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ITERATIVE ALGORITHMS FOR A MULTIPLE-SETS SPLIT FEASIBILITY PROBLEM IN BANACH SPACES

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Abstract. In this paper, we consider a multiple-sets split feasibility problem in Banach spaces. By converting it to a fixed point problem, we propose two new iterative algorithms for solving the problem considered. Under some mild assumptions, we prove that the proposed algorithms are strongly convergent provided that the involved spaces are smooth and uniformly convex.

Keywords. Split feasibility problem; Bregman projection; Metric projection; Duality mapping.

1. Introduction

The split feasibility problem (SFP) is formulated as finding a point $x^* \in \mathbb{R}^n$ satisfying the property:

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subset of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and A is an $m \times n$ matrix (i.e., a linear operator from $\mathbb{R}^n$ into $\mathbb{R}^m$). The SFP was first introduced by Censor and Elfving [7] and has been proved very useful in dealing with problems arising from various applied disciplines; see e.g [5, 6, 9, 10, 22].

There are various entensions of the SFP recently. In this paper, we mainly focus on the multiple-sets split feasibility problem (MSSFP) in Banach spaces: For $m \in \mathbb{N}$ and for two Banach spaces X and Y, find $x^* \in \bigcap_{i=1}^m C_i$ such that

$$A_i x^* \in Q_i, \text{ for each } i = 1, 2, \cdots, m, \tag{1.2}$$

where C_i and Q_i are nonempty closed convex subset of X and Y, respectively, and A_i is a linear operator from X into Y.

In the setting of Hilbert spaces, various algorithms were invented to solve the SFP (1.1) (see [4, 9, 10, 11, 16, 19, 20, 21] and reference therein). In 2002, Byrne introduced his CQ algorithm:

$$x_{n+1} = P_C[x_n - rA^*(I - P_Q)Ax_n], \tag{1.3}$$

where $0 < r_n < \frac{2}{\|A\|^2}$, A^* is the transpose of A, I is the identity operator, P_C denote the metric projection onto C, and P_Q denotes the metric projection onto Q. By using the Polyak's gradient

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method, Wang [17] recently proposed another iterative algorithm:

$$x_{n+1} = x_n - r_n[(I - P_C)x_n + A^*(I - P_Q)Ax_n], \tag{1.4}$$

where $0 < r_n < \frac{2}{1+\|A\|^2}$ and I denotes the identity operaotr.

In the setting of Banach spaces, Schöpfer, Schuster and Louis [12] considered the SFP in Banach spaces and they extended the CQ method as:

$$x_{n+1} = \Pi_C J_{X^*}[J_X x_n - rA^* J_Y(I - P_Q)Ax_n], \tag{1.5}$$

where J_{X^*}, J_X, J_Y are the duality mappings, Π_C denotes the Bregman projection and P_Q denotes the metric projection. Then the weak convergence of (1.5) is guaranteed if X is p-uniformly convex, uniformly smooth, and J_X is sequentially weak-to-weak continuous; see [12] for more details.

Recently, Takahashi [14] suggested a novel way for the SFP:

$$\begin{cases} z_n = x_n - rJ_{X^*}A^*J_Y(I - P_Q)Ax_n, \\ C_n = \{z \in C : \langle z_n - z, J_X(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in C : \langle x_n - z, J_X(x_0 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.6}$$

Instead of the weak convergence, Takahashi proved the strong convergence of method (1.6) under the assumption that X is uniformly convex and smooth, which is clearly weaker than that used in [12].

Motivated by this method, Wang [18] recently introduced another new method as

$$\begin{cases} z_n = x_n - rJ_{X^*}[J_X(I - P_C)x_n + A^*J_Y(I - P_Q)Ax_n], \\ C_n = \{z \in X : \langle z_n - z, J_X(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in X : \langle x_n - z, J_X(x_0 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.7}$$

It is clear that C_n and Q_n constructed in (1.7) are halfspaces. Since the projection onto the intersection of two half spaces has closed-form soulution, it is easy to implement the method (1.7). For some related works on this topic, we refer to [11, 15, 16].

The aim in this paper is to continue the above works by constructing new iterative methods for solving the MSSFP. By converting it to an equivalent fixed point problem, we propose two iterative algorithms and prove their strong convergence to a solution of the MSSFP. It is also worth noting that the parameter in one algorithm is chosen in a way that no priori knowledge of the operator norms are required.

2. PRELIMINARIES

NOTATION: "$\to$" denotes strong convergence, "$\rightharpoonup$" denoes weak convergence and $\omega_w\{x_n\}$ denotes the set of weak cluster points of a sequence $\{x_n\}$. Let $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \leq 1\}$, respectively, be the unit sphere and unit ball of X. $\mathrm{Fix}(T) = \{x \in X : Tx = x\}$ denotes the fixed-point set of T.

Definition 2.1. Let X be a Banach space.

(1) The *modulus of convexity* $\delta_X(\varepsilon) : [0,2] \to [0,1]$ is defined as

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in B_X,\ \|x - y\| \geq \varepsilon\right\}.$$

(2) The *modulus of smoothness* $\rho_X(t) : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_X(t) = \sup\left\{ \frac{\|x+ty\|+\|x-ty\|}{2} - 1 : x,y \in S_X \right\}.$$

(3) The *duality mapping* $J_X : X \to 2^{X^*}$ is defined by

$$J_X x = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Definition 2.2. Let X be a Banach space.

(1) X is called *strictly convex* if $\delta_X(2) = 1$.
(2) X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$ for any $\varepsilon \in (0,2]$.
(3) X is called *p-uniformly convex* if there exist $p \geq 2$ and a constant $c > 0$ such that $\delta_X(\varepsilon) \geq c\varepsilon^p$, $\forall \varepsilon \in (0,2]$.

Definition 2.3. Let X be a Banach space.

(1) X is called *smooth* if $\lim\limits_{t\to 0} \dfrac{\|x+ty\|-\|x\|}{t}$ exists for each $x,y \in S_X$.
(2) X is called *uniformly smooth* if $\lim\limits_{t\to 0} \dfrac{\rho_X(t)}{t} = 0$.

Lemma 2.1. [8, 13] *Let X be a uniformly convex Banach space. Then X^* is uniformly smooth and X is strictly convex and reflexive.*

Lemma 2.2. [8, 13] *Let X be a uniformly convex Banach space. Let $\{x_n\}$ be a sequence in X. If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then the sequence $\{x_n\}$ converges strongly to x.*

Lemma 2.3. [8, 13] *If X is reflexive, smooth and strictly convex, then the duality mapping J_X is one-to-one, single-valued and $J_X^{-1} = J_{X^*}$, where J_{X^*} is the duality mapping of X^*.*

Definition 2.4. Let C be a nonempty closed convex subset of X. The metric projection $P_C : X \to C$ is defined as

$$P_C x := \arg\min_{y\in C} \|x-y\|, \quad x \in X.$$

Lemma 2.4. [11, 12] *Let $\{x_n\}$ be a sequence in X, and $C \subseteq X$ a nonempty closed convex subset. Then, for $x \in X$,*

(1) $\langle J_X(x-P_Cx), z-P_Cx \rangle \leq 0$, $\forall z \in C$;
(2) $\|x-P_Cx\|^2 \leq \langle J_X(x-P_Cx), x-z \rangle$, $\forall z \in C$;
(3) if $x_n \rightharpoonup x$ and $\|x_n - P_Cx_n\| \to 0$, then $x \in C$.

The Bregman distance with respect to $\|\cdot\|$ is given by

$$\Delta(x,y) = \frac{1}{2}\|x\|^2 - \langle J_X x, y \rangle + \frac{1}{2}\|y\|^2.$$

This notion goes back to Bregman [3] and now is successfully used in various optimization problems in Banach spaces; see, e.g., [1, 2]. In general, the Bregman distance is not a metric due to the absence of symmetry, but it has some distance-like properties.

Definition 2.5. Let C be a nonempty closed convex subset of X. The Bregman projection $\Pi_C : X \to C$ is defined as

$$\Pi_C x = \arg\min_{y\in C} \triangle(x,y), \quad x \in X.$$

In Hilbert spaces, the metric and Bregman projections are the same, but in general they are completely different. More importantly, the metric projection can not share the decent property:

$$\triangle(\Pi_C x, z) \leq \triangle(x, z) - \triangle(x, \Pi_C x), \ \forall z \in C, \tag{2.1}$$

as the Bregman projection in Banach spaces.

3. Reformulation for MSSFP

To construct our algorithm, we first convert the MSSFP to an equivalent fixed point problem. As a matter of fact, for each $i \in \{1, 2, \cdots, m\}$, let us define

$$W := I - rJ_X^* \left[\sum_{i=1}^{m} (J_X U_i + A_i^* J_Y T_i A_i) \right], \tag{3.1}$$

where $r > 0$, $U_i = I - P_{C_i}$ and $T_i = I - P_{Q_i}$.

Lemma 3.1. *Assume that both X and Y are smooth, reflexive, and strictly convex. If the MSSFP is consistent, that is, its solution set $\mathscr{S}$ is nonempty, then $\mathscr{S} = \mathrm{Fix}(W)$.*

Proof. By our assumption, both J_X and J_Y are one-to-one and single-valued. It is readily seen that $\mathscr{S} \subseteq \mathrm{Fix}(W)$. To see the converse, let $x^* \in \mathrm{Fix}(W)$ and choose any element z is in $\mathscr{S}$. By Lemma 2.4, for each $i \in \{1, 2, \cdots, m\}$, we have

$$\|U_i x^*\|^2 \leq \langle J_X U_i x^*, x^* - z \rangle,$$
$$\|T_i A_i x^*\|^2 \leq \langle A_i^* J_Y T_i A_i x^*, x^* - z \rangle.$$

Combining these inequalities, we have

$$\begin{aligned} & r \sum_{i=1}^{m} \left(\|U_i x^*\|^2 + \|T_i A_i x^*\|^2 \right) \\ & \leq r \left\langle \sum_{i=1}^{m} (J_X U_i x^* + A_i^* J_Y T_i A_i x^*), x^* - z \right\rangle \\ & = \langle J_X(x^* - W x^*), x^* - z \rangle = 0, \end{aligned}$$

which yields $x^* \in \mathscr{S}$. Thus the desired result follows. □

4. Algorithms for MSSFP

As an application of Lemma 3.1, we propose a new algorithm for solving the MSSFP, which is indeed an extension of algorithm (1.7).

Algorithm 4.1. Choose an arbitrary initial guess $x_0 \in X$. Given x_n, update x_{n+1} by the iteration formula:

$$\begin{cases} C_n = \{z \in X : \langle W x_n - z, J_X(x_n - W x_n) \rangle \geq 0\}, \\ Q_n = \{z \in X : \langle x_n - z, J_X(x_0 - x_n) \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases}$$

where W is defined as in (3.1) and the parameter r_n is chosen so that

$$0 < r \leq r_n \leq \frac{1}{1 + \max\limits_{1 \leq i \leq m} \|A_i\|^2}. \tag{4.1}$$

Lemma 4.1. *Assume that both X and Y are smooth, reflexive, and strictly convex Banach spaces. If the MSSFP is consistent, then, for each each $n \in \mathbb{N}$, the set $C_n \cap Q_n$ is nonempty, closed and convex. Moreover the sequence $\{x_n\}$ generated by Algorithm 4.1 is well defined.*

Proof. For any $z \in \mathscr{S}$, we have

$$\begin{aligned}
&\langle x_n - z, J_X(x_n - Wx_n)\rangle \\
&= r_n \langle x_n - z, \sum_{i=1}^{m} (J_X U_i x_n + A_i^*(J_Y(T_i(A_i x_n))))\rangle \\
&= r_n \sum_{i=1}^{m} (\langle x_n - z, J_X U_i x_n\rangle + \langle A_i x_n - A_i z, J_Y(T_i(A_i x_n))\rangle).
\end{aligned}$$

It then follows from Lemma 2.4 that

$$\langle x_n - z, J_X(x_n - Wx_n)\rangle \geq r_n \sum_{i=1}^{m} (\|U_i x_n\|^2 + \|T_i(A_i x_n)\|^2), \tag{4.2}$$

which implies

$$\begin{aligned}
&\langle Wx_n - z, J_X(x_n - Wx_n)\rangle \\
&= \langle Wx_n - x_n, J_X(x_n - Wx_n)\rangle + \langle x_n - z, J_X(x_n - Wx_n)\rangle \\
&= -\|Wx_n - x_n\|^2 + \langle x_n - z, J_X(x_n - Wx_n)\rangle \\
&\geq r_n \sum_{i=1}^{m} (\|U_i x_n\|^2 + \|T_i(A_i x_n)\|^2) - \|x_n - Wx_n\|^2.
\end{aligned} \tag{4.3}$$

On the other hand, by Young's inequality,

$$\begin{aligned}
\|x_n - Wx_n\|^2 &= r_n^2 \|\sum_{i=1}^{m} (J_X U_i x_n + A_i^*(J_Y(T_i(A_i x_n))))\|^2 \\
&\leq r_n^2 \left(\sum_{i=1}^{m} (\|U_i x_n\| + \|A_i\| \|T_i A_i x_n\|) \right)^2 \\
&\leq r_n^2 \sum_{i=1}^{m} (1 + \|A_i\|^2)(\|U_i x_n\|^2 + \|T_i(A_i x_n)\|^2) \\
&\leq r_n^2 \left(1 + \max_{1 \leq i \leq m} \|A_i\|^2\right) \sum_{i=1}^{m} (\|U_i x_n\|^2 + \|T_i(A_i x_n)\|^2).
\end{aligned}$$

Substituting this into (4.3), we have

$$\begin{aligned}
&\langle Wx_n - z, J_X(x_n - Wx_n)\rangle \\
&\geq r_n \left(1 - r_n \left(1 + \max_{1 \leq i \leq m} \|A_i\|^2\right)\right) \sum_{i=1}^{m} (\|U_i x_n\|^2 + \|T_i(A_i x_n)\|^2).
\end{aligned}$$

By condition (4.1), we have $\langle Wx_n - z, J_X(x_n - Wx_n)\rangle \geq 0$. Hence, we deduce that $z \in C_n$. Since z is chosen in $\mathscr{S}$ arbitrarily, we conclude that $\mathscr{S} \subseteq C_n$ for all $n \in \mathbb{N}$.

We next show $\mathscr{S} \subseteq Q_n$ by induction. It is easy to check that $\mathscr{S} \subseteq Q_0$. Now assume that $\mathscr{S} \subseteq Q_k$ for some $k \geq 1$, which implies $\mathscr{S} \subseteq C_k \cap Q_k$. Since x_{k+1} lies in $C_k \cap Q_k$, this gives

$$\langle x_{k+1} - z, J_X(x_0 - x_{k+1})\rangle \geq 0, \forall z \in C_k \cap Q_k.$$

In particular, we have

$$\langle x_{k+1}-z, J_X(x_0-x_{k+1})\rangle \geq 0, \forall z \in \mathscr{S},$$

which implies $\mathscr{S} \subseteq Q_{k+1}$. Hence, $\mathscr{S} \subseteq Q_n$ for all $n \in \mathbb{N}$.

Altogether, $\mathscr{S} \subseteq C_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that the set $C_n \cap Q_n$ is nonempty, closed and convex, and therefore iterative sequence $\{x_n\}$ is well defined. Thus, the proof is complete. □

Now let us state the convergence of $\{x_n\}$ generated by Algorithm 4.1.

Theorem 4.1. *Assume that X is a smooth and uniformly convex Banach space, and Y is a smooth, reflexive, and strictly convex Banach space. If the MSSFP is consistent, then the sequence $\{x_n\}$ generated by Algorithm 4.1 converges strongly to the solution $P_{\mathscr{S}}(x_0)$.*

Proof. We first show that $\lim_n \|x_{n+1}-x_n\| = 0$. Let $z \in S$. By Lemma 4.1, we have

$$\|x_1 - x_{n+1}\| = \|x_1 - P_{C_n\cap Q_n}x_1\| \leq \|x_1 - z\|.$$

Hence $\{x_n\}$ is bounded. On the other hand, using Lemma 4.1, we have

$$\|x_0 - x_n\| = \|x_0 - P_{Q_n}x_0\| \leq \left\| x_0 - \frac{x_n + x_{n+1}}{2} \right\|,$$

due to $(x_n + x_{n+1})/2 \in Q_n$. Similarly, $\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|$, which implies that $\{\|x_0 - x_n\|\}$ is nondecreasing. This together with its boundness yields that $\lim_n \|x_0 - x_n\|$ exists. Let M be such a limit. It is clear that $M > 0$ and $\|x_0 - x_n\| \leq M$ for all $n \in \mathbb{N}$.

From definition of the modulus of convexity, we have

$$\begin{aligned}
\delta_X\left(\frac{\|x_n - x_{n+1}\|}{M}\right) &= \delta_X\left(\|\frac{x_n - x_0}{M} - \frac{x_{n+1}-x_0}{M}\|\right) \\
&\leq 1 - \frac{1}{2}\left\|\frac{x_n - x_0}{M} + \frac{x_{n+1}-x_0}{M}\right\| \\
&= 1 - \frac{1}{M}\left\| x_0 - \frac{x_n + x_{n+1}}{2}\right\| \\
&\leq 1 - \frac{\|x_0 - x_n\|}{M} \to 0.
\end{aligned}$$

Since X is uniformly convex, we assert that $\lim_n \|x_{n+1} - x_n\| = 0$.

We next show that $\omega_w\{x_n\} \subseteq \mathscr{S}$. It follows from Lemma 4.1 that

$$\|x_n - Wx_n\| = \|x_n - P_{C_n}x_n\| \leq \|x_n - x_{n+1}\|.$$

It then follows that $\lim_n \|x_n - Wx_n\| = 0$, which together with (4.2) and (4.1) yields, for $i \in \{1, 2, \cdots, m\}$,

$$\lim_{n\to\infty} \|x_n - P_{C_i}x_n\| = \lim_{n\to\infty} \|A_i x_n - P_{Q_i}(A_i x_n)\| = 0. \tag{4.4}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to x^*. By Lemma 2.4, $x^* \in C_i$ for each $i \in \{1, 2, \cdots, m\}$. On the other hand, since $A_i x_{n_k} \rightharpoonup A_i x^*$, it follows from (4.4) and Lemma 2.4 that $A_i x^* \in Q_i$ for each $i \in \{1, 2, \cdots, m\}$. Altogether, we have $x^* \in \mathscr{S}$.

Fixing $x \in \omega_w(x_n)$, we see $x^* \in \mathscr{S}$, and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to x. It then follows from (i) that

$$\begin{aligned}\|x_0 - P_{\mathscr{S}}(x_0)\| \le \|x_0 - x\| &\le \lim_{k\to\infty} \|x_0 - x_{n_k}\| \\ &= \lim_{k\to\infty} \|x_0 - P_{C_{n_k-1}\cap Q_{n_k-1}}(x_0)\| \\ &\le \|x_0 - P_{\mathscr{S}}(x_0)\|,\end{aligned}$$

where the first and the last inequalities follow from the property of metric projections, and the second one from the lower semi-continuity of the norm. Hence, $P_{\mathscr{S}}(x_0) = x$ and

$$\lim_{k\to\infty} \|x_0 - x_{n_k}\| = \|x_0 - P_{\mathscr{S}}(x_0)\|.$$

Since x is chosen arbitrarily, this implies that $\omega_w(x_n)$ is exactly single-point set, that is, $\{x_n\}$ converges weakly to $P_{\mathscr{S}}(x_0)$. Note that $x_0 - x_{n_k} \rightharpoonup x_0 - P_{\mathscr{S}}(x_0)$. By Lemma 2.3, $\lim_k x_{n_k} = P_{\mathscr{S}}(x_0)$. Since $\{x_n\}$ converges weakly, this yields $\lim_n x_n = P_{\mathscr{S}}(x_0)$ as desired. □

Remark 4.1. In Hilbert spaces, the above algorithm is reduced to:

$$\begin{cases} C_n = \{z \in X : \langle Wx_n - z, x_n - Wx_n\rangle \ge 0\}, \\ Q_n = \{z \in X : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0). \end{cases}$$

Remark 4.2. It worth noting that C_n and Q_n in our algorithm are half-spaces, while those in (1.6) are not in general. Our condition that ensures the convergence is weaker than that employed in [12].

As we see from Algorithm 4.1, the choice of r_n is related to $\|A_i\|$, thus to implement this algorithm, one has to compute the norm $\|A_i\|$, which is generally not easy in practice. In what follows, we introduce another choice of r_n, which ultimately has no relation with $\|A_i\|$.

Algorithm 4.2. Choose an arbitrary initial guess $x_0 \in X$. Given x_n, if

$$\|\sum_{i=1}^m (J_X(U_i x_n) + A_i^*(J_Y T_i(A_i x_n)))\| = 0,$$

then x_n is a solution; otherwise, update x_{n+1} by the iteration formula:

$$\begin{cases} C_n = \{z \in X : \langle Wx_n - z, J_X(x_n - Wx_n)\rangle \ge 0\}, \\ Q_n = \{z \in X : \langle x_n - z, J_X(x_0 - x_n)\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \end{cases}$$

where W is defined as in (3.1) and the parameter r_n is chosen as

$$r_n = \frac{\sum_{i=1}^m (\|U_i x_n\|^2 + \|T_i(A_i x_n)\|^2)}{\|\sum_{i=1}^m (J_X(U_i x_n) + A_i^*(J_Y T_i(A_i x_n)))\|^2}. \tag{4.5}$$

Lemma 4.2. *Assume that both X and Y are smooth, reflexive, and strictly convex Banach spaces. If the MSSFP is consistent, then, for each $n \in \mathbb{N}$, the set $C_n \cap Q_n$ is nonempty, closed and convex. Moreover, the sequence $\{x_n\}$ generated by Algorithm 4.2 is well defined.*

Proof. From the proof of Lemma 4.1, it suffices to show that $\langle Wx_n - z, J_X(x_n - Wx_n)\rangle \geq 0$ for each $n \in \mathbb{N}$. To see this, let $z \in \mathscr{S}$. It then follows from (4.3) that

$$\begin{aligned}
&\langle Wx_n - z, J_X(x_n - Wx_n)\rangle \\
&\geq r_n(\sum_{i=1}^{m}(\|U_ix_n\|^2 + \|T_i(A_ix_n)\|^2) - \|x_n - Wx_n\|^2 \\
&= r_n(\sum_{i=1}^{m}(\|U_ix_n\|^2 + \|T_i(A_ix_n)\|^2) - r_n^2\|\sum_{i=1}^{m}(J_X(U_ix_n) + A_i^*(J_YT_i(A_ix_n)))\|^2.
\end{aligned}$$

By our choice of r_n, we have the desired inequality immediately. □

Now let us state the convergence of $\{x_n\}$ generated by Algorithm 2.

Theorem 4.2. *Assume that X is a smooth and uniformly convex Banach space, and Y is a smooth, reflexive, and strictly convex Banach space. Let $\{x_n\}$ be generated by Algorithm 4.2. If the MSSFP is consistent, then $\{x_n\}$ converges strongly to the solution $P_{\mathscr{S}}(x_0)$.*

Proof. From the proof of Theorem 4.1, it suffices to verify that (4.4) still holds true. As a matter of fact, by inequality (4.2), it follows that

$$\lim_{n\to\infty} r_n(\sum_{i=1}^{m}(\|U_ix_n\|^2 + \|T_i(A_ix_n)\|^2) = 0. \tag{4.6}$$

On the other hand, we see that

$$\begin{aligned}
r_n &= \frac{\sum_{i=1}^{m}(\|U_ix_n\|^2 + \|T_i(A_ix_n)\|)^2}{\|\sum_{i=1}^{m}(J_X(U_ix_n) + A_i^*(J_YT_i(A_ix_n)))\|^2} \\
&\geq \frac{\sum_{i=1}^{m}(\|U_ix_n\|^2 + \|T_i(A_ix_n)\|)^2}{(\|J_X(x_n - P_Cx_n)\| + \|A\|\|J_Y(Ax_n - P_{Q_j}(Ax_n))\|)^2} \\
&\geq \frac{\sum_{i=1}^{m}(\|U_ix_n\|^2 + \|T_i(A_ix_n)\|)^2}{(1 + \max\limits_{1\leq i\leq m}\|A_i\|^2)(\sum_{i=1}^{m}(\|U_ix_n\|^2 + \|T_i(A_ix_n)\|^2)} \\
&\geq \frac{1}{1 + \max\limits_{1\leq i\leq m}\|A_i\|^2} \\
&> 0.
\end{aligned}$$

This together with (4.6) yields (4.4) as desired. □

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