Appl. Set-Valued Anal. Optim. 2 (2020), No. 2, pp. 175-181
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.2.2020.2.04

FIXED POINT RESULTS FOR MULTI-VALUED LOCALLY CONTRACTIVE OPERATORS

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Abstract. In this paper, we present several fixed point results for multi-valued (ε,α)-uniformly locally contractive operators in complete ε-chainable metric spaces. Connections to the mathematics of multi self-similar sets are given.

Keywords. Multi-valued operator; ε-chainable metric space; Fixed point; Multi-valued contraction; Multi-valued locally contractive operator.

1. Introduction

Let (X,d) be a metric space. We denote by $P(X)$ the family of all nonempty subsets of X, by $P_{cl}(X)$ the family of all nonempty closed subsets of X, by $P_b(X)$ the family of all nonempty bounded subsets of X, and by $P_{cp}(X)$ the family of all nonempty compact subsets of X.

The following notations will be used throughout this paper:

(1) the gap functional generated by d,

$$D_d(A,B) := \inf\{d(a,b) \mid a \in A,\ b \in B\};$$

(2) the excess functional of A over B generated by d,

$$e_d(A,B) := \sup\{D_d(a,B) \mid a \in A\};$$

(3) the Hausdorff-Pompeiu functional generated by d,

$$H_d(A,B) = \max\{e_d(A,B), e_d(B,A)\}.$$

The diameter of a set $Y \in P(X)$ will be denoted by

$$diam_d(Y) := \sup_{x,y \in Y} d(x,y).$$

We will avoid the subscript d if no confusion occurs.

A metric space (X,d) is called ε-chainable if $\varepsilon > 0$ and, for every $x,\tilde{x} \in X$, there exists an ε-chain, i.e., a finite set of points $x = x_0, x_1, ..., x_n = \tilde{x}$ (where the natural number $n \geq 1$ may depend on both u and v) from X such that $d(x_{i-1}, x_i) < \varepsilon$, for each $i \in \{1,2,...,n\}$. A well-chainable

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Received April 20, 2020; Accepted May 31, 2020.

metric space is an ε-chainable metric space with respect to any $\varepsilon > 0$. For other details, we refer to [10].

In 1969, Nadler proved a fixed point result in complete ε-chainable metric spaces for multi-valued locally contractive operators and extended some previous theorems given by Edelstein in [4] for the single-valued case. See also [7] for local fixed point theorems of single-valued operators.

A multi-valued operator $F : X \to P(X)$ is said to be a multi-valued (ε, α)- uniformly locally contractive operator if, $\varepsilon > 0$, $\alpha \in [0,1[$ and the following implication holds

$$x, y \in X \text{ and } d(x,y) < \varepsilon \Rightarrow H(F(x), F(y)) \leq \alpha d(x,y).$$

A fixed point of operator F is an element $x^* \in X$ such that $x^* \in F(x^*)$. A strict fixed point of F is an element $x^* \in X$ such that $F(x^*) = \{x^*\}$. From now on, we will denote by $Fix(F)$ the fixed point set of F and by $SFix(F)$ the set of all strict fixed points of F.

Theorem 1.1. [6] *Let $\varepsilon > 0$ and (X,d) be a complete metric ε-chainable space. Suppose that $F : X \to P_{cp}(X)$ is a multi-valued (ε, α)- uniformly locally contractive operator. Then $Fix(F) \neq \emptyset$.*

For the case of well-chained metric space, we refer to [5]. For the case of the sequence of multi-valued operators and fixed points, we refer to [2]. For other interesting fixed point theorems for multi-valued locally contractive type operators, we refer to [11] and [12]. For other fixed point results for multi-valued operators, see [1] and [8].

In 1983, Barcz extended the above result for the case of nonlinear contractions with bounded and closed values. Denote by Ψ the family of all functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, which are increasing, continuous from the right and satisfy the following two conditions:

(i) $\sum_{n \geq 0} \psi^n(t) < \infty$;

(ii) $\psi(t) = 0$ implies $t = 0$.

A multi-valued operator $F : X \to P(X)$ is said to be a multi-valued nonlinear (ε, ψ)-uniformly locally contractive operator if, $\varepsilon > 0$, $\psi \in \Psi$ and the following implication holds

$$x, y \in X \text{ and } d(x,y) < \varepsilon \Rightarrow H(F(x), F(y)) \leq \psi(d(x,y)).$$

Theorem 1.2. [3] *Let $\varepsilon > 0$ and (X,d) be a complete metric ε-chainable space. Suppose that $F : X \to P_{b,cl}(X)$ is a multi-valued (ε, ψ)-uniformly locally contractive operator. Then $Fix(F) \neq \emptyset$.*

The purpose of this paper is to give several fixed point results for multi-valued (ε, α)-uniformly locally contractive operators in complete ε-chainable metric spaces. Connections to the mathematics of multi self-similar sets are also given.

2. Main results

Let (X,d) be a metric space and $F : X \to P(X)$ be a multi-valued operator. We say that $(x_n)_{n \in \mathbb{N}} \subset X$ is a sequence of successive approximations for F starting from x_0 if $x_{n+1} \in F(x_n)$, for every $n \in \mathbb{N}$. For related notions, we refer to [6], [9] and [12].

The following result is a fixed point theorem for an (ε, α)-uniformly locally contractive operator in a complete metric space (X,d). It follows immediately from [3, Theorem 3.1]. We notice that the boundedness of the values of the multi-valued operator can be omitted.

Theorem 2.1. *Let (X,d) be a complete metric space and let $F : X \to P_{cl}(X)$ be a multi-valued (ε,α)-uniformly locally contractive operator. Suppose there exists $x_0 \in X$ such that $D(x_0,F(x_0)) < \varepsilon$. Then, the following conclusions hold:*

(a) $Fix(F) \neq \emptyset$;

(b) there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for F starting from $x_0 \in X$, which converges to a fixed point $x^*(x_0)$ of F.

(c) the following apriori estimation holds

$$d(x_{n+1},x^*(x_0)) \leq \frac{(q\alpha)^n}{1-q\alpha} d(x_0,x_1), \text{ for every } n \in \mathbb{N},$$

where $q \in]1, \frac{1}{\alpha}[$ can be arbitrary chosen.

Proof. Lettting $\psi(t) = \alpha t$ for every $t \in \mathbb{R}_+$ in [3, Theorem 3.1], we obtain (a) and (b). For (c), by a classical approach, we have

$$d(x_n,x_{n+p}) \leq \frac{(q\alpha)^n}{1-q\alpha} d(x_0,x_1), \text{ for every } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*.$$

Letting $p \to \infty$, we get the desired conclusion immediately. □

Using the properties of the excess functional e, we can prove the following slight extension of the above theorem in a similar approach.

Theorem 2.2. *Let (X,d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multi-valued operator for which there exist $\varepsilon > 0$ and $\alpha \in [0,1[$ such that the following implication holds*

$$x,y \in X \text{ with } d(x,y) < \varepsilon \ \Rightarrow \ e(F(x),F(y)) \leq \alpha d(x,y). \tag{2.1}$$

Then, the following conclusions hold:

(a) $Fix(F) \neq \emptyset$;

(b) there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for F starting from x_0, which converges to a fixed point of F and we have the following apriori estimation

$$d(x_{n+1},x^*(x_0)) \leq \frac{(q\alpha)^n}{1-q\alpha} d(x_0,x_1), \text{ for every } n \in \mathbb{N},$$

where $q \in]1, \frac{1}{\alpha}[$ can be arbitrary chosen.

If (X,d) is a metric space and $F : X \to P_{cp}(X)$ is a multi-valued upper semi-continuous operator, then the fractal operator generated by F is denoted by $\hat{T} : P_{cp}(X) \to P_{cp}(X)$ and defined by

$$\hat{T}(Y) := F(Y) = \bigcup_{y\in Y} F(y), \text{ for all } Y \in P_{cp}(X).$$

A fixed point of $\hat{T}$ is called a multi self-similar set for T. In the same context, if $F_i : X \to P_{cp}(X)$ (where $i \in \{1,2,\cdots,m\}$) are multi-valued upper semi-continuous operators, then we denote by

$$\hat{T}_F : P_{cp}(X) \to P_{cp}(X),$$

given by

$$\hat{T}_F(Y) := \bigcup_{i=1}^{m} F_i(Y), \text{ for } Y \in P_{cp}(X),$$

the fractal operator generated by the iterated multi-function system

$$F = (F_1, \cdots, F_m).$$

By the upper semi-continuity of each F_i $(i \in \{1, 2, \cdots, m\})$, the operator $\hat{T}_F$ is well-defined. It is easy to observe that any multi-valued (ε, α)-uniformly locally contractive operator $F : X \to P_{cp}(X)$ is upper semi-continuous.

Recall that a single-valued operator $f : X \to X$ is said to be an (ε, α)-uniformly locally contraction if, $\varepsilon > 0$, $\alpha \in [0, 1[$ and the following implication holds

$$x, y \in X,\ x, y < \varepsilon \ \Rightarrow\ d\left(f(x), f(y)\right) \leq \alpha d(x, y). \tag{2.2}$$

In the single-valued case, the following result was proved by Nadler [6]

Theorem 2.3. [6] *Let (X, d) be a complete ε-chainable metric space and let $f : X \to X$ be an (ε, α)-uniformly locally contraction. Then, there exists a unique fixed point $x^* \in X$ of f and, for each $x \in X$, the sequence of Picard iterations $(f^n(x))_{n \in \mathbb{N}}$ converges to x^* as $n \to \infty$.*

In the multi-valued case, we have the following result.

Theorem 2.4. *Let (X, d) be a metric space and let $F : X \to P_{cp}(X)$ be an (ε, α)-uniformly locally contractive operator. Then, the fractal operator $\hat{T} : P_{cp}(X) \to P_{cp}(X))$ generated by F is an (ε, α)-uniformly locally contraction on $P_{cp}(X)$.*

Proof. We show that if $A, B \in P_{cp}(X)$ with $H(A, B) < \varepsilon$, then

$$H(\hat{T}(A), \hat{T}(B)) \leq \alpha H(A, B).$$

For this purpose, let $u \in \hat{T}(A)$ be arbitrary. Then, there exists $a \in A$ such that $u \in F(a)$. For $a \in A$, by the compactness of B, there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) < \varepsilon.$$

By the compactness of the values of F, let $v \in F(b) \subset \hat{T}(B)$ such that

$$d(u, v) = D(u, F(b)).$$

Thus, using the fact that $F : X \to P_{cp}(X)$ is (ε, α)-uniformly locally contractive, we conclude

$$\begin{aligned} d(u, v) &= D(u, F(b)) \\ &\leq H(F(a), F(b)) \\ &\leq \alpha d(a, b) \\ &\leq \alpha H(A, B). \end{aligned}$$

By a similar method, we can also show that, for every $v \in \hat{T}(B)$, there exists $u \in \hat{T}(A)$ such that $d(u, v) \leq \alpha H(A, B)$. Hence

$$H(\hat{T}(A), \hat{T}(B)) \leq \alpha H(A, B).$$

□

Before our next result, we recall the following important theorem proved by Xu in [12].

Theorem 2.5. *If (X, d) is an ε-chainable metric space, then $(P_{cp}(X), H_d)$ is also an ε-chainable metric space.*

Using Theorem 2.3 and the above two Theorems, we immediately obtain the following existence, uniqueness and approximation result.

Theorem 2.6. *Let (X,d) be a complete ε-chainable metric space and let $F : X \to P_{cp}(X)$ be an (ε,α)-uniformly locally contractive operator. Then, there exists a unique $A^* \in P_{cp}(X)$ such that $A^* = F(A^*)$ (i.e., a multi self-similar set) and the sequence $(A_n)_{n\in\mathbb{N}} \subset P_{cp}(X)$, defined by $A_{n+1} := F(A_n)$, $n \in \mathbb{N}$ converges to A^* as $n \to \infty$, for every $A_0 \in P_{cp}(X)$.*

Proof. By Theorem 2.4, the operator $\hat{T} : P_{cp}(X) \to P_{cp}(X))$ is an (ε,α)-uniformly locally contraction. Using Theorem 2.3 and Theorem 2.5, we get that $\hat{T}$ has a unique fixed point $A^* \in P_{cp}(X)$ and the corresponding sequence of successive approximations starting from any $A_0 \in P_{cp}(X)$ converges to A^*. This completes the proof. □

Example 2.1. Let $X := \{(x,y) \in \mathbb{R}^2 : x = \cos t, y = \sin t, t \in [0, \frac{3\pi}{2}]\}$ and $F : X \to P_{cp}(X)$ be a mapping defined by

$$F(\cos t, \sin t) = \{(1,0), (\cos\frac{t}{2}, \sin\frac{t}{2})\}.$$

Then, F is a multi-valued (ε,λ)-uniformly locally contractive operator and it is not a global contraction. Since X is complete ε-chainable, Theorem 2.6 applies and F has a unique fixed point $A^* = \{(1,0)\} \in X \times X$ and the sequence of Picard iteration given by

$$F^n(\cos t, \sin t) = \{(1,0), (\cos\frac{t}{2^n}, \sin\frac{t}{2^n})\}$$

converges to $\{(1,0)\}$ as $n \to \infty$, for every $t \in [0, \frac{3\pi}{2}]$.

Let us consider now an iterated multi-function system $F = (F_1, \cdots, F_m)$, where $F_i : X \to P_{cp}(X)$ (for $i \in \{1,2,\cdots,m\}$) are multi-valued (ε,α)-uniformly locally contractive operators.

Theorem 2.7. *Let (X,d) be a metric space. Suppose that $F_i : X \to P_{cp}(X)$ are (ε,α_i)-uniformly locally contractive operators, for $i \in \{1,2,\cdots,m\}$. Then, the fractal operator $\hat{T}_F : P_{cp}(X) \to P_{cp}(X)$, generated by the iterated multi-function system $F = (F_1,\cdots,F_m)$ is an (ε,α)-uniformly locally contraction, where $\alpha := \max\{\alpha_1,\cdots,\alpha_m\}$.*

Proof. First, notice that, by the upper semi-continuity of each F_i, the operator $\hat{T}_F$ is well defined. We have to show that if $A,B \in P_{cp}(X)$ with $H(A,B) < \varepsilon$, then

$$H(\hat{T}_F(A), \hat{T}_F(B)) \le \lambda H(A,B).$$

For this purpose, let $u \in \hat{T}_F(A)$. Then, there exists $k \in \{1,2,\cdots,m\}$ and $a \in A$ such that $u \in F_k(a)$. For $a \in A$, by the compactness of the set B, there exists $b \in B$ such that

$$d(a,b) \le H(A,B) < \varepsilon.$$

Let $v \in F_k(b) \in \hat{T}_F(B)$ be such that

$$d(u,v) = D(u, F_k(b)).$$

Then, using the fact that $F_k : X \to X$ is (ε,λ_k)-uniformly locally contractive, we get

$$\begin{aligned} d(u,v) &= D(u, F_k(b) \\ &\le H(F_k(a), F_k(b)) \\ &\le \alpha_k d(a,b) \\ &\le \alpha H(A,B). \end{aligned}$$

Using a similar method, we can also show that, for every $v \in \hat{T}_F(B)$, there exists $u \in \hat{T}_F(A)$ such that $d(u,v) \leq \alpha H(A,B)$. Hence

$$H(\hat{T}_F(A), \hat{T}_F(B)) \leq \alpha H(A,B).$$

This completes the proof. □

Using the above results, we get the following existence, uniqueness and approximation result.

Theorem 2.8. *Let (X,d) be a complete ε-chainable metric space and let $F_i : X \to P_{cp}(X)$ be (ε, α_i)-uniformly locally contractive operators, for $i \in \{1,2,\cdots,m\}$. Then, there exists a unique $A^* \in P_{cp}(X)$ such that $A^* = \bigcup_{i=1}^{m} F_i(A^*)$ and the sequence $(A_n)_{n\in\mathbb{N}} \subset P_{cp}(X)$, defined by*

$$A_{n+1} := \bigcup_{i=1}^{m} F_i(A_n),$$

$n \in \mathbb{N}$ converges to A^* as $n \to \infty$, for every $A_0 \in P_{cp}(X)$.

Proof. From Theorem 2.7, we have that the operator $\hat{T}_F : P_{cp}(X) \to P_{cp}(X))$, given by

$$\hat{T}_F(Y) := \bigcup_{i=1}^{m} F_i(Y),$$

is an (ε, α)-uniformly locally contraction. Using Theorem 2.3 and Theorem 2.5, we get that the operator $\hat{T}_F$ has a unique fixed point A^* in $P_{cp}(X)$ and the corresponding sequence of successive approximations starting from any $A_0 \in P_{cp}(X)$ converges to A^*. This completes the proof. □

Remark 2.1. It is an open question to prove the similar results for the case of multi-valued nonlinear (ε, ψ)-uniformly locally contractive operators or to other classes of contractive type multi-valued operators. Another question is to extend the above results for multi-valued operators with closed values.

Acknowledgments

The authors were supported by the Babeş-Bolyai University Cluj-Napoca under ATC grants.

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