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ON SPLIT GENERALIZED MIXED EQUALITY EQUILIBRIUM AND SPLIT EQUALITY FIXED POINT PROBLEMS

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Abstract. For $p \geq 2$, a new iterative algorithm is introduced and used to approximate a common element of the set of solutions of a split generalized mixed equality equilibrium problem and the set of solutions of a split equality fixed point problem for quasi-ϕ-nonexpansive mappings in p-uniformly convex and uniformly smooth real Banach spaces. A strong convergence theorem is proved without any compactness-type assumption on the mappings. Furthermore, our theorem, which is applicable, in particular, in L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$ for $2 \leq p < \infty$, complements several important recent results that were established in 2-uniformly convex and uniformly smooth real Banach spaces.

Keywords. Fixed point; Quasi-ϕ-nonexpansive; p-uniformly convex; Uniformly smooth.

1. Introduction

Let C be a nonempty closed and convex subset of a real Banach space E whose dual space is denoted by E^*. Let $\varphi : C \to \mathbb{R}$ be a mapping, $A : C \to E^*$ be a mapping and $f : C \times C \to \mathbb{R}$ be a bifunction. The *generalized mixed equilibrium problem* (GMEP) is a problem of finding

$$u^* \in C \quad \text{such that} \quad f(u^*, y) + \varphi(y) - \varphi(u^*) + \langle y - u^*, Au^* \rangle \geq 0, \ \forall y \in C. \tag{1.1}$$

We denote the set of solutions of (1.1) by GMEP and it is given by

$$\text{GMEP} = \{u^* \in C : f(u^*, y) + \varphi(y) - \varphi(u^*) + \langle y - u^*, Au^* \rangle \geq 0, \ \forall y \in C\}.$$

It is well known that this class of problems contains the class of equilibrium problems, optimization problems, fixed point problems, variational inequality problems, and so on. These classes of nonlinear problems have been studied extensively by various authors in the setting of real Hilbert spaces and more general real Banach spaces (see, e.g., [3, 9, 10, 11, 19, 21] and the references therein).

Let H_1 and H_2 be two real Hilbert spaces, and let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be two nonlinear mappings with nonempty fixed point sets, $F(T) := \{x \in H_1 : Tx = x\}$ and $F(S) := \{u \in H_2 : Su = u\}$, respectively. Let H_3 be an arbitrary real Hilbert space. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear mappings with adjoints A^* and B^*, respectively. Moudafi [17] recently studied the following problem:

$$\text{find} \quad u^* \in F(T), \quad v^* \in F(S) \quad \text{such that} \quad Au^* = Bv^*.$$

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This problem, which is the so-called *split equality fixed point problem* (SEFPP), has recently attracted much attention and interest of lots of researchers due to its numerous applications in, for example, game theory, intensity modulated therapy treatment planning, decomposition methods for partial differential equations, fully discretized models of inverse problems which arise from phase retrievals and in medical image reconstructions (see, e.g., [1, 4, 7] and the references therein). We next denote the set of solutions of the split equality fixed point problem by SEFPP. If $H_2 = H_3$ and B is the identity mapping on H_2, the SEFPP is reduced to the so-called *split common fixed point problem* (SCFPP), introduced by Censor and Segal [8]. This problem has also been extensively studied by various authors (see, e.g., [6, 12, 24] and the references therein).

In 2014, Bnouchachem [5] introduced the following split equilibrium problem in real Hilbert spaces. Let $f : C\times C \to \mathbb{R}$ and $g : Q\times Q \to \mathbb{R}$ be bifunctions, where C and Q are nonempty closed convex subsets of real Hilbert spaces, H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear mapping. The *split equilibrium problem* (SEQP) is as the following:

$$\begin{gathered}\text{find } u^*\in C \text{ such that } f(u^*,y)\ge 0,\ \forall y\in C \text{ and}\\ v = Au^*\in Q \text{ solves } g(v,z)\ge 0,\ \forall z\in Q.\end{gathered}$$

In 2015, Zhao *et al.* [16] studied the following more general problem called the *split equality equilibrium problem* (SEEP), which is a problem of finding $(u^*, v^*) \in C\times Q$ such that

$$f(u^*,y)+\vartheta(y)-\vartheta(u^*)\ge 0,\ \forall y\in C,\ g(v^*,z)+\varphi(z)-\varphi(v^*)\ge 0 \text{ and } Au^* = Bv^*,$$

where $f : C\times C\to\mathbb{R}$ and $g : Q\times Q\to\mathbb{R}$ are bifunctions, $\vartheta : C\to\mathbb{R}\cup\{\infty\}$ and $\varphi : Q\to\mathbb{R}\cup\{\infty\}$ are proper lower semi-continuous functions, $A : C\subset H_1\to H_3$ and $B : Q\subset H_2\to H_3$ are bounded linear mappings, where H_1 H_2 and H_3 are real Hilbert spaces. They proposed an iterative algorithm under their setting and proved that the sequence generated by their algorithm converges weakly to an element in $F(T)\cap F(S)\cap(SEFPP)$. Under additional assumption that T and S are semi-compact, they established the strong convergence of the sequence generated by their algorithm.

Very recently, Monday [18] studied the much more general problem, called the *split generalized mixed equality equilibrium problem* (SGMEEP) in real Banach spaces.

Setting 1.

(1) E_1 and E_2 are 2-uniformly convex and uniformly smooth real Banach spaces with dual spaces, E_1^* and E_2^*, respectively. E_3 is an arbitrary smooth real Banach space.
(2) C and M are nonempty closed and convex subsets of E_1 and E_2, respectively. $f : C\times C$ and $g : M\times M\to\mathbb{R}$ are bifunctionals.
(3) $\vartheta : C\to\mathbb{R}\cup\{\infty\}$ and $\varphi : M\to\mathbb{R}\cup\{\infty\}$ are proper lower semi-continous and convex functions.
(4) $U : C\to E_1^*$ and $V : M\to E_2^*$ are continuous monotone mappings.
(5) $A : E_1\to E_3$ and $B : E_2\to E_3$ are bounded linear mappings.

The SGMEEP is a problem of finding $(u^*, v^*) \in C\times M$ such that

$$\begin{gathered}f(u^*,y)+\psi(y)-\psi(u^*)+\langle y-u^*, Uu^*\rangle\ge 0,\ \forall y\in C,\\ g(v^*,z)+\varphi(z)-\varphi(v^*)+\langle z-v^*, Vv^*\rangle\ge 0,\ \forall z\in M, \text{ and } Au^* = Bv^*.\end{gathered}$$

Monday [18] studied the SGMEEP in conjunction with the SEFPP for quasi-ϕ-nonexpansive mappings under the above setting. He constructed an iterative algorithm and proved that the sequence generated by his algorithm converges strongly to an element in the intersection of the solution set of the SGMEEP and the solution set of the SEFPP. The quasi-ϕ-nonexpansive operators were not assumed to be semi-compact. Monday's theorem improves and complements several important recent results.

Remark 1.1. 2-uniformly convex real Banach spaces are more general than Hilbert spaces (for example, they include L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$, for $1 < p \leq 2$), while L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$, for $2 < p < \infty$ are not 2-uniformly convex.

It is our purpose in this paper to study the SGMEEP and the SEFPP for quasi-ϕ-nonexpansive mappings in p-uniformly convex and uniformly smooth real Banach spaces, $2 \leq p < \infty$. We propose a new iterative algorithm under our setting and prove that the sequence generated by our algorithm converges strongly to an element in the intersection of the solution set of SGMEEP and the solution set of the SEFPP. Moreover, our operators are not assumed to be semi-compact. Our theorem is, in particular, applicable in $L_p,\ l_p$ and the Sobolev spaces $W_p^m(\Omega)$, for all p such that $2 \leq p < \infty$.

2. Preliminaries

Let E be a strictly convex and smooth real Banach space. For $p > 1$, define $J_p : E \to 2^{E^*}$ by

$$J_p(x) := \{u^* \in E^* : \langle x, u^* \rangle = \|x\| \|u^*\|,\ \|u^*\| = \|x\|^{p-1}\}.$$

J_p is called the *generalized duality mapping on* E. If $p = 2$, J_2 is called the *normalized duality map* and is denoted by J. In a real Hilbert space H, J is the identity map on H. It is easy to see from the definition that

$$J_p(x) = \|x\|^{p-2} Jx, \quad \text{and} \quad \langle x, J_p x \rangle = \|x\|^p,\ \forall x \in E.$$

It is well-known that if E is smooth, then J is single-valued and if E is strictly convex, J is one-to-one, and J is surjective if E is reflexive.

Let E be a reflexive, strictly convex and smooth real Banach space with dual space E^*. For $p > 1$, Chidume [13] defined the following functionals: $\phi_p : E \times E \to \mathbb{R}^+$ by

$$\phi_p(x,y) := \|x\|^p - p\langle x, J_p y \rangle + \|J_p y\|^p,\ \forall x, y \in E,$$

and $V_p : E \times E^* \to \mathbb{R}^+$ by

$$V_p(x, x^*) := \|x\|^p - p\langle x, x^* \rangle + \|x^*\|^p,\ \forall x \in E,\ x^* \in E^*.$$

It is clear from these definitions that

$$V_p(x, x^*) = \phi_p(x, J_p^{-1} x^*),\ \forall x \in E,\ x^* \in E^*.$$

Remark 2.1. If $p = 2$, we denote $\phi_2(x,y)$ simply as $\phi(x,y)$. So,

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2,\ \forall x, y \in E.$$

Definition 2.1. Let C be a nonempty subset of a real normed space E and let $T : C \to C$ be a mapping. Then,

(i) T is called *quasi-ϕ-nonexpansive* [20] if $F(T) := \{x \in C : Tx = x\} \neq \emptyset$, and

$$\phi_p(x^*, Tx) \leq \phi_p(x^*, x),\ \forall x \in C,\ x^* \in F(T).$$

(ii) T is said to be closed if for any sequence $\{x_n\} \subset C$ and $x_n \to x$ and $Tx_n \to y$, then $y = Tx$.

In the sequel, we shall need the following lemmas, which was recently established by Chidume [13].

Lemma 2.1. *Let E be a reflexive, strictly convex and smooth real Banach space. Then, for $p>1$,*

$$V_p(u,u^*)+p\langle J_p^{-1}u^*-u,v^*\rangle \le V_p(u,u^*+v^*),\ \forall u\in E,\ u^*,v^*\in E^*.$$

Lemma 2.2. *For $p>1$, let E be a p-uniformly convex and smooth real Banach space. Let D be a nonempty closed and convex subset of E. Let $x_1\in E$ and $P_D:E\to D$ be the metric projection of E onto D. Then*

$$x^*=P_Dx_1,\ \Leftrightarrow\ \langle x^*-z,J_p(x_1-x^*)\rangle\ge 0,\ \forall z\in D.$$

Lemma 2.3. *Let E be a p-uniformly convex and smooth real Banach space with dual space E^*. For $p>1$, let $J_p:E\to E^*$ be the generalized duality map. Then,*

$$\|J_p^{-1}x-J_p^{-1}y\|\le \kappa_p\|x-y\|^{\frac{1}{p-1}},\ \forall x,y\in E,$$

where $\kappa_p=\left(\frac{1}{c_2}\right)^{\frac{1}{p-1}}$ for some constant $c_2>0$.

Lemma 2.4. *Let E be a reflexive, strictly convex and smooth real Banach space. Then, for $p>1$, there exists a constant $c_p>0$ such that, for all $x,u,v\in E$,*

$$\phi_p\big(x,J_p^{-1}(\lambda J_pu+(1-\lambda)J_pv)\big)\le\lambda\phi_p(x,u)+(1-\lambda)\phi_p(x,v)-c_pw_p(\lambda)\|J_pu-J_pv\|^p,$$

where $w_p(\lambda)=\lambda^p(1-\lambda)+\lambda(1-\lambda)^p$.

We shall also need the following well known lemmas.

Lemma 2.5. [2] *Let D be a nonempty closed and convex subset of a reflexive, strictly convex and smooth real Banach space E. Then,*

$$\phi(u,\Pi_Dy)+\phi(\Pi_Dy,y)\le\phi(u,y),\ \forall\ u\in D,y\in E,$$

where Π_D is the generalized projection of E onto D.

Lemma 2.6. [14] *Let E be a uniformly convex and uniformly smooth real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n,y_n)\to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n-y_n\|\to 0$, as $n\to\infty$.*

Lemma 2.7. [23] *For $p>1$, let E be a p-uniformly convex real Banach space. Then, there exists a constant $c_p>0$ such that for all $x,y\in E$, the following inequality holds:*

$$\|\alpha x+(1-\alpha)y\|^p\le\alpha\|x\|^p+(1-\alpha)\|y\|^p-c_pw_p(\alpha)\|x-y\|^p. \tag{2.1}$$

2.1. **Analytical representations of generalized duality mappings in L_p, l_p, and W_m^p, spaces, $1<p<\infty$.** Using the analytic representation of the *normalized* duality mappings in L_p, l_p, and W_m^p, $1<p<\infty$ (see, e.g., Lindenstrauss and Tzafriri [15]) and the relation $J_p(x)=||x||^{p-2}J(x)$, we obtain the analytical representations of generalized duality mappings in these spaces as follows:

$$J_pz=y\in l_q,\ y=\{|z_1|^{p-2}z_1,|z_2|^{p-2}z_2,...\},\ z=\{z_1,z_2,...\},$$
$$J_p^{-1}z=y\in l_p,\ y=\{|z_1|^{q-2}z_1,|z_2|^{q-2}z_2,...\},\ z=\{z_1,z_2,...\},$$
$$J_pz=|z(s)|^{p-2}z(s)\in L_q(G),\ s\in G,$$

$$J_p^{-1}z = |z(s)|^{q-2}z(s) \in L_p(G),\ s \in G,$$

and

$$J_p z = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha}(|D^{\alpha}z(s)|^{p-2}D^{\alpha}z(s)) \in W^q_{-m}(G), \quad m>0, s\in G.$$

3. MAIN RESULTS

We are now ready to prove our main theorem. Before we go, we need to prove the following new lemma, which will play a key role in the proof of our theorem and which is also of independent interest.

Lemma 3.1. *Let E be a reflexive, strictly convex and smooth real Banach space and let C be a nonempty closed and convex subset of E. Let $r>0$, $p>1$ and $x\in E$. Define a mapping $T_r : E \to C$ by*

$$T_r(x) := \{z \in C : \Theta(z,y) + \frac{1}{r}\langle y - z, J_p z - J_p x\rangle \ge 0,\ \forall y \in C\},$$

where Θ is any bifunction. Then, T_r is single valued. For $p\ge 2$, the following inequality holds:

$$\phi_p(q, T_r(x)) + \phi_p(T_r(x), x) \le \phi_p(q,x),\ \forall q \in F(T_r).$$

Proof. From [22], we have that T_r is single valued. Using the definition of ϕ_p and the definition of T_r, we have

$$\begin{aligned}
\phi_p(q,T_r(x)) + \phi_p(T_r(x),x) &= \|q\|^p - p\langle q, J_pT_r(x)\rangle + \|J_pT_rx\|^p + \|T_r(x)\|^p \\
&\quad - p\langle T_r(x), J_px\rangle + \|J_px\|^p \\
&= \|q\|^p - p\langle q, J_pT_r(x)\rangle + \|J_pT_rx\|^p + \|T_r(x)\|^p - p\langle q, J_px\rangle \\
&\quad - p\langle T_r(x) - q, J_px\rangle + \|J_px\|^p \\
&= \phi_p(q,x) - p\langle q, J_pT_r(x)\rangle + 2\langle T_r(x), J_pT_r(x)\rangle \\
&\quad - p\langle T_r(x) - q, J_px\rangle \\
&\le \phi_p(q,x) - p\langle q - T_r(x), J_pT_r(x)\rangle + p\langle q - T_r(x), J_px\rangle \\
&= \phi_p(q,x) - p\langle q - T_r(x), J_pT_r(x) - J_px\rangle \\
&\le \phi_p(q,x).
\end{aligned}$$

This completes the proof. □

Basic assumption. Let C be a nonempty closed and convex subset of a real Banach space E with dual space E^*. Let $g : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and let $A : C \to E^*$ be continuous and monotone. For solving the generalized mixed equality equilibrium problem, we assume that each bifunction $f : C\times C \to \mathbb{R}$ satisfies the following conditions:

- (A1) $f(u,u) = 0,\ \forall u \in C$;
- (A2) f is monotone, i.e., $f(u,v) + f(v,u) = 0,\ \forall u, v \in C$;
- (A3) $\limsup_{t\downarrow 0} f(u + t(z-u), v) \le f(u,v),\ \forall u, v, z \in C$;
- (A4) $f(u,\cdot)$ is convex and lower semi-continous, $\forall u \in C$.

For Theorem 3.1 below, we have the following setting.
Setting 2.

(1) For $p \geq 2$, E_1 and E_2 are p-uniformly convex and uniformly smooth real Banach spaces with dual spaces, E_1^* and E_2^*, respectively. E_3 is an arbitrary smooth real Banach space with dual space, E_3^*.
(2) C and M are nonempty closed and convex subsets of E_1 and E_2, respectively.
(3) $\varphi : C \to \mathbb{R} \cup \{\infty\}$ and $\vartheta : M \to \mathbb{R} \cup \{\infty\}$ are proper lower semi-continuous functions.
(4) $f : C \times C \to \mathbb{R}$ and $g : M \times M \to \mathbb{R}$ are bifunctionals satisfying (A1)-(A4).
(5) $U : C \to E^*$ and $V : M \to E_2^*$ are continuous and monotone mappings.
(6) $T : E_1 \to E_1$ and $S : E_2 \to E_2$ are closed quasi-ϕ-nonexpansive mappings.
(7) $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear mappings with adjoints A^* and B^*, respectively.
(8) $\alpha \in (0,1)$ and μ is such that $0 < \mu < \left[\frac{1}{\kappa_p\left(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}}\right)}\right]^{p-1}$, $p \geq 2$.
(9) J_{pE_i} denotes the generalized duality map on E_i, $i = 1,2,3$, respectively.

Algorithm 3.1.

$$\begin{cases} x_1 \in E_1,\ y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2, \\ u_n = T_r x_n,\ v_n = T_r y_n,\ e_n = J_{pE_3}(Au_n - Bv_n), \\ \theta_n = J_{pE_1}^{-1}(J_{pE_1} u_n - \mu A^* e_n),\ \delta_n = J_{pE_2}^{-1}(J_{pE_2} v_n + \mu B^* e_n), \\ z_n = J_{pE_1}^{-1}\big(\alpha J_{pE_1} x_n + (1-\alpha) J_{pE_1} T\theta_n\big), \\ w_n = J_{pE_2}^{-1}\big(\alpha J_{pE_2} y_n + (1-\alpha) J_{pE_2} S\delta_n\big), \\ C_{n+1} = \{c \in C_n : \phi_p(c, z_n) \leq \phi_p(c, x_n)\}, \\ Q_{n+1} = \{q \in Q_n : \phi_p(q, w_n) \leq \phi_p(q, y_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad y_{n+1} = \Pi_{Q_{n+1}} y_1,\ \forall n \geq 1, \end{cases} \tag{3.1}$$

where Π denotes the generalized projection of Alber [2].

Theorem 3.1. *Let $\{(x_n, y_n)\} \subset E_1 \times E_2$ be a sequence generated by Algorithm 3.1. Assume $\Omega := SGMEEP \cap SEFPP \neq \emptyset$. Then, $\{(x_n, y_n)\}$ converges strongly to a point $(x^*, y^*) \in \Omega$.*

Proof. We divide the proof into *four* steps.
Step 1. Show that the sequences $\{x_n\}$ and $\{y_n\}$ are well defined.

First, we show that C_n and Q_n are closed and convex. Clearly, $C_1 = E_1$ and $Q_1 = E_2$ are closed and convex. Assume C_n and Q_n are closed and convex for some $n \geq 1$. From the definition of C_{n+1}, we obtain that

$$C_{n+1} = \big\{c \in C_n : p\langle c, J_{pE_1} x_n - J_{pE_1} z_n \rangle \leq \|J_p x_n\|^p - \|J_p z_n\|^p\big\}.$$

Thus, C_{n+1} is closed and convex. Similarly, Q_{n+1} is closed and convex. Hence, C_n and Q_n are closed and convex, for all $n \geq 1$.

Note that $\Omega \subset C_1 \times Q_1$. Assume that $\Omega \subset C_n \times Q_n$ for some integer $n \geq 1$. Let $(c, q) \in \Omega$. Then, by Lemma 2.4 and the fact that T is quasi-ϕ-nonexpansive, we have

$$\begin{aligned} \phi_p(c, z_n) &= \phi_p(c, J_{pE_1}^{-1}\big(\alpha J_{pE_1} x_n + (1-\alpha) J_{pE_1} T\theta_n\big)) \\ &\leq \alpha \phi_p(c, x_n) + (1-\alpha)\phi_p(c, T\theta_n) - c_p w_p(\alpha)\|J_{pE_1} x_n - J_{pE_1} T\theta_n\|^p \\ &\leq \alpha \phi_p(c, x_n) + (1-\alpha)\phi_p(c, \theta_n) - c_p w_p(\alpha)\|J_{pE_1} x_n - J_{pE_1} T\theta_n\|^p. \end{aligned} \tag{3.2}$$

Now, using definition of V_p, Lemmas 2.1 and 3.1, we have

$$\begin{aligned}
\phi_p(c,\theta_n) &= \phi_p(c, J_{pE_1}^{-1}(J_{pE_1}u_n - \mu A^* e_n)) \\
&= V_p(c, J_{pE_1}u_n - \mu A^* e_n) \\
&\leq V_p(c, J_{pE_1}u_n) - p\mu \langle J_{pE_1}^{-1}\big(J_{pE_1}u_n - \mu A^* e_n\big) - c, A^* e_n\rangle \\
&= \phi_p(c,u_n) - p\mu \langle \theta_n - c, A^* e_n\rangle \\
&= \phi_p(c,u_n) - p\mu \langle A(\theta_n - c), e_n\rangle \\
&\leq \phi_p(c,x_n) - p\mu \langle A(\theta_n - c), e_n\rangle.
\end{aligned} \tag{3.3}$$

Substituting this inequality into inequality (3.2), we obtain

$$\phi_p(c,z_n) \leq \phi_p(c,x_n) - p\mu(1-\alpha)\langle A(\theta_n - c), e_n\rangle - c_p w_p(\alpha)\|J_{pE_1}x_n - J_{pE_1}T\theta_n\|^p. \tag{3.4}$$

Similarly, we obtain that

$$\phi_p(q,w_n) \leq \phi_p(q,y_n) - p\mu(1-\alpha)\langle B(q - \delta_n), e_n\rangle - c_p w_p(\alpha)\|J_{pE_2}y_n - J_{pE_2}S\delta_n\|^p. \tag{3.5}$$

Adding inequalities (3.4) and (3.5) and using the fact that $Ac = Bq$, we have

$$\begin{aligned}
\phi_p(c,z_n) + \phi_p(q,w_n) \leq{}& \phi_p(c,x_n) + \phi_p(q,y_n) - p\mu(1-\alpha)\langle A\theta_n - B\delta_n, e_n\rangle \\
& - c_p w_p(\alpha)\big[\|J_{pE_1}x_n - J_{pE_1}T\theta_n\|^p + \|J_{pE_2}y_n - J_{pE_2}S\delta_n\|^p\big].
\end{aligned} \tag{3.6}$$

Using this inequality and Lemma 2.3, we arrive at

$$\begin{aligned}
-(1-\alpha)p\mu\langle A\theta_n - B\delta_n, e_n\rangle &= -(1-\alpha)p\mu\big[\langle Au_n - Bv_n, e_n\rangle - \langle Au_n - A\theta_n, e_n\rangle \\
&\quad - \langle B\delta_n - Bv_n, e_n\rangle\big] \\
&\leq -(1-\alpha)p\mu\|Au_n - Bv_n\|^p + (1-\alpha)p\mu\big[\|A\|\|u_n - \theta_n\|\|e_n\| \\
&\quad + \|B\|\|\delta_n - v_n\|\|e_n\|\big].
\end{aligned} \tag{3.7}$$

But

$$\begin{aligned}
\|\theta_n - u_n\| &= \|J_{pE_1}^{-1}(J_{pE_1}u_n - \mu A^* e_n) - J_{pE_1}^{-1}J_{pE_1}u_n\| \\
&\leq \kappa_p\|\mu A^* e_n\|^{\frac{1}{p-1}} \leq \kappa_p\mu^{\frac{1}{p-1}}\|A\|^{\frac{1}{p-1}}\|e_n\|^{\frac{1}{p-1}},
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
\|\delta_n - v_n\| &= \|J_{pE_2}^{-1}(J_{pE_2}v_n + \mu B^* e_n) - J_{pE_1}^{-1}J_{pE_1}v_n\| \\
&\leq \kappa_p\|\mu B^* e_n\|^{\frac{1}{p-1}} \leq \kappa_p\mu^{\frac{1}{p-1}}\|B\|^{\frac{1}{p-1}}\|e_n\|^{\frac{1}{p-1}}.
\end{aligned} \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7), we obtain

$$\begin{aligned}
-(1-\alpha)\mu\langle A\theta_n - B\delta_n, e_n\rangle &\leq -(1-\alpha)p\mu\|Au_n - Bv_n\|^p + (1-\alpha)p\mu^{\frac{p}{p-1}}\kappa_p\big[\|A\|^{\frac{p}{p-1}} \\
&\quad + \|B\|^{\frac{p}{p-1}}\big]\|e_n\|^{\frac{p}{p-1}} \\
&= -(1-\alpha)p\mu\big[1 - \mu^{\frac{1}{p-1}}\kappa_p(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}})\big]\times \\
&\quad \|Au_n - Bv_n\|^p \leq 0,
\end{aligned} \tag{3.10}$$

due to

$$0 < \mu < \Big[\frac{1}{\kappa_p\big(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}}\big)}\Big]^{p-1}.$$

Let $\zeta = (1-\alpha)p\mu\big[1-\mu^{\frac{1}{p-1}}\kappa_p(\|A\|^{\frac{p}{p-1}}+\|B\|^{\frac{p}{p-1}})\big]$. Then, from inequality (3.6), and using inequality (3.10), we obtain that

$$\begin{aligned}\phi_p(c,z_n)+\phi_p(q,w_n) &\leq \phi_p(c,x_n)+\phi_p(q,y_n)-\zeta\|Au_n-Bv_n\|^p\\ &\quad -c_pw_p(\alpha)\big[\|J_{pE_1}x_n-J_{pE_1}T\theta_n\|^p+\|J_{pE_2}y_n-J_{pE_2}S\delta_n\|\big] \qquad (3.11)\\ &\leq \phi_p(c,x_n)+\phi_p(q,y_n). \qquad (3.12)\end{aligned}$$

This implies that $(c,q)\in C_{n+1}\times Q_{n+1}$. Hence, $\Omega\subset C_n\times Q_n$, $\forall n\geq 1$. These conclusions imply that C_n and Q_n are nonempty closed and convex, for all $n\geq 1$. Therefore, $\{x_n\}$ and $\{y_n\}$ are well defined.

Step 2. Prove that the sequences $\{x_n\}$ and $\{y_n\}$ are convergent.

This is standard by means of Lemmas 2.6 and 2.7. It follows exactly as in [18] that $x_n\to x^*$ and $y_n\to y^*$, as $n\to\infty$, for some $x^*\in E_1$ and $y^*\in E_2$. Furthermore, $z_n\to x^*$ and $w_n\to y^*$, as $n\to\infty$.

Step 3. Show that $\lim\limits_{n\to\infty}\|u_n-x_n\|=0$ and $\lim\limits_{n\to\infty}\|v_n-y_n\|=0$.

From inequality (3.11), we have

$$\begin{aligned}c_pw_p(\alpha)\big[\|J_{pE_1}x_n-J_{pE_1}T\theta_n\|^p+\|J_{pE_2}y_n-J_{pE_2}S\delta_n\|^p\big]+\zeta\|Au_n-Bv_n\|\\ \leq \phi_p(c,x_n)-\phi_p(c,z_n)+\phi_p(q,y_n)-\phi_p(q,w_n).\end{aligned}$$

This implies that

$$\begin{aligned}\lim_{n\to\infty}\|J_{pE_1}x_n-J_{pE_1}T\theta_n\|=0,\quad \lim_{n\to\infty}\|J_{pE_2}y_n-J_{pE_2}S\delta_n\|=0,\\ \text{and}\quad \lim_{n\to\infty}\|Au_n-Bv_n\|=0.\end{aligned} \qquad (3.13)$$

Furthermore, using the uniform continuity of J_p^{-1} on bounded sets, we have

$$\lim_{n\to\infty}\|x_n-T\theta_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|y_n-S\delta_n\|=0. \qquad (3.14)$$

Using Lemma 2.3 and equation (3.13) we have that

$$\begin{aligned}\|u_n-\theta_n\| &\leq \kappa_p\|\mu A^*e_n\|^{\frac{1}{p-1}}\\ &\leq \kappa_p\mu^{\frac{1}{p-1}}\|A\|^{\frac{1}{p-1}}\|Au_n-Bv_n\|,\end{aligned}$$

which implies that $\lim\limits_{n\to\infty}\|u_n-\theta_n\|=0$. Similarly, $\lim\limits_{n\to\infty}\|v_n-\delta_n\|=0$. Also, using equations (3.13) and Lemma 2.3, we have

$$\|z_n-x_n\|\leq\kappa_p(1-\alpha)^{\frac{1}{p-1}}\|J_{pE_1}T\theta_n-J_{pE_1}x_n\|^{\frac{1}{p-1}}\to 0,\ \text{as } n\to\infty.$$

Thus, $\lim\limits_{n\to\infty}\|z_n-x_n\|=0$. Similarly, we have

$$\lim_{n\to\infty}\|w_n-y_n\|=0.$$

Next, we show that $\lim\limits_{n\to\infty}\phi_p(u_n,x_n)=0$ and $\lim\limits_{n\to\infty}\phi_p(v_n,y_n)=0$. By Lemma 3.1, we have that

$$\phi_p(c,u_n)\leq\phi_p(c,x_n)-\phi_p(u_n,x_n) \qquad (3.15)$$

$$\text{and}\qquad \phi_p(q,v_n)\leq\phi_p(q,y_n)-\phi_p(v_n,y_n). \qquad (3.16)$$

From inequalities (3.2), (3.3) and (3.15), we have that

$$\begin{aligned}\phi_p(c,z_n) &\le \alpha\phi_p(c,x_n)+(1-\alpha)\phi_p(c,\theta_n)-c_p w_p(\alpha)\|J_{pE_1}x_n-J_{pE_1}T\theta_n\|^p\\ &\le \alpha\phi_p(c,x_n)+(1-\alpha)\big[\phi_p(c,u_n)-p\mu\langle A(\theta_n-c),e_n\rangle\big]\\ &\le \phi_p(c,x_n)-(1-\alpha)\phi_p(u_n,x_n)-p\mu(1-\alpha)\langle A(\theta_n-c),e_n\rangle. \end{aligned}\tag{3.17}$$

Similarly,

$$\phi_p(q,w_n)\le\phi_p(q,y_n)-(1-\alpha)\phi_p(v_n,y_n)-p\mu(1-\alpha)\langle B(q-\delta_n),e_n\rangle. \tag{3.18}$$

Using inequalities (3.17) and (3.18), the fact that $Ac=Bq$, inequality (3.10) and Step 2, we obtain that

$$\phi_p(u_n,x_n)+\phi_p(v_n,y_n)\le\frac{1}{(1-\alpha)}\big(\phi_p(c,x_n)-\phi_p(c,z_n)+\phi_p(q,y_n)-\phi_p(q,w_n)\big).$$

Thus,

$$\lim_{n\to\infty}\phi_p(u_n,x_n)+\phi_p(v_n,y_n)=0.$$

By use of Lemma 2.6, we obtain that

$$\lim_{n\to\infty}\|x_n-u_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|y_n-v_n\|=0.$$

Step 4. Show that $(x^*,y^*)\in\Omega$ and $Ax^*=By^*$.

Since, for $p\ge 2$, each p-uniformly convex and uniformly smooth space is strictly convex, reflexive and smooth, we can apply the established result involving the functional ϕ instead of the functional, ϕ_p. With this, the proof of Step 4 follows immediately by use of the same argument as in [18]. □

Remark 3.1. The condition on μ involves the norms, $\|A\|$ and $\|B\|$, respectively. This is not a drawback on implementing the algorithm because, for the computational purposes, one does not need to compute these norms. The norms can be replaced with two constants associated with the mappings, A and B, which are easily obtained. To assert that a linear mapping A, is bounded, one has to show that $\|Ax\|\le K\|x\|$, $\forall x\in E$, and some constant $K>0$. This constant $K>0$, which is an upper bound for $\|A\|$, is generally easy to obtain (since it is not unique) for any bounded linear mapping. Similarly, to assert that a linear map B is bounded, one has to show $\|Bx\|\le L\|x\|$, $\forall x\in E$, and some constant $L>0$. Again, this constant $L>0$ is easily obtained. It is easy to see from the proof of Theorem 3.1 that the condition

$$0<\mu<\left[\frac{1}{\kappa_p\big(\|A\|^{\frac{p}{p-1}}+\|B\|^{\frac{p}{p-1}}\big)}\right]^{p-1},$$

can be replaced with the condition

$$0<\mu<\left[\frac{1}{\kappa_p\big(K^{\frac{p}{p-1}}+L^{\frac{p}{p-1}}\big)}\right]^{p-1},$$

where K and L are easily obtained.

4. THE CONCLUSION

The theorem of Monday [18] provides an algorithm, which is applicable in L_p, l_p and the Sobolev spaces, $W_p^m(\Omega)$, where $1 < p \leq 2$, for the problem studied because these spaces are 2-uniformly convex and uniformly smooth. The theorem is not applicable in L_p, l_p and the Sobolev spaces, $W_p^m(\Omega)$, where $2 < p < \infty$ because these spaces are not 2-uniformly convex. Our results provide an algorithm, which is applicable in these spaces which are p-uniformly convex and uniformly smooth. Consequently, our result complement the Monday's theorem and provide applicable algorithms in L_p, l_p and the Sobolev spaces, $W_p^m(\Omega)$, for all p such that $1 < p < \infty$ for the problem.

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