniformly convergent to $h(x)$ on E, we have $\lim_{n\to\infty}(h_n(x^*) - x^*) = h(x^*) - x^*$. Meanwhile, we have

$$\limsup_{k\to\infty}\langle h_{n_k}(x^*) - x^*, x_{n_k} - x^*\rangle = \langle h(x^*) - x^*, \tilde{x} - x^*\rangle, \quad \forall \tilde{x} \in S. \tag{3.13}$$

Since x^* is the unique solution to variational inequality problem (3.1). We get

$$\limsup_{k\to\infty}\langle h_{n_k}(x^*) - x^*, x_{n_k} - x^*\rangle \le 0$$

and hence $\limsup_{k\to\infty}\delta_{n_k} \le 0$. This completes the proof. $\qquad\square$

Consider the bounded perturbation of (3.4) by the following iterative form:

$$\begin{cases} y_n = x_n + \beta_n v_n, \\ x_{n+1} = \alpha_n h_n(y_n) + (1 - \alpha_n)J_\lambda^{B_1}((I - \tau_n A^*(I - J_\lambda^{B_2})A(y_n) + e(y_n)), \end{cases} \tag{3.14}$$

where $\tau_n \in (0, \frac{2}{L})$. Similarly, we put

$$\tilde{e}_n = J_\lambda^{B_1}((I - \tau_n A^*(I - J_\lambda^{B_2})A(y_n) + e(y_n)) - J_\lambda^{B_1}(I - \tau_n A^*(I - J_\lambda^{B_2})A(y_n)). \tag{3.15}$$

Theorem 3.2. *Assume that the sequences $\{\beta_n\}$ and $\{v_n\}$ satisfy the condition (1.4). Let H_1, H_2 be two real Hilbert spaces and A be a bounded linear operator with $L = \|A^*A\|$, where A^* is the adjoint of A. Suppose that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone operators and $S \ne \emptyset$. Let h_n be a ρ_n-contractive self-maps of H_1 with $0 \le \rho_l = \liminf_{n\to\infty}\rho_n \le \limsup_{n\to\infty}\rho_n = \rho_u < 1$. Suppose the contractive mapping sequence $\{h_n(x)\}$ is uniformly convergent for any $x \in E$, where E is any bounded subset of H_1. Given $x_0 \in H_1$, define the sequence $\{x_n\}$ by (3.14). Suppose that*

   (i) $\lim_{n\to\infty}\alpha_n = 0$;
   (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$;
   (iii) $0 < \liminf_{n\to\infty}\tau_n \le \limsup_{n\to\infty}\tau_n < \frac{2}{L}$;
   (iv) $\sum_{n=0}^{\infty}\|e(y_n)\| < \infty$.

Then $\{x_n\}$ converges strongly to x^*, where x^* is a solution of the SVIP (1.1)-(1.2), which is also the unique solution of variational inequality problem (3.1).

Proof. We can rewrite (3.14) as

$$x_{n+1} = \alpha_n h_n(x_n) + (1 - \alpha_n)(J_\lambda^{B_1}(I - \tau_n A^*(I - J_\lambda^{B_2})A(x_n)) + \tilde{e}_n) + \bar{e}_n, \tag{3.16}$$

where

$$\bar{e}_n = \alpha_n(h_n(y_n) - h_n(x_n)) + (1 - \alpha_n)(J_\lambda^{B_1}(I - \tau_n A^*(I - J_\lambda^{B_2})A)y_n - J_\lambda^{B_1}(I - \tau_n A^*(I - J_\lambda^{B_2})A)x_n), \tag{3.17}$$

In fact, it is not hard to get

$$\begin{aligned}
\|\bar{e}_n\| &\leq \alpha_n\|h_n(y_n) - h_n(x_n)\| + (1 - \alpha_n)\|y_n - x_n - \tau_n(A^*(I - J_\lambda^{B_2})A(y_n) - A^*(I - J_\lambda^{B_2})A(x_n))\| \\
&\leq \alpha_n\rho_n\beta_n\|v_n\| + (1 - \alpha_n)(\beta_n\|v_n\| + \lambda_n L\|y_n - x_n)\|) \\
&\leq (\alpha_n\rho_u + (1 - \alpha_n)(1 + \lambda_n L))\beta_n\|v_n\|. \tag{3.18}
\end{aligned}$$

From condition (1.4), we have $\sum_{n=0}^{\infty}\|\bar{e}_n\| < \infty$. Thus, using the Theorem 3.1, we get the algorithm (3.4) is bounded perturbation resilient. □

Remark 3.1. Theorem 3.1 is a outer perturbation algorithm. We replace it by the following algorithm:

$$x_{n+1} = \alpha_n h_n(x_n + e(x_n)) + (1 - \alpha_n)J_\lambda^{B_1}(I - \tau_n A^*(I - J_\lambda^{B_2})A)(x_n + e(x_n)), \tag{3.19}$$

It is obvious that it is a inner perturbation version. Similarly, we rewrite it as

$$x_{n+1} = \alpha_n h_n(x_n) + (1 - \alpha_n)J_\lambda^{B_1}(I - \tau_n A^*(I - J_\lambda^{B_2})A)(x_n) + \hat{e}_n. \tag{3.20}$$

It is easy to prove

$$\|\hat{e}_n\| \leq \|e(x_n)\|.$$

Under appropriate conditions, we can obtain strong convergence theorems.

4. NUMERICAL RESULTS

In this section, we consider two simple numerical examples to demonstrate the effectiveness, realization, and convergence of the Theorem 3.1.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}^2$. Define $h_n(x) = \frac{n}{10(n+1)}x$. Take $B_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $B_2 : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$B_1(x) = T_1(x) \ and \ B_2(x) = T_2(x),$$

where

$$T_1 = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}.$$

Observe T_1 and T_2 are positive linear operators, and then they are maximal monotone. So, we obtain the resolvent operators $J_\lambda^{B_1} = (I + \lambda B_1)^{-1}$ and $J_\lambda^{B_2} = (I + \lambda B_2)^{-1}$, respectively. The detailed expressions are calculated as follows:

$$J_\lambda^{B_1} = \frac{1}{(8\lambda + 1)(2\lambda + 1)}\begin{pmatrix} 2\lambda + 1 & 0 \\ 0 & 8\lambda + 1 \end{pmatrix}$$

and

$$J_\lambda^{B_2} = \frac{1}{(3\lambda + 1)(6\lambda + 1)}\begin{pmatrix} 6\lambda + 1 & 0 \\ 0 & 3\lambda + 1 \end{pmatrix}.$$

Generate a 2×2 random matrix A, and compute the Lipschitz constant $L = \|A^T A\|$, where A^T represents the transpose of A. Take $\lambda = 0.5$, $\tau_n = \tau = \frac{1}{100*L}$ and $\alpha_n = \frac{1}{n+1}$. According to the iterative process of Theorem 3.1, the sequence $\{x_n\}$ is defined by

$$x_{n+1} = \frac{1}{n+1} * \frac{1}{10} x_n + (1 - \frac{1}{n+1}) J_\lambda^{B_1} (x_n - \frac{1}{100L} A^T (I - J_\lambda^{B_2}) A x_n). \tag{4.1}$$

As $n \to \infty$, we have $x_n \to x^*$. By taking random initial guess x_0, and using software MATLAB, we obtain the numerical experiment results in Table 1.

TABLE 1. $x_0 = rand(2,1)$.

$\tau = \frac{1}{100*L}$	n	$time(s)$	x_n	$err(\|x_{n+1} - x_n\|)$
0.0051	16	0.000064	$10^{-6} * [-0.0014\ 0.9087]^T$	$6.1812 * 10^{-7}$
0.0143	23	0.000087	$10^{-7} * [-0.0002\ 0.1537]^T$	$7.9865 * 10^{-9}$
0.0067	26	0.000098	$10^{-8} * [-0.0004\ 0.1602]^T$	$8.2823 * 10^{-10}$

Next, we consider the algorithm with bounded perturbation resilience. Choose the bounded sequence $\{v_n\}$ and the summarable nonnegative real sequence $\{\beta_n\}$ as follows:

$$v_n = \begin{cases} -\dfrac{d_n}{\|d_n\|}, & if\ 0 \neq d_n \in B_1(x_n), \\ 0, & if\ 0 \in B_1(x_n). \end{cases} \tag{4.2}$$

$\beta_n = c^n$, for some $c \in (0,1)$. Set $c = 0.5$. The numerical results can be seen in Table 2.

TABLE 2. $x_0 = rand(2,1)$.

$\tau = \frac{1}{100*L}$	n	$time(s)$	x_n	$err(\|x_{n+1} - x_n\|)$
0.0056	22	0.002426	$10^{-6} * [0.1072\ -0.0003]^T$	$6.9097 * 10^{-8}$
0.0045	25	0.002636	$10^{-7} * [0.1071\ 0.0000]^T$	$8.6093 * 10^{-9}$
0.0046	29	0.004276	$10^{-8} * [0.6693\ 0.0002]^T$	$5.3830 * 10^{-10}$

As we have seen, the error of the solution becomes smaller as the increasing iterative numbers. And, the sequence $\{x_n\}$ converges $(0,0)$, which is the solution of the numerical example. Of course, it is also the unique solution of the variational inequality $\langle (I - h)x^*, x - x^* \rangle \geq 0$.

Example 4.2. Let $H_1 = \mathbb{R}^3$, $H_2 = \mathbb{R}^2$. Define $h_n(x) = h(x) = \frac{1}{15}x$. Take $B_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $B_2 : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$B_1(x) = N_C(x)\ and\ B_2(x) = N_Q(x),$$

where N_C denote the normal cone to C defined by

$$N_C(x) = \begin{cases} \{u \in H : \langle u, x - y \rangle \geq 0, \forall y \in C\}, & if\ x \in C, \\ \emptyset, & if\ x \notin C. \end{cases}$$

Suppose

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 4 & 0 \end{pmatrix},$$

$$C = \{x \in \mathbb{R}^3 : \|x\|_2 \le 1\}$$

and

$$Q = \{y \in \mathbb{R}^2 : 1 \le y(i) \le 2, \ i = 1, 2\},$$

where $y(i)$ denotes the *ith* element of y. Given the parameters $\alpha_n = 1/\sqrt{n+1}$, $\tau_n = 1/L$, we obtain the following iterative form:

$$x_{n+1} = \frac{1}{\sqrt{n+1}} * \frac{1}{10} x_n + (1 - \frac{1}{\sqrt{n+1}}) P_C(x_n - \frac{1}{L} A^T (I - P_Q) A x_n). \tag{4.3}$$

Taking random initial vector $x_0 \in \mathbb{R}^3$, we get the numerical results in Table 3.

TABLE 3. $x_0 = rand(3, 1)$.

$\tau = \frac{1}{L}$	n	$time(s)$	x_n	y_n	eps
0.056	62	0.000001	$[0.1862\ 0.2001\ 0.2031]^T$	$[0.9817\ 0.9866]^T$	10^{-4}
0.056	597	0.00012	$[0.1886\ 0.2025\ 0.2072]^T$	$[0.9988\ 0.9986]^T$	10^{-6}
0.056	6028	0.2105	$[0.1889\ 0.2028\ 0.2074]^T$	$[1.0000\ 1.0000]^T$	10^{-8}

Consider the bounded perturbation of (4.3). The definitions of v_n and β_n are similar to Example 4.1. The corresponding numerical results are shown in Table 4.

TABLE 4. $x_0 = rand(3, 1)$.

$\tau = \frac{1}{L}$	n	$time(s)$	x_n	y_n	eps
0.056	62	0.000001	$[0.1859\ 0.2001\ 0.2031]^T$	$[0.9811\ 0.9862]^T$	10^{-4}
0.056	568	0.0001	$[0.1886\ 0.2025\ 0.2070]^T$	$[0.9981\ 0.9986]^T$	10^{-6}
0.056	6005	0.1980	$[0.1889\ 0.2028\ 0.2074]^T$	$[1.0000\ 1.0000]^T$	10^{-8}

From the point of view of calculation, the above algorithms in the examples are easy to implement.

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