Appl. Set-Valued Anal. Optim. 2 (2020), No. 2, pp. 143-157
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.2.2020.2.02

FIXED POINTS FOR MULTIVALUED NON-EXPANSIVE MAPPINGS

T. DOMÍNGUEZ BENAVIDES*, P. LORENZO RAMÍREZ

Departamento de Análisis Matemático, Universidad de Sevilla, Spain

Dedicated to H.K. Xu on the occasion of his 60th birthday and celebrating his deep contribution in nonlinear analysis and optimization

Abstract. We revise some fixed point results for multivalued non-expansive mappings in Banach spaces and modular spaces and we state some new results depending either on the Opial modulus and or on the Partington modulus in modular spaces.

Keywords. Multivalued non-expansive mappings; Fixed point; Modular spaces; Noncompact uniform convexity.

1. Historical background

Besides the Contraction Mapping Principle, stated by Banach in 1922 [2], the first relevant results in metric fixed point theory were stated in 1965 when Browder [3, 4], Göhde [17] and Kirk [22] respectively proved the existence of a fixed point for any non-expansive single-valued mapping defined from a convex closed bounded set C of a Banach space X into C whenever X is a Hilbert space, a uniformly convex space or a reflexive space with normal structures. Since the fixed point theory for multivalued mappings has many useful applications in applied sciences. In particular, in game theory and mathematical economics, the extension of the fixed point results for single-valued mappings to the setting of multivalued mappings is a natural problem. Nadler's Theorem [30] is commonly understood as the extension of the Banach Contraction Principle to the setting of multivalued mappings. Further extensions of Nadler's Theorem consider more general contractive conditions [29, 31]. But the most interesting problems concerning some possible extensions of Nadler's Theorem focus on non-expansive multivalued mappings (see [33] and the references therein). The main difficulty to obtain a fixed point result in this setting comes from the fact that the "classic" technique for single-valued mappings, based upon Goebel-Karlovitz' Lemma on minimal invariant sets, does not work in the multivalued case. In fact, it is still an open question if Kirk fixed point theorem for single-valued non-expansive mapping in reflexive Banach spaces with normal structure also works for set-valued mappings (see [6] for a discussion about this problem). The first positive results were obtained by Lim [24] and Kirk and Massa [23] by using the Edelstein's technique on asymptotic centers. To revise these results, $K(X)$ (resp. $KC(X)$) will denote the family of all nonempty compact (resp. compact convex) subsets of X. We recall that a multivalued mapping $T : X \to K(X)$ is

*Corresponding author.
E-mail addresses: tomasd@us.es (T. Domínguez Benavides), ploren@us.es (P. Lorenzo Ramírez).
Received July 15, 2020; Accepted August 16, 2020.

said to be nonexpansive if $H(Tx,Ty) \leq \|x-y\|$ for every $x,y \in X$, where $H(\cdot,\cdot)$ denotes the Hausdorff metric given by

$$H(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for every bounded subsets A and B of X.

Theorem 1.1. (Lim 1974) [24] *Let X be a uniformly convex Banach space, let C be a closed bounded convex subset of X and let $T : C \to K(C)$ be a nonexpansive mapping. Then T has a fixed point.*

Theorem 1.2. (Kirk and Massa 1990) [23] *Let E be a nonempty closed bounded convex subset of a Banach space X and $T : E \to KC(E)$ a nonexpansive mapping. Suppose that the asymptotic center in E of each bounded sequence of X is nonempty and compact. Then T has a fixed point.*

The compactness condition of the asymptotic centers is satisfied, in particular, if X is uniformly convex. In this case, Xu [32, 33] extended Lim's result for non-selfmappings which satisfy an inward condition. Nominally:

Theorem 1.3. (Xu 1991) [32] *Assume that X is a uniformly convex Banach space, E is a closed bounded convex subset of X, and $T : E \to K(X)$ is a non-expansive mapping satisfying the weak inwardness condition: $Tx \subset \overline{I_E(x)}$ for every $x \in E$, where $I_E(x) = \{x+\lambda(y-x) : y \in E, \lambda \geq 0\}$. Then T has a fixed point.*

Replacing the weak inwardness condition by a stronger condition, Kirk-Massa's Theorem can be directly extended [33].

Theorem 1.4. (Xu 2000) [33] *Let E be a nonempty closed bounded convex subset of a Banach space X and $T : E \to KC(X)$ a non-expansive non-self-mapping which satisfies the inwardness condition: $Tx \subset I_E(x)$ for every $x \in E$. Suppose that the asymptotic center in E of each bounded sequence of X is nonempty and compact. Then T has a fixed point.*

In [33], some open questions were listed. In particular, the author wondered about the possibility of extending the above theorems to the case of a nearly uniformly convex Banach space. Such a question was solved for self-mappings as well as for non-self-mappings in [9, 10, 11]. We will revise the main results in these papers.

A Banach space X is said to be nearly uniformly convex (NUC) if and only if

$$\Delta_{X,\phi}(\varepsilon) := \inf\{1-d(0,A) : A \subset B_X \text{ convex }, \phi(A) > \varepsilon\} > 0$$

for each $\varepsilon > 0$, or equivalently

$$\varepsilon_\phi(X) := \sup\{\varepsilon \geq 0 : \Delta_{X,\phi}(\varepsilon) = 0\} = 0,$$

where ϕ is a measure of noncompactness.

Assume that α is the Kuratowski measure, β is the separation measure and χ is the Hausdorff measure (for the definitions see, for instance, [1] or [15]). We have the following relationships among the different moduli:

$$\Delta_{X,\alpha}(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon)$$

and consequently

$$\varepsilon_\alpha(X) \geq \varepsilon_\beta(X) \geq \varepsilon_\chi(X).$$

We list the most general results appearing in [11] for multivalued non-expansive mappings satisfying an inwardness condition.

Theorem 1.5. *Let X be a Banach space such that $\varepsilon_\alpha(X) < 1$ and let C be a closed bounded convex subset of X. If $T : C \to KC(X)$ is a non-expansive and 1-χ-contractive mapping such that $T(C)$ is a bounded set, and satisfies*

$$Tx \subset I_C(x) \quad \forall x \in C,$$

then T has a fixed point.

Theorem 1.6. *Let X be a Banach space such that $\varepsilon_\beta(X) < 1$ and let C be a closed bounded convex and separable subset of X. If $T : C \to KC(X)$ is a non-expansive and 1-χ-contractive mapping such that $T(C)$ is a bounded set, and satisfies*

$$Tx \subset I_C(x) \quad \forall x \in C,$$

then T has a fixed point.

For self-mappings, a more general result can be obtained. We recall [1] that X is NUC if and only if X is reflexive and

$$\Delta_X(\varepsilon) := \inf\{1 - \|x\| : \{x_n\} \subset B_X, x_n \rightharpoonup x, \liminf_n \|x_n - x\| \geq \varepsilon\} > 0$$

for each $\varepsilon > 0$, or equivalently

$$\Delta_0(X) := \sup\{\varepsilon > 0 : \Delta_X(\varepsilon) = 0\} = 0.$$

It is not difficult to check the following inequalities:

$$\Delta_{X,\alpha}(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_X(\varepsilon)$$

and so

$$\varepsilon_\beta(X) \geq \Delta_0(X)$$

when X is reflexive. Since X is reflexive when $\varepsilon_\beta(X) < 1$, the following result in [5] extends Theorem 1.5 and Theorem 1.6 in the case of a self-mapping.

Theorem 1.7. *Let C be a nonempty weakly compact convex subset of a Banach space X such that $\Delta_0(X) < 1$ and $T : C \to KC(C)$ is a nonexpansive mapping. Then T has a fixed point.*

2. MEASURE OF NONCOMPACTNESS AND FIXED POINT FOR MULTIVALUED NON-EXPANSIVE MAPPINGS IN MODULAR SPACES

In this section, we are going to show a recent result about the existence of fixed points for multivalued non-expansive mappings in modular spaces, by using noncompactness measures. We start recalling some notions and facts concerning modular spaces. For more details, the reader is referred to [20, 21, 25, 26, 27].

Definition 2.1. Let X be an arbitrary vector space.

(a) A functional $\rho : X \to [0, \infty]$ is called a convex modular if, for arbitrary $x, y \in X$,
  (i) $\rho(x) = 0$ if and only if $x = 0$;
  (ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$;
  (iii) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.

(b) A modular ρ defines a corresponding modular space, i.e. the vector space X_ρ given by

$$X_\rho = \{x \in X : \rho(x/\lambda) < \infty \text{ for some } \lambda > 0\}.$$

The formula

$$\|x\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \leq 1\right\},$$

defines a norm, which is frequently called the Luxemburg norm. Thus, any convex modular space can be simultaneously studied as a normed space with the Luxemburg norm and any topic on these spaces can be divided in two parts corresponding either to the modular space or the normed space. We should notice that, in non-trivial cases, the Luxemburg norm is particularly difficult to compute and, as a consequence, the results for the normed space should be deduced from the corresponding for the modular.

Definition 2.2.
(a) The sequence $\{x_n\}$ in X_ρ is said to be ρ-convergent to $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(b) A subset C of X_ρ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.
(c) A subset C of X_ρ is ρ-bounded if

$$diam_\rho(C) = \sup\{\rho(x-y); x, y \in C\} < \infty.$$

Given a subset A of X_ρ, we denote $co(A)$ its convex hull, i.e., the smallest convex subset of X_ρ containing A. From the convexity of the modular ρ, we easily deduce that $diam_\rho(co(A)) = diam_\rho(A)$.

From the ρ-convergence, it is possible to define a topology. Since any norm-convergent sequence in X_ρ is ρ-convergent, every ρ-closed set in X_ρ is norm-closed.

We say that the modular satisfies the Δ_2-condition if there exists $K > 0$ such that $\rho(2x) \leq K\rho(x)$ for any $x \in X_\rho$. In this case, it is easy to prove that ρ-convergence is identical to norm-convergence. Thus, ρ-compact (ρ-closed) and norm-compact (norm-closed) sets are identical and also ρ-bounded and norm-bounded sets are identical. From now on, we will assume that the modular space satisfies the Δ_2-condition. Hence, we will drop the prefix ρ for the notions in Definition 2.2. Note that, in this case, a kind of "triangular inequality" is satisfied, because for every $x, y \in X_\rho$ we have $\rho(x+y) \leq (K/2)(\rho(x) + \rho(y))$. As a consequence X_ρ satisfies the Fatou's Property, i.e., $\rho(x) \leq \liminf_n \rho(x_n)$ whenever $\lim_n \rho(x_n - x) = 0$. Therefore, the closed ρ-balls are closed and the ρ-diameter of a set is the same as the ρ-diameter of its ρ-closure.

Definition 2.3. The growth function $\omega_\rho : [0,\infty) \to [0,\infty)$ of a modular ρ is defined as follows

$$\omega_\rho(t) := \sup\left\{\frac{\rho(tx)}{\rho(x)} : 0 < \rho(x) < \infty\right\} \quad \text{for all } t \geq 0.$$

Lemma 2.1. [8] *Let ρ be a convex modular such that ρ satisfies the Δ_2-condition. Then the growth function ω_ρ has the following properties:*

(1) $\omega_\rho(t) < \infty$ for every $t \in [0,\infty)$.
(2) $\omega_\rho(t) = 0$ if and only if $t = 0$.
(3) $\omega_\rho : [0,\infty) \to [0,\infty)$ is a convex, strictly increasing function. So, it is continuous.
(4) $\omega_\rho(\alpha\beta) \leq \omega_\rho(\alpha)\omega_\rho(\beta)$ for all $\alpha, \beta \geq 0$.
(5) $\omega_\rho^{-1}(\alpha)\omega_\rho^{-1}(\beta) \leq \omega_\rho^{-1}(\alpha\beta)$ for all $\alpha, \beta \geq 0$, where ω_ρ^{-1} is the function inverse of ω.

(6) $\|x\|_\rho \leq \frac{1}{\omega_\rho^{-1}(1/\rho(x))}$ *for every* $x \in X_\rho \backslash 0$.

Many fixed point theorems for nonlinear mappings defined in modular spaces require a kind of uniform continuity of the modular.

Definition 2.4. A modular ρ is said to be uniformly continuous on bounded sets if, for every bounded subset C of X_ρ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\rho(x+y) - \rho(x)| < \varepsilon$$

whenever $x \in C$, $y \in X_\rho$, $\rho(y) < \delta$.

The following result can be found in [13] (see also [19]).

Lemma 2.2. *Assume that* X_ρ *is a convex modular space satisfying the* Δ_2*-condition. Then*

(1) ρ *is uniformly continuous on bounded sets.*
(2) For any bounded sequence $\{x_n\} \subset X_\rho$ *and any bounded subset* C *of* X_ρ*, the function* $\Phi : C \to \mathbb{R}$ *defined by* $\Phi(x) = \limsup_n \rho(x_n - x)$ *is weakly lower semicontinuous.*

The definition of the Kuratowski measure of noncompactness was introduced in modular function spaces by Khamsi and Kozlowski [20]. Similarly, we can extend such a concept to modular abstract spaces in the following way.

Definition 2.5. Let X_ρ be a convex modular satisfying the Δ_2-condition and $\mathscr{B}$ the family of nonempty ρ-bounded subset of X_ρ. Define the Kuratowski measure of noncompactness of $A \in \mathscr{B}$ by

$$\begin{aligned}\alpha(A) = \inf\{\varepsilon > \quad 0 \quad &: A \text{ can be covered by a finite number of sets} \\ &\text{with } \rho-\text{diameter smaller than } \varepsilon\},\end{aligned}$$

and the Hausdorff measure of noncompactness of A by

$$\begin{aligned}\chi(A) = \inf\{\varepsilon > \quad 0 \quad &: A \text{ can be covered by a finite number of} \\ &\rho-\text{balls of radius smaller than } \varepsilon\}.\end{aligned}$$

It is clear that $\alpha(A) \geq \chi(A)$ for every bounded set A. The functions $\alpha(\cdot)$ and $\chi(\cdot)$ satisfy the usual properties of a measure of noncompactness (see, [12, Proposition 3.2]). By using these properties, it is easy to prove that, as in the case of a normed space, $\chi(B_\rho(0,r)) = r$ for every $r > 0$.

Similarly as Goebel and Sekowski [16] did for Banach spaces, we can define a scaling for modular spaces using a measure of noncompactness.

Definition 2.6. [20] Let X_ρ be a convex modular. The ρ-modulus of the noncompact convexity associated to α is defined in the following way

$$\begin{aligned}\Delta_\alpha(r,\varepsilon) = \sup\{c > 0 \quad : \quad &\text{for any } \rho\text{-bounded convex } A \subset B_\rho(0,r) \text{with} \\ &\alpha(A) \geq r\varepsilon, \text{ then } d_\rho(0,A) \leq (1-c)r\},\end{aligned}$$

for every $r > 0$, $\varepsilon \in (0, \alpha_r)$, where $\alpha_r =: \alpha(B_\rho(0,r))/r \geq 1$.

The ρ-characteristic of the noncompact convexity of X_ρ associated with the measure of noncompactness α is defined for every $r > 0$ by

$$\varepsilon_\alpha(r) = \sup\{0 \le \varepsilon \le \alpha_r : \Delta_\alpha(r,\varepsilon) = 0\}.$$

Definition 2.7. Let X_ρ be a convex modular. The modular space is said to be uniformly ρ-noncompact convex (UNC in short) if and only if $\varepsilon_\alpha(r) = 0$ for every $r > 0$.

Theorem 2.1. [12] *Let ρ be a convex modular satisfying the Δ_2-condition. Assume that X_ρ is UNC and for some $\varepsilon_0 \in (0,1)$ and there exists $\gamma > 0$ such that $\lim_{r\to 0} \Delta_\alpha(r,\varepsilon_0) \ge \gamma$. Let C be a nonempty ρ-closed ρ-bounded convex subset of the space X_ρ and $T : C \to KC_\rho(C)$ a ρ-nonexpansive mapping. Then T has a fixed point.*

3. Opial modulus and Partington modulus for modular spaces

In this section, we are going to introduce an Opial modulus for modular spaces. The Opial property for modular spaces has already been considered (see [20, section 4.4] when ρ-a.e. convergence is considered). However, we must highlight that our modulus concerns weak convergence, which does not imply the ρ-a.e. convergence (see, for instance, [18, 13.43(a)]). We also define a modulus, inspired by the Partington [28] modulus in linear normed spaces, that, in reflexive spaces, is an UNC modulus. We will prove some properties of these moduli and a strong connection between both of them which will be used to obtain some fixed point results in next section.

Definition 3.1. Let ρ be a convex modular satisfying the Δ_2-conditions and such that the space X_ρ does not satisfy the Schur property. For $s > 0$, $c > 0$ we define

$$r(s,c) = \inf\{(1/s)\liminf_n \rho(x_n + x) - 1\},$$

where the infimum is running over all $x, x_n \in X_\rho$, $\rho(x) \ge sc$, $x_n \rightharpoonup 0$, and $\liminf_n \rho(x_n) \ge s$.

Note that the function $r(s,\cdot)$ is non-decreasing in c. Furthermore, from Lemma 2.2, this modulus satisfies $r(s,c) \ge c-1$ for all $s,c > 0$. In particular, $r(s,c) \ge 0$ for all $s > 0$ and $c \ge 1$.

We will prove that, as in the case of linear normed spaces, the Opial modulus is continuous. In this case, since it depends on two variables, we will check that it is jointly continuous on both variables. We need a previous lemma.

Lemma 3.1. *Let $r(\cdot,\cdot)$ the Opial modulus. Then,*

$$r(s,c) = \inf\{(1/s)\liminf_n \rho(x_n + x) - 1\},$$

where $x, x_n \in X_\rho$, $M \ge \rho(x) \ge sc$, $x_n \rightharpoonup 0$, $M \ge \liminf_n \rho(x_n) \ge s$, with M being equal to $s(1 + c)\omega(2)^2$.

Proof. Taking an arbitrary weakly null sequence such that $\liminf_n \rho(x_n) = s$ and an arbitrary x such that $\rho(x) = cs$, we have $\liminf_n \rho(x_n + x) \le \frac{\omega(2)}{2}(\liminf_n \rho(x_n) + \rho(x)) \le \omega(2)s(1+c)/2$. Thus $1 + r(s,c) \le \omega(2)(1+c)/2$. By convexity, it is clear that we can assume that either $\rho(x) = sc$ or $\liminf_n \rho(x_n) = s$ in the definition of $r(s,c)$. In the first case, we choose x_n, x such that $s(r(s,c)+1)+s \ge \liminf_n \rho(x_n + x)$. We have

$$\liminf_n \rho(x_n) \le \frac{\omega(2)}{2}(\liminf_n \rho(x_n + x) + \rho(x)) \le \frac{\omega(2)}{2} s(r(s,c) + 2 + c)) \le M$$

The second case is similar.

□

Proposition 3.1. *The function $(s,c) \to r(s,c)$ is continuous at any point.*

Proof. Let $s_0, c_0 > 0$ be fixed. By Lemma 3.1, we can assume that x_n, x in the definition of $r(s,c)$ belong to a bounded set C for every $s \leq 2s_0$ and every $c \leq 2c_0$. We choose $M \geq 1$ such that $\rho(x+y) \leq M$ for every $x,y \in C$. For an arbitrary $\varepsilon > 0$, we choose $\delta > 0$ as in Definition 2.4. Since $s \to 1/s$ is a continuous function, it suffices to prove that the function $a(s,c) = \inf\{\liminf_n \rho(x_n + x)\}$, x_n, x as above, is continuous. We only consider values of $(s,c) \in (s_0/2, 2s_0) \times (c_0/2, 2c_0)$.

Claim 3.1. (a) Let $c > c_0$ such that $c - c_0 < c_0(\delta/M)$. Then,

$$a(s_0,c) \geq a(s_0,c_0) \geq a(s_0,c) - \varepsilon.$$

(b) As a consequence, we have $|a(s_0,c) - a(s_0,c_0)| < \varepsilon$ if $|c - c_0| < (c_0/2)(\delta/M)$.

Indeed, choose an arbitrary $x \in C$ such that $\rho(x) \geq c_0 s$. Denote $y = \frac{c}{c_0}x$. We have $\rho(y) \geq \frac{c}{c_0}\rho(x) \geq cs_0$ and, for any weakly null sequence x_n such that $\liminf_n \rho(x_n) \geq s_0$, we have

$$\liminf_n \rho(x + x_n) \geq \liminf_n \rho(y + x_n) - \varepsilon \geq a(s_0,c) - \varepsilon$$

because $\rho(x-y) \leq \delta$. Thus, $a(s_0,c_0) \geq a(s_0,c) - \varepsilon$.

Claim 3.2. (a) Let $s > 0$ be such that

$$|s - s_0| < s_0 \max\left\{\frac{\delta}{\omega(2)M}, \left(\omega\left(\frac{M\omega(2)}{M\omega(2)-\delta}\right) - 1\right)\right\}.$$

Then, $|a(s_0,c) - a(s,c)| < \varepsilon$ for every $c \in (c_0/2, 2c_0)$.

(b) As a consequence, we have $|a(s_0,c) - a(s,c)| < \varepsilon$ if

$$|s - s_0| < \frac{s_0}{2} \max\left\{\frac{\delta}{\omega(2)M}, \left(\omega\left(\frac{M\omega(2)}{M\omega(2)-\delta}\right) - 1\right)\right\}.$$

Indeed, let $x \in C$ be such that $\rho(x) \geq cs_0$ and let x_n be a weakly null sequence in C such that $\liminf_n \rho(x_n) \geq s_0$. Denote $y = \frac{s}{s_0}x$, $y_n = \frac{s}{s_0}x_n$. We have $\rho(y) \geq cs$; $\rho(y_n) \geq s$ and

$$\liminf_n \rho(x + x_n) \geq \liminf_n \rho(y + y_n) - \varepsilon \geq a(s,c) - \varepsilon$$

because $\rho(x + x_n - y - y_n) \leq \frac{\omega(2)M(s-s_0)}{s_0} \leq \delta$. Thus, $a(s_0,c) \geq a(s,c) - \varepsilon$.

Conversely, choose $x \in C$ such that $\rho(x) \geq sc$ and x_n weakly null such that $\liminf_n \rho(x_n) \geq s$. Denote $y = kx$, and $y_n = kx_n$, where $\omega(1/k) = s/s_0 > 1$ (and so $k < 1$). We have $\rho(y) \geq cs_0$; $\rho(y_n) \geq s_0$ and

$$\liminf_n \rho(x + x_n) \geq \liminf_n \rho(y + y_n) - \varepsilon \geq a(s_0,c) - \varepsilon$$

because $\rho(x + x_n - y - y_n) \leq \omega(2)(1-k)M < \delta$. Thus,

$$a(s,c) \geq a(s_0,c) - \varepsilon.$$

Finally, for any pair $s,c > 0$ satisfying the condition as in Claim 3.1 (b) and 3.2 (b), we have

$$|a(s,c) - a(s_0,c_0)| \leq |a(s,c) - a(s_0,c)| + |a(s_0,c) - a(s_0,c_0)| \leq 2\varepsilon.$$

□

Definition 3.2. Assume that ρ is a convex modular satisfying the Δ_2-condition. For every $s>0$, we define

$$\Delta_{X_\rho}(s,\varepsilon)=\inf\left\{1-\frac{\rho(x)}{s}:\{x_n\}\subset B_\rho(0,s),x_n\rightharpoonup x,\liminf_n\rho(x_n-x)>s\varepsilon\right\},$$

for every $\varepsilon>0$ such that the above set is non-empty.

Observe that the function $\Delta_{X_\rho}(s,\cdot)$ can be defined for $\varepsilon\in(0,d(s)]$, where $d(s)=\sup\{\varepsilon>0:$ there exists a weakly convergent sequence, say to x, in $B_\rho(0,s)$ satisfying $\liminf_n\rho(x_n-x)>s\varepsilon\}$. We will define $\Delta_{X_\rho}(s,\varepsilon)=1$ if $\varepsilon>d(s)$. Note that for any weakly convergent sequence $\{x_n\}$ say to x, such that $\lim_n\rho(x_n-x)>s\varepsilon$, taking a subsequence and using [12, Proof of Theorem 3.12], we can assume that $\lim_{n,m;n\neq m}\rho(x_n-x_m)=ds$ exists. By the weak lower continuity of the modular (Lemma **??**), we have $d\geq\varepsilon$, which implies $\alpha_s\geq d(s)$ for every $s>0$. We will prove that $d(s)\geq 1$ in reflexive spaces.

Theorem 3.1. *Let X_ρ be a modular infinite-dimensional reflexive space. Then, there exists a weakly convergent sequence $\{x_n\}$ in $B_\rho(0,s)$ such that* $\lim\rho(x_n-x)\geq s$.

Proof. Choose an infinite-dimensional separable subspace Y of X_ρ and let B be the set $\{x\in Y:\rho(x)\leq s\}$. We have $\chi(B)=s$. Let S be a countable dense subset of B and B_n a sequence formed by all balls centered at any point of S with radius any rational number less than s. Denote $M_n=\cup_{m=1}^n B_m$. Since $\chi(M_n)<s$, there exists $x_n\in B\setminus M_n$ for every $n\in\mathbb{N}$. Note that for every $n\in\mathbb{N}$, there exists n' such that the set $\{x_i:i=1,\dots,n\}\subset M_{n'}$. Thus, $x_{n'}\neq x_i$, $i=1,\dots,n$ and the range of $\{x_n\}$ contains infinitely many elements. Let $\{y_n\}$ be a weakly convergent subsequence of $\{x_n\}$, say to x, formed by distinct points and such that $\lim_n\rho(y_n-x)=d$ does exist. Note that $d\geq\chi(\{y_n:n\in N\})$. We have that $\{y_n:n\in\mathbb{N}\}\cap M_n$ is a finite set. We claim that $\chi(\{y_n:n\in N\})=s$. Indeed, otherwise, we could assume that $\{y_n:n\in\mathbb{N}\}$ is contained in a ball $B_\rho(x_0,rs)$ for some $r<1$. By use of Lemma 2.2, we obtain $m\in\mathbb{N}$ such that $\{y_n:n\in\mathbb{N}\}\subset M_m$, which is a contradiction. □

Next, we will compare this modulus with the α-modulus for noncompact uniform convexity defined before (Definition 2.6).

Theorem 3.2. *For every $s>0$, and $\varepsilon\in(0,d(s))$, we have $\Delta_\alpha(s,\varepsilon)\leq\Delta_{X_\rho}(s,\varepsilon)$ for $s>0$*

Proof. For any fixed $s>0$ and $\varepsilon\in(0,d(s))$, we choose an arbitrary $a>\Delta_{X_\rho}(s,\varepsilon)$. We can find a sequence $\{x_n\}$, which is weakly convergent to x, such that $x_n\in B_\rho(s)$, $\lim_n\rho(x_n-x)\geq s\varepsilon$, and $\rho(x)>1-a$. Taking a subsequence and using [12, Proof of Theorem 3.12], we can assume that $\lim_{n,m;n\neq m}\rho(x_n-x_m)=ds$ exists. By the weak lower continuity of the modular (Lemma **??**), we have $d\geq\varepsilon$. Thus, $\alpha(\overline{\text{co}}\{x_n:n\geq k\})\geq\varepsilon s$ for every $k\in\mathbb{N}$, which implies $d(0,\overline{\text{co}}\{x_n:n\geq k\})\leq 1-\Delta_\alpha(s,\varepsilon)$. For a given $\eta>0$, we can choose $z_k\in\text{co}\,\{x_n:n\geq k\}$ such that $\rho(z_k)\leq 1-\Delta_\alpha(s,\varepsilon)+\eta$. Since the sequence $\{z_k\}$ lies in a weakly compact set, we can find a weakly convergent subsequence $\{y_k\}$ of $\{z_k\}$. Its weak limit must be x because $\cap_{k=1}^\infty\overline{\text{co}}\{x_n:n\geq k\}=\{x\}$. The weak lower semicontinuity of the modular implies $\rho(x)\leq 1-\Delta_\alpha(s,\varepsilon)+\eta$. Since η is arbitrary, we obtain $\Delta_\alpha(s,\varepsilon)\leq a$. □

Denote $r_{X_\rho}(c)=\inf\limits_{0<s}r(s,c)$. Next, we show a strong connection between the Opial modulus and the Partington modulus in modular spaces.

Theorem 3.3. *Let ρ be a convex modular satisfying the Δ_2-condition. Then, the following conditions are equivalent:*

(1) $r_{X_\rho}(1) > 0$*;*
(2) $\inf_{s>0} \Delta_{X_\rho}(s,\varepsilon_0) > 0$ *for some* $\varepsilon_0 \in (0,1)$.

Proof. Assume that $r_{X_\rho}(1) > 0$. Suppose that $\inf_{0<s} \Delta_{X_\rho}(s,\varepsilon) = 0$ for all $\varepsilon \in (0,1)$. Then, for all $n \geq 1$, there exists $s_n > 0$ such that

$$\Delta_{X_\rho}(s_n,\varepsilon_n) < \frac{1}{2(n+1)},$$

where $\varepsilon_n = \dfrac{n}{n+1}$. On the other hand, there exists a sequence $\{x_k\} \subset B_\rho(0,s_n)$, which is weakly convergent to a point x with $\liminf_k \rho(x_k - x) \geq s_n\varepsilon_n$ and

$$1 - \frac{\rho(x)}{s_n} < \Delta_{X_\rho}(s_n,\varepsilon_n) + \frac{1}{2(n+1)}.$$

Consider the weakly sequence $x_k - x$. Since

$$\rho(x) \geq s_n\varepsilon_n \frac{1-\Delta_{X_\rho}(s_n,\varepsilon_n) - \frac{1}{2(n+1)}}{\varepsilon_n},$$

we have

$$\begin{aligned} r\left(\varepsilon_n s_n, \frac{1-\Delta_{X_\rho}(s_n,\varepsilon_n) - \frac{1}{2(n+1)}}{\varepsilon_n}\right) &\leq \frac{\liminf_k \rho(x_k)}{\varepsilon_n s_n} - 1 \\ &\leq \frac{1}{\varepsilon_n} - 1 \\ &= \frac{1}{n}. \end{aligned}$$

Bearing in mind that

$$\frac{1-\Delta_{X_\rho}(s_n,\varepsilon_n) - \frac{1}{2(n+1)}}{\varepsilon_n} > 1,$$

we have

$$r(\varepsilon_n s_n, 1) \leq \frac{1}{n}.$$

Thus, $r_{X_\rho}(1) = 0$, which is a contradiction.

Conversely, we assume that $\inf_{0<s} \Delta_{X_\rho}(s,\varepsilon_0) > 0$ for some $\varepsilon_0 \in (0,1)$ and suppose that $r_{X_\rho}(1) = 0$. Then, for any $n \geq 1$, we can find $s_n > 0$ such that

$$r(s_n,1) < \frac{1}{2n}.$$

Since $\varepsilon_0 \in (0,1)$, we can choose $n \geq 1$ large enough such that $\varepsilon_0 < \frac{n}{n+1}$. Take a weakly null sequence $\{x_k\}$ with $\liminf_k \rho(x_k) \geq s_n$ and $x \in X_\rho$ with $\rho(x) \geq s_n$ such that

$$\liminf_k \frac{\rho(x_k + x)}{s_n} < 1 + r(s_n,1) + \frac{1}{2n}.$$

Without loss of generality, we can assume that

$$x_k + x \in B_\rho\left(0, s_n\left(1 + r(s_n,1) + \frac{1}{2n}\right)\right).$$

Since

$$\rho(x) \geq s_n \frac{1 + r(s_n,1) + \frac{1}{2n}}{1 + r(s_n,1) + \frac{1}{2n}},$$

we obtain

$$\begin{aligned} 1 - \frac{1}{1 + r(s_n,1) + \frac{1}{2n}} &\geq \Delta_{X_\rho}\left(s_n\left(1 + r(s_n,1) + \frac{1}{2n}\right), \frac{1}{1 + r(s_n,1) + \frac{1}{2n}}\right) \\ &\geq \Delta_{X_\rho}\left(s_n\left(1 + r(s_n,1) + \frac{1}{2n}\right), \varepsilon_0\right). \end{aligned}$$

Therefore

$$\frac{1}{n+1} \geq \Delta_{X_\rho}\left(s_n\left(1 + r(s_n,1) + \frac{1}{2n}\right), \varepsilon_0\right) \geq \inf_{0<s} \Delta_{X_\rho}(s, \varepsilon_0),$$

which is a contradiction. □

4. Main results

The method of asymptotic centers play an important role in the fixed point theory for nonexpansive multivalued mappings on Banach spaces. Some definitions and results concerning asymptotic centers can be adapted to modular spaces in a straightforward way.

Let C be a nonempty ρ-closed ρ-bounded of the space X_ρ and $\{x_n\}$ be a bounded sequence in X_ρ. We define

$$r_\rho(C, \{x_n\}) = \inf\{\limsup_n \rho(x_n - x) : x \in C\},$$

and

$$A_\rho(C, \{x_n\}) = \{x \in C : \limsup_n \rho(x_n - x) = r_\rho(C, \{x_n\})\}.$$

The number $r_\rho(C, \{x_n\})$ and the (possible empty) set $A_\rho(C, \{x_n\})$ are called, respectively, the ρ-*asymptotic radius* and the ρ-asymptotic center of $\{x_n\}$ in C.

Obviously, $A_\rho(C, \{x_n\})$ is a convex set as C is. Furthermore, the set $A_\rho(C, \{x_n\})$ is nonempty and closed whenever C is weakly compact. Indeed, for any $m \geq 1$, consider the set

$$A_m = \left\{y \in C : \limsup_n \rho(x_n - y) \leq r + \frac{1}{m}\right\},$$

where $r = r_\rho(C, \{x_n\})$. Clearly A_m is nonempty and convex. Also, A_m is closed by Lemma 2.2. It follows from the weak compactness of C that $A_\rho(C, \{x_n\}) = \cap_{m \geq 1} A_m \neq \emptyset$.

The sequence $\{x_n\}$ is said to be *regular* relative to C if the asymptotic radii of all subsequences of $\{x_n\}$ (relative to C) are the same. If, in addition, $A_\rho(C, \{y_n\}) = A_\rho(C, \{x_n\})$ for every subsequence $\{y_n\}$ of $\{x_n\}$, we say that $\{x_n\}$ is *asymptotically uniform* relative to C.

Similarly as Goebel [14], Kirk [22] and Lim [24] did for Banach spaces, the following lemma can be proved.

Lemma 4.1. *Let ρ be a convex modular satisfying the Δ_2-condition. Let C be a nonempty closed bounded separable subset of the space X_ρ and $\{x_n\}$ be a bounded sequence in X_ρ. Then $\{x_n\}$ contains a regular and asymptotically uniform subsequence relative to C.*

If D is a ρ-bounded subset of X_ρ, the ρ-Chebyshev radius of D relative to C is defined by

$$r_\rho(C,D) := \inf\{\sup\{\rho(x-y) : y \in D\} : x \in C\}.$$

Notice that if $D \subset C$, then $r_\rho(C,D) \leq diam_\rho(D) \leq \omega(2) r_\rho(C,D)$.

In the following theorem, we find a connection between the asymptotic center of a sequence and the Opial modulus of the modular space. As in the case of normed spaces [5], it will provide a sufficient condition for a modular space to have the fixed point property for multivalued ρ-nonexpansive mappings.

Theorem 4.1. *Let ρ be a convex modular satisfying the Δ_2-condition. Let C be a nonempty weakly compact convex subset of the space X_ρ and let $\{x_n\}$ be a sequence in C, which is regular relative to C. Then*

$$r_\rho(C,A) \leq \frac{1}{1+r_{X_\rho}(1)} r_\rho(C,\{x_n\}).$$

where $A := A_\rho(C,\{x_n\})$

Proof. Denote $r = r_\rho(C,\{x_n\})$. Since C is a weakly compact set, we can assume, by passing to a subsequence if necessary, that $\{x_n\}$ is weakly convergent to a point $x \in C$ and $\lim_n \rho(x_n - x)$ exists. Since $\{x_n\}$ is regular relative to C, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. Let $z \in A$, $z \neq x$. From Lemma 2.2, we obtain

$$\lim_n \rho(x_n - x) \geq \limsup_n \rho(x_n - z) = r \geq \rho(x-z)$$

Now, we consider $y_n = x_n - x$, which is weakly null with $\lim_n \rho(x_n - x) \geq \rho(x-z)$ and $y = x - z$. Then

$$1 + r(\rho(x-z),1) \leq \frac{\liminf_n \rho(x_n - z)}{\rho(x-z)} = \frac{r}{\rho(x-z)}.$$

Therefore

$$\rho(x-z) \leq \frac{r}{1+r(\rho(x-z),1)} \leq \frac{r}{1+r_{X_\rho}(1)}$$

and we deduce the inequality in the statement. □

Next, we state a fixed point theorem for multivalued ρ-nonexpansive mappings. To this end, we will make use of the following results which were proved in [12].

Proposition 4.1 (Proposition 6.1, [12]). *Let ρ be a convex modular satisfying the Δ_2-condition. Let C be a nonempty weakly compact separable subset of the space X_ρ and $T : C \to KC_\rho(C)$ a ρ-nonexpansive mapping. Suppose that $\{x_n\} \subset C$ is an approximate fixed point sequence for T, i.e., $\lim_n d_\rho(x_n, Tx_n) = 0$. Then, there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*

$$Tx \cap A \neq \emptyset, \text{ for all } x \in A := A_\rho(C,(z_n)).$$

Theorem 4.2 (Theorem 6.5, [12]). *Let ρ be a convex modular satisfying the Δ_2-condition and C a nonempty closed convex subset of X_ρ. Assume that $T : C \to KC_\rho(C)$ is a $k-$contractive mapping. Suppose that A is a closed bounded convex subset of C such that $Tx \cap A \neq \emptyset$ for every $x \in A$. Then, T has a fixed point in A.*

Now we state the main fixed point theorem in this paper.

Theorem 4.3. *Let ρ be a convex modular satisfying the Δ_2-condition. Assume that $r_{X_\rho}(1) > 0$. Let C be a nonempty weakly compact bounded convex subset of the space X_ρ and $T : C \to KC_\rho(C)$ a ρ-nonexpansive mapping. Then T has a fixed point.*

Proof. The proof of this theorem is analogous in some parts to that of Theorem 6.6 in [12]. Since the Δ_2-condition is satisfied and T is a continuous compact valued self-mapping, we can construct a closed convex subset of C, which is separable and invariant under T (see [15]). Thus, we can suppose that C is separable. For a fixed element $x_0 \in C$ and for each $n \geq 1$, we define the mapping

$$T_n x := \frac{1}{n} x_0 + \left(1 - \frac{1}{n}\right) T x, \quad x \in C.$$

Then T_n is a multivalued self ρ-contraction mapping and hence has a fixed point $\{x_n\} \in C$ by Theorem 4.2. It is easily seen that $\lim_n d_\rho(x_n, Tx_n) = 0$. Without loss of generality, we may assume that $\{x_n\}$ is regular and asymptotically uniform with respect to C.

According to the previous proposition, we can also assume that

$$Tx \cap A_1 \neq \emptyset, \quad \forall x \in A_1 := A_\rho(C, \{x_n\}).$$

On the other hand, we can apply Theorem 4.1 to obtain

$$r_\rho(C, A_\rho(C, \{x_n\})) \leq \lambda r_\rho(C, \{x_n\}),$$

where $\lambda := \frac{1}{1 + r_{X_\rho}(1)} < 1$.

Fix $z_0 \in A_1$ and define a contraction

$$T_n x := \frac{1}{n} z_0 + \left(1 - \frac{1}{n}\right) T x, \quad x \in C.$$

The convexity of A_1 implies $T_n x \cap A_1 \neq \emptyset$ for all $x \in A_1$. By use of Theorem 4.2, we have that T_n has a fixed point in A_1, say x_n^1. Thus, we can find an approximate fixed point sequence x_n^1 in A_1 for T, which is regular and asymptotically uniform with respect to C. Again we apply Theorem 4.1 and Proposition 4.1 to have

$$Tx \cap A_2 \neq \emptyset, \quad \forall x \in A_2 := A_\rho(C, \{x_n^1\})$$

and

$$r_\rho(C, A_2) \leq \lambda r_\rho(C, \{x_n^1\}).$$

Continuing a inductive process, we can get an approximate fixed point sequence $\{x_n^m\}$ in $A_m = A_\rho(C, \{x_n^{m-1}\})$ such that

$$r_\rho(C, A_m) \leq \lambda r_\rho(C, \{x_n^{m-1}\}.$$

Consequently,

$$r_\rho(C, A_m) \leq \lambda^{m-1} r_\rho(C, A_1).$$

Choosing $x_m \in A_m$, we proceed as in the proof of Theorem 6.6 in [12] to deduce that $\{x_m\}$ is a ρ-convergent to a point $x \in C$, which is in fact a fixed point of T. □

Corollary 4.1. *Let ρ be a convex modular satisfying the Δ_2-condition. Assume that $\inf_{0<s} \Delta_{X_\rho}(s, \varepsilon_0) > 0$ for some $\varepsilon_0 \in (0, 1)$. Let C be a nonempty weakly compact bounded convex subset of the space X_ρ and $T : C \to KC_\rho(C)$ a ρ-nonexpansive mapping. Then T has a fixed point.*

In [12], the authors proved an analogous result to the Kirk-Massa's theorem in modular spaces. With the following example considered in [12], we show that the above corollary extends in some sense Theorem 6.2 in [12].

Example 4.1. Let $\{p_n\}$ be a sequence in $[1,\infty)$ such that $1 < p =: \liminf_n p_n \leq \limsup_n p_n < \infty$. Consider the Musielak-Orlicz space ℓ^{p_n} for the modular

$$\rho(x) = \sum_{n=1}^{\infty} |x(n)|^{p_n}.$$

It is well known that (ℓ^{p_n}, ρ) is a modular sequence space, which satisfies the Δ_2-condition. Furthermore, $(\ell^{p_n}, \|\cdot\|_\rho)$ is a reflexive Banach space whenever $\|\cdot\|_\rho$ is the corresponding Luxemburg norm [7, Theorem 18].

Define $\sigma : \ell^{p_n} \to [0,\infty)$ by

$$\sigma(x) = \sup_n \{|x(n)|^{p_n} + \frac{1}{2} \sum_{k=n+1}^{\infty} |x(k)|^{p_k}\}.$$

It is clear that σ is a convex modular and (ℓ^{p_n}, σ) satisfies the Δ_2-condition. Furthermore,

$$\frac{1}{2}\rho(x) \leq \sigma(x) \leq \rho(x),$$

which implies that the Luxemburg norms $\|\cdot\|_\sigma$ and $\|\cdot\|_\rho$ are equivalent. (Nominally,

$$\|x\|_\sigma \leq \|x\|_\rho \leq 2^{1/P}\|x\|_\sigma,$$

where $P = \limsup_n p_n$). Thus, $(\ell^{p_n}, \|\cdot\|_\sigma)$ is a reflexive Banach.

Assume that $\{x_n\}$ is a sequence in the closed ball $B_\sigma(0,s)$ weakly sequence convergent to a point x such that $\lim_n \sigma(x_n - x) \geq s\varepsilon$. Passing to a subsequence if necessary, we can assume that the limits $\lim_n \sigma(x_n - x)$ and $\lim_{n\neq m} \sigma(x_n - x_m)$ exist. Then $\lim_{n\neq m} \sigma(x_n - x_m) \geq s\varepsilon$. Now we follow the same argument as in Example 6.7 in [12] to prove that $\Delta_{X_\rho}(s,\varepsilon) \geq \varepsilon/2$. Therefore $\inf_{s>0} \Delta_{X_\rho}(s,\varepsilon) > 0$ for all $\varepsilon \in (0,1)$. for all $\varepsilon \in (0,1)$.

Considering the sequence $\{e_n\}$, it is clear that $\liminf_n \sigma(e_n - x) \geq 1$ for every $x \in \ell^{p_n}$. Thus, $r_\rho(B_\sigma(0,1), \{e_n\}) = 1$. Note that $\sigma(e_n - (1/2)e_k) = 1$ for every $k < n$, which implies that the sequence $\{e_n/2\}$ lies in the asymptotic center of $\{e_n\}$ with respect to the unit ball. Therefore, $A_\rho(\{e_n\}, B_\sigma(0,1))$ is not compact.

Open problems.

(1) Another question listed as Problem 1 in [33] was the following: Let X be a uniformly smooth Banach space, E a nonempty closed bounded convex subset of X, and $T : E \to K(E)$ a nonexpansive mapping. Does T have a fixed point? This problem was solved in [5]. Nominally, the following result was proved:

Theorem 4.4. *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\rho_X'(0) < 1/2$, and $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

As far as we know, the notion of the uniform smoothness has not been considered in modular spaces. We think that a suitable notions of smoothness in modular spaces could provide new fixed point theorems for multivalued non-expansive mappings in this setting.

(2) It would be interesting to compare Corollary 4.1 in this paper and Theorem **??** from [12]. It is known that every NUC modular space satisfies that the intersection of any decreasing sequence $\{C_n\}_{n\geq 1}$ of closed bounded convex subsets of X_ρ is non-empty [12, Theorem 4.8]. This property implies the reflexivity of X_ρ (see, for instance, [1, Lemma V.1.5]). Thus, any bounded closed convex set in a UNC space is weakly compact. Moreover, it is not difficult to check that the assumptions in Theorem **??** from [12] imply that for some $\varepsilon_0 \in (0,1)$ there exists $\gamma > 0$ such that $\Delta_\alpha(s,\varepsilon_0) \geq \gamma/2$ for every $s \in (0,s_0]$. From Theorem 3.2, we obtain that $\Delta_{X_\rho}(s,\varepsilon_0) \geq \gamma/2$ for every $s \in (0,s_0]$. By using the similar arguments to those in the proof of Theorem 3.3, it can be proved that $\inf\{r(s,1) : 0 < s < s_0\} > 0$. However, we do not know if this condition (and the continuity of the Opial modulus) implies that $r_{X_\rho}(1) > 0$.

Acknowledgements

The authors were partially supported by MICINN, Grant PGC2018-098474-B-C21 and Andalusian Regional Government Grant FQM-127.

References

[1] J.M. Ayerbe, T. Domínguez Benavides, G. López Acedo, Measures of Noncompactness in Metric Fixed Point Theory, Springer, Birkhäuser, 1997.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications, Fund. Math. 3 (1922), 133-181.

[3] F.E. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, Proc. Nat. Acad. Sci. USA 43 (1965), 1272-1276.

[4] F.E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. USA 54 (1965), 1041-1044.

[5] T. Domínguez Benavides, B. Gavira, The fixed point property for multivalued non-expansive mappings, J. Math. Anal. Appl. 328 (2007), 1471-1483.

[6] T. Domínguez Benavides, B. Gavira, Does Kirk's Theorem holds for multivalued non-expansive mappings?, Fixed Point Theory and Appl. 2010 (2010), Article ID 546761.

[7] T. Domínguez Benavides, M. Japón, Reflexivity and fixed points on variable Lebesgue spaces, preprint.

[8] T. Domínguez Benavides, M. A. Khamsi, S. Samadi, Asymptotically regular mappings in modular function spaces, Sci. Math. Jpn. 53 (2001), 295-304.

[9] T. Domínguez Benavides, P. Lorenzo, Fixed point theorems for multivalued nonexpansive mappings without uniform convexity, Abs. Appl. Anal. 2003 (2003), 375-386.

[10] T. Domínguez Benavides, P. Lorenzo, Asymptotic centers and fixed points for multivalued nonexpansive mappings, Ann. Univ. Mariae Curie-Sklodowska Sect. A 58 (2004), 37-45.

[11] T. Domínguez Benavides, P. Lorenzo, Fixed point theorems for multivalued nonexpansive mappings satisfying inwardness conditions, J. Math. Anal. Appl. 291 (2004), 100-108.

[12] T. Domínguez Benavides, P. Lorenzo, Measures of noncompactness in modular spaces and fixed point theorems for multivalued nonexpansive mappings, preprint.

[13] T. Domínguez-Benavides, S. M. Moshtaghioun, A. Sadeghi Hafshejani, Fixed points for several classes of mappings in variable Lebesgue spaces, Optimization, DOI: 10.1080/02331934.2019.1711086.

[14] K. Goebel, On a fixed point theorem for multivalued nonexpansive mappings, Ann. Univ. Marie Curie-Sklodowska 29 (1975) 70-72.

[15] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[16] K. Goebel, T. Sekowski, The modulus of noncompact convexity, Ann. Univ. Mariae Curie-Sklodowska Sect. A 38 (1984), 41-48.
[17] D. Göhde, Zum Prinzip der kontraktiven Abbildung, Math. Nachr. 30 (1965), 251-258.
[18] E. Hewitt, K. Stromberg, Real and Abstract Analysis, Springer-Verlag: Berlin, New York, 1965.
[19] A. Kamińska, On uniform convexity of Orlicz spaces, Nederl. Akad. Wetensch. Indag. Math. 44 (1982), 27-36.
[20] M.A. Khamsi, W.M. Kozlowski, Fixed Point Theory in Modular Function Spaces, Birkäuser: Basel, Switzerland, 2015.
[21] W.M. Kozlowski, Modular Function Spaces, Dekker, New York, Basel, 1988.
[22] W.A. Kirk, A fixed point theorem for mappings which do not increase the distances, Amer. Math. Monthly 72 (1965), 1004-1006.
[23] W.A. Kirk, S. Massa, Remarks on asymptotic and Chebyshev centers, Houston J. Math. 16 (1990), 357-364.
[24] T.C. Lim, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space, Bull. Amer. Math. Soc. 80 (1974), 1123-1126.
[25] J. Musielak, W. Orlicz, On modular spaces, Studia Math, 18 (1959) 591-597.
[26] H. Nakano, Modulared Semi-ordered Linear Spaces, Maruzen Co., Tokyo, 1950.
[27] H. Nakano, Topology of Linear Topological Spaces. Maruzen Co., Ltd., Tokyo, 1951.
[28] J.P. Partington, On nearly uniformly convex Banach spaces, Math. Proc. Camb. Phil. Soc. 93 (1983), 127-129.
[29] S. Reich, Fixed points of contractive funcitons, Boll. Un. Mat. Ital. 5 (1972), 26–42.
[30] S.B. Nadler, Jr., Multi-valued contraction mappings, Pacific J. Math. 30 (1969), 475-488.
[31] H.K. Xu, Fixed point theorems for single-valued and set-valued mappings, Thesis, Zhejiang Univ., 1985.
[32] H.K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal. 16 (1991), 1127–1138.
[33] H.K. Xu, Metric fixed point theory for multivalued mappings, Dissertationes Math. (Rozprawy Mat.) 389 (2000), 39.