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THE ULAM-HYERS STABILITY OF AN ORDINARY DIFFERENTIAL EQUATION VIA GRONWALL LEMMAS

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Abstract. Following [I.A. Rus, Gronwall lemma approach to the Hyers-Ulam-Rassias stability of an integral equation, in: Nonlinear Analysis and Variational Problems, Panos Pardalos, Themistocles M. Rassias, Akhtar A. Khan (ed.), pp. 147-152, Springer, 2009], we study the Ulam-Hyers stability of some ordinary differential equations based on Gronwall lemmas.

Keywords. Cauchy problem; Differential equation; Integral equation; Gronwall lemma; Ulam-Hyers stability.

1. Introduction

For the study of different Ulam types stability, there are various techniques in the literatures: exact solution approaches, fixed point approaches (weakly Picard operators,...), Gronwall's lemma approach, and so on. For the details on these approaches, we refer to [6, 8, 14, 15, 17, 18, 19] and the references therein. In this paper, based on the results presented in [14] and [17], we study the Ulam-Hyers stability of some ordinary differential equations by Gronwall lemma techniques.

2. Preliminaries

2.1. **Basic notions in the Ulam-Hyers stability of differential equations.** Let $(\mathbb{B}, |\cdot|)$ be a Banach space. Let $\varepsilon > 0$, $a \in \mathbb{R}$, and $b \in \mathbb{R} \cup \{+\infty\}$ with $a < b$. We denote by I an interval of the real axis ($I := [a,b]$ or $[a,b[$, depending on the context).

Let $f \in C(I \times \mathbb{B}, \mathbb{B})$ be a given continuous operator. We consider the following problems:

1) the differential equation

$$x'(t) = f(t, x(t)), t \in I, \tag{2.1}$$

with its solution set denoted by S_0;

2) the differential inequation

$$|x'(t) - f(t, x(t))| \leq \varepsilon, t \in I, \tag{2.2}$$

with the solution set (for each $\varepsilon > 0$) denoted by S_ε. Notice that a function $y \in C^1(I, \mathbb{B})$ is a solution of (2.2) if and only if there exists a function $g \in C(I, \mathbb{B})$ (which may depend on y) such that

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(i) $|g(t)| \leq \varepsilon$, for all $t \in I$;
(ii) $y'(t) = f(t, y(t)) + g(t)$, for every $t \in I$.
A solution to inequation (2.2) is called an ε-solution of equation (2.1).
The following concepts are important for our study in this paper (see [10, 18]).

Definition 2.1. The differential equation (2.1) is said to be Ulam-Hyers stable if there exists a real number $C_f > 0$ such that, for each $\varepsilon > 0$, there exists a retraction $r_\varepsilon : S_\varepsilon \to S_0$ satisfying

$$|y(t) - r_\varepsilon(y)(t)| \leq C_f \varepsilon, \text{ for each solution } y \in C^1(I, \mathbb{B}) \text{ of (2.2) and every } t \in I. \quad (2.3)$$

Definition 2.2. Differential equation (2.1) is said to be generalized Ulam-Hyers stable if there exists a function $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\theta(0) = 0$ such that, for each $\varepsilon > 0$, there exists a retraction $r_\varepsilon : S_\varepsilon \to S_0$ satisfying

$$|y(t) - r_\varepsilon(y)(t)| \leq \theta_f(\varepsilon), \text{ for each solution } y \in C^1(I, \mathbb{B}) \text{ of (2.2) and every } t \in I. \quad (2.4)$$

2.2. **Gronwall lemmas.** In this paper, we need the following Gronwall lemmas (see [2, 3, 4, 5, 7, 12, 13, 14]).

Lemma 2.1. *(Gronwall Lemma I.) Let J be an interval of the real axis, $t_0 \in J$ and the functions $h, k, u \in C(J, \mathbb{R}_+)$. If*

$$u(t) \leq h(t) + \left| \int_{t_0}^{t} k(s)u(s)ds \right|, \text{ for all } t \in J,$$

then

$$u(t) \leq h(t) + \left| \int_{t_0}^{t} h(s)k(s)e^{\left|\int_s^t k(\sigma)d\sigma\right|}ds \right|, \text{ for all } t \in J.$$

If, in addition, h is increasing, then $(J := [a, b[)$

$$u(t) \leq h(t)e^{\int_a^t k(s)ds}, \ t \in [a, b[.$$

Lemma 2.2. *(Gronwall Lemma II.) Let $h \in C([a, b], \mathbb{R}_+)$ and $\beta > 0$ with $\beta(b - a) < 1$. If $u \in C([a, b], \mathbb{R}_+)$ satisfies the relation*

$$u(t) \leq h(t) + \beta \int_a^b u(s)ds, \text{ for } t \in [a, b],$$

then

$$u(t) \leq h(t) + \frac{\beta}{1 - \beta(b - a)} \int_a^b h(s)ds, \text{ for every } t \in [a, b].$$

3. An heuristic point of view on the Ulam-Hyers stability of a differential equation

Let $f \in C([a, b] \times \mathbb{B}, \mathbb{B})$, $g \in C([a, b], \mathbb{B})$ and $\varepsilon \in \mathbb{R}_+^*$.
We consider the differential equations

$$x'(t) = f(t, x(t)), \ t \in [a, b], \quad (3.1)$$

$$x'(t) = f(t, x(t)) + g(t), \ t \in [a, b], \quad (3.2)$$

and the differential inequation

$$|x'(t) - f(t, x(t))| \leq \varepsilon, \ t \in [a, b]. \quad (3.3)$$

For a function $x \in C^1([a,b],\mathbb{B})$ and for given $x_0 \in \mathbb{B}$, we consider the Cauchy condition

$$x(a) = x_0. \tag{3.4}$$

We suppose that (see [2, 4, 7, 9, 12]):

(C) For each $x_0 \in \mathbb{B}$, Cauchy problem (3.1)+(3.4) has a unique solution, denoted by $x(\cdot,x_0) \in C^1([a,b],\mathbb{B})$;

(PC) For each $g \in C([a,b],\mathbb{B})$ and each $x_0 \in \mathbb{B}$, Cauchy problem (3.2)+(3.4) has a unique solution, denoted by $x(\cdot,x_0,g) \in C^1([a,b],\mathbb{B})$.

Let us consider the following subsets of $C^1([a,b],\mathbb{B})$:

$$S_0 := \{x(\cdot,x_0) \mid x_0 \in \mathbb{B}\},$$
$$S_\varepsilon := \{x(\cdot;x_0,g) \mid g \in C([a,b],\mathbb{B}),\ |g(t)| \le \varepsilon,\ \forall\, t \in [a,b],\ x_0 \in \mathbb{B}\}.$$

It is clear that, in the conditions (C) and (PC),

1) S_0 is the solution set of differential equation (3.1);

2) S_ε is the solution set of differential inequation (3.3).

Let $y \in C^1([a,b],\mathbb{R})$ be a solution of (3.3). Then, there exists $g \in C^1([a,b],\mathbb{B})$ with $|g(t)| \le \varepsilon$, $\forall t \in [a,b]$, such that y is a solution of differential equation (3.2). Let $x(\cdot,y(a))$ be the unique solution to (3.1) with $x(a) = y(a)$.

For y and $x(\cdot,y(a))$, we have

$$y(t) = y(a) + \int_a^t f(s,y(s))ds + \int_a^t g(s)ds,\ t \in [a,b],$$

and

$$x(t;y(a)) = y(a) + \int_a^t f(s,x(s;y(a))ds,\ t \in [a,b].$$

The above relations imply that

$$|y(t) - x(t;y(a))| \le \int_a^t |f(s,y(s)) - f(s,x(s;y(a)))|ds + (b-a)\varepsilon,\ t \in [a,b]. \tag{3.5}$$

In some conditions on f, we can obtain estimations for $|y(t) - x(t;y(a))|$. For example, let us suppose that

(Lip) There exists $L > 0$ such that

$$|f(t,u) - f(t,v)| \le L|u-v|,\ \forall\, t \in [a,b],\ \forall\, u,v \in \mathbb{B}.$$

With this condition, we have that

$$|y(t) - x(t;y(a))| \le L \int_a^t |y(s) - x(s;y(a))|ds + (b-a)\varepsilon,\ t \in [a,b].$$

From Gronwall Lemma I (see Lemma 2.1), the estimate follows

$$|y(t) - x(t;y(a))| \le (b-a)e^{L(b-a)}\varepsilon,\ t \in [a,b].$$

From the above considerations, for each $\varepsilon > 0$, we have a retraction, $r_\varepsilon : S_\varepsilon \to S_0$, defined by $y \mapsto x(\cdot,y(a))$, for which

$$|y(t) - r_\varepsilon(y)(t)| \le c\varepsilon,\ t \in [a,b],$$

where $c = (b-a)e^{L(b-a)}$.

Since condition (Lip) implies the conditions (C) and (PC), we have the following result.

Theorem 3.1. *Let us consider the differential equation (3.1). Suppose that the condition* (Lip) *holds. Then, for each* $\varepsilon > 0$*, there exists a retraction* $r_\varepsilon : S_\varepsilon \to S_0$ *such that*

$$|y(t) - r_\varepsilon(y)(t)| \le (b-a)e^{L(b-a)}\varepsilon,\ \forall\, y \in S_\varepsilon \text{ and } \forall\, t \in [a,b],$$

i.e., equation (3.1) is Ulam-Hyers stable.

Remark 3.1. Let us take $\mathbb{B} := \mathbb{R}^m$ and denote a norm on $\mathbb{R}^m$ by $|\cdot|$. In this case, since all norms on $\mathbb{R}^m$ are equivalent, if equation (3.1) is Ulam-Hyers stable with respect to a norm on $\mathbb{R}^m$, then it is Ulam-Hyers stable with respect to each other norm on $\mathbb{R}^m$ (see [11, 16, 18]). If we denote by $L_{|\cdot|}$ the Lipschitz constant of f with respect to $|\cdot|$, then we have, in terms of the norm $|\cdot|$, the following estimation

$$|y(t) - r_\varepsilon(y)(t)| \le (b-a)e^{L_{|\cdot|}(b-a)}\varepsilon, \text{ for all } y \in S_\varepsilon \text{ and } t \in [a,b].$$

For example, in the case that $f(t,y(t)) = A(t)x(t)$, where $A \in C([a,b],\mathbb{R}^{m\times m}$ is a matrix function and $|A(t)|$ is the corresponding matrix norm of $A(t)$ with respect to the vector norm $|\cdot|$, we have

$$L_{|\cdot|} = \max_{t\in[a,b]} |A(t)|.$$

From Theorem 3.1, we have the following.

Theorem 3.2. *If* $A \in C([a,b],\mathbb{R}^{m\times m})$*, then the linear system of differential equations*

$$x'(t) = A(t)x(t),\ t \in [a,b],$$

is Ulam-Hyers stable.

We have a similar result in the case $\mathbb{B} := \mathbb{C}^m$.

Remark 3.2. Let $(\mathbb{B},|\cdot|) := (l_2(\mathbb{R}),|\cdot|_2)$. In this case, equation (3.1) is a countable system of differential equations (see [20]). In the corresponding (Lip) condition, this system is Ulam-Hyers stable.

Remark 3.3. Instead of condition (Lip), some authors (see, for example, [6]) considered the condition

$$(Lip(t)) \qquad |f(t,u(t)) - f(t,v(t))| \le L|u(t)-v(t)|, \text{ for every } u,v \in C([a,b],\mathbb{B}),\ t \in [a,b].$$

It is clear that conditions (Lip) and $(Lip(t))$ are equivalent.

4. Second order differential equations

For a better understanding of the Ulam-Hyers stability of n-order differential equations, we consider, for simplicity, the following second order differential equation

$$-x''(t) = f(t,x(t)),\ t \in [a,b], \tag{4.1}$$

where $f \in C([a,b]\times\mathbb{R})$.

For $g \in C[a,b]$, we consider the differential equation

$$-x''(t) = f(t,x(t)) + g(t),\ t \in [a,b], \tag{4.2}$$

and for $\varepsilon > 0$, the following differential inequation

$$|-x''(t) - f(t,x(t))| \le \varepsilon,\ t \in [a,b]. \tag{4.3}$$

Let us suppose that:

(C) for all $x_0, x_0' \in \mathbb{R}$, there exists a unique solution, denoted by $x(\cdot; x_0, x_0') \in C^2[a,b]$ of (4.1), such that

$$x(a) = x_0,\ x'(a) = x_0',$$

(PC) for all $g \in C[a,b]$, $x_0, x_0' \in \mathbb{R}$, there exists a unique solution, denoted by $x(\cdot; x_0, x_0', g) \in C^2[a,b]$ of (4.2), such that

$$x(a) = x_0,\ x'(a) = x_0'.$$

In conditions (C) and (PC),

$$S_0 := \{x(\cdot; x_0, x_0') \mid x_0, x_0' \in \mathbb{R}\}$$

is the solution set of (4.1) and

$$S_\varepsilon := \{x(\cdot; x_0, x_0', g) \mid x_0, x_0' \in \mathbb{R},\ g \in C[a,b],\ |g(t)| \le \varepsilon,\ \forall\, t \in [a,b]\}$$

is the solution set of (4.3).

Now, let $y \in C^1[a,b]$ be a solution of (4.3). Then, there exists $g \in C[a,b]$, $|g(t)| \le \varepsilon$, $t \in [a,b]$, such that y is a solution of (4.2). Let $x(\cdot; y(a), y'(a))$ be the unique solution to (4.1). For y and $x(\cdot; y(a), y'(a))$, we have

$$y(t) = y(a) + y'(a)(t-a) + \int_a^t (s-t) f(s, y(s)) ds + \int_a^t (s-t) g(s) ds$$

and

$$x(t; y(a), y'(a)) = y(a) + y'(a)(t-a) + \int_a^t (s-t) f(s, x(s; y(a), y'(a))) ds.$$

These imply that

$$|y(t) - x(t; y(a), y'(a))| \le \int_a^t |f(s, y(s)) - f(s, x(s; y(a), y'(a)))| + \frac{1}{2}(b-a)^2 \varepsilon.$$

If we suppose, for example, the condition (Lip) for f, then

$$|y(t) - x(t; y(a), y'(a))| \le \frac{1}{2}(b-a)^2 e^{L(b-a)} \varepsilon,$$

for all $y \in S_\varepsilon$ and $t \in [a,b]$.

In the above conditions, we get the retractions

$$r_\varepsilon : S_\varepsilon \to S_0,\ y \mapsto x(\cdot; y(a), y'(a)),$$

with

$$|y(t) - r_\varepsilon(y)(t)| \le \frac{1}{2}(b-a)^2 e^{L(b-a)} \varepsilon,$$

for all $y \in S_\varepsilon$ and $t \in [a,b]$.

Thus, by the above observations, we have the following result.

Theorem 4.1. *If condition (Lip) takes place, then differential equation (4.1) is Ulam-Hyers stable.*

Now, let us work in more restrictive conditions:

(B) for all $\alpha, \beta \in \mathbb{R}$, equation (4.1) has a unique solution, denoted by $x(\cdot;\alpha,\beta) \in C^2[a,b]$, such that

$$x(a) = \alpha,\ x(b) = \beta.$$

(PB) for all $g \in C[a,b]$ and all $\alpha, \beta \in \mathbb{R}$, the equation (4.2) has a unique solution, denoted by $x(\cdot;\alpha,\beta,g) \in C^2[a,b]$, such that

$$x(a) = \alpha,\ x(b) = \beta.$$

In these conditions, we denote by

$$S_0 = \{x(\cdot,\alpha,\beta) \mid \alpha,\beta \in \mathbb{R}\}$$

the solution set of (4.1) and by

$$S_\varepsilon = \{x(\cdot,\alpha,\beta,g) \mid \alpha,\beta \in \mathbb{R},\ g \in C[a,b],\ |g(t)| \leq \varepsilon,\ t \in [a,b]\}$$

the solution set of (4.3).

We remark that the operator $\rho_\varepsilon : S_\varepsilon \to S_0$, defined by $y \mapsto x(\cdot;y(a),y(b))$ is a retraction of S_ε on S_0. Let us estimate $|y(t) - \rho_\varepsilon(y)(t)|$.

We have, in the terms of the Green function $G(s,t)$ of the operator $-x''$, the following relations for y and $x(\cdot;y(a),y(b))$,

$$y(t) = \frac{b-t}{b-a}y(a) + \frac{t-a}{b-a}y(b) + \int_a^b G(t,s)f(s,y(s))ds + \int_a^b G(t,s)g(s)ds,$$
$$t \in [a,b]$$

and

$$x(t;y(a),y(b)) = \frac{b-t}{b-a}y(a) + \frac{t-a}{b-a}y(b) + \int_a^b G(t,s)f(s,x(s;y(a),y(b)))ds,$$
$$t \in [a,b].$$

Recall that the Green function of the operator $-x''$ is given by

$$G(t,s) = \begin{cases} \dfrac{(s-a)(b-t)}{b-a} & \text{if} \quad s \leq t, \\ \dfrac{(t-a)(b-s)}{b-a} & \text{if} \quad s \geq t. \end{cases}$$

The above relations imply that

$$|y(t) - x(t;y(a),y(b))| \leq \int_a^b G(t,s)|f(s,y(s)) - f(s,x(s;y(a),y(b)))|ds + \int_a^b G(t,s)|g(s)|ds \leq$$
$$\int_a^b G(t,s)|f(s,y(s)) - f(s,x(s;y(a),y(b)))| + \frac{(b-a)^2}{8}\varepsilon.$$

If we suppose, for example, the condition (Lip) for f, then

$$|y(t) - x(t;y(a),y(b))| \leq L\int_a^b G(t,s)|y(s) - x(s;y(a),y(b))|ds + \frac{(b-a)^2}{8}\varepsilon.$$

for all $y \in S_\varepsilon$ and $t \in [a,b]$. Thus, by Gronwall Lemma II, (see Lemma 2.2), we get

$$|y(t) - x(t;y(a),y(b))| \leq \frac{(b-a)^2}{8}\varepsilon + \frac{(b-a)^3}{8}\varepsilon\frac{L\frac{b-a}{4}}{1 - L\frac{(b-a)^2}{4}} = \frac{(b-a)^2}{8}\left(1 + \frac{L(b-a)^2}{4 - L(b-a)^2}\right)\varepsilon.$$

In view of the above, we get the following result immediately.

Theorem 4.2. *If condition* (Lip) *takes place and* $\frac{L(b-a)^2}{4} < 1$, *then differential equation (4.1) is Ulam-Hyers stable.*

Remark 4.1. For related results, we refer to [15, pp. 131-132].

Let us also notice that, in general, $r_\varepsilon \neq \rho_\varepsilon$, as we will see in the following example.

Example 4.1. Let us consider the differential equations

$$-x'' = -x,\ t \in [0,1]$$

and

$$-x'' = -x+1, t \in [0,1].$$

Here $g(t) = 1$. Then, for $\varepsilon \geq 1$, the function $y(t) = e^t + 1$ is a solution of the differential inequation

$$|-x''+x| \leq \varepsilon.$$

If we consider the initial conditions

$$\begin{cases} y(0) = 2, \\ y'(0) = 1, \end{cases}$$

then, by an easy calculation, we get

$$x(t;2,1) = \frac{3}{2}e^t + \frac{1}{2}e^{-t} := r_\varepsilon(y)(t).$$

On the other hand, if we consider the bilocal conditions

$$\begin{cases} y(0) = 2, \\ y(1) = e+1, \end{cases}$$

then

$$x(t;2,e+1) = \frac{2e^{-1}-e-1}{e^{-1}-e}e^t + \frac{1-e}{e^{-1}-e}e^{-t} := \rho_\varepsilon(y)(t).$$

5. Differential equations with solutions in $C^1([a,+\infty[,\mathbb{B})$

Let $f \in C([a,\infty[,\times\mathbb{B},\mathbb{B})$, $g \in C([a,+\infty[,\mathbb{B})$, $\varepsilon > 0$ and $\varphi \in C([a,+\infty[,\mathbb{R}_+^*)$. We consider the differential equations

$$x'(t) = f(t,x(t)),\ t \in [a,+\infty[, \tag{5.1}$$

$$x'(t) = f(t,x(t)) + g(t),\ t \in [a,+\infty[, \tag{5.2}$$

and the differential inequations

$$|x'(t) - f(t,x(t))| \leq \varepsilon,\ \forall\, t \in [a,+\infty[, \tag{5.3}$$

$$|x'(t) - f(t,x(t)) \leq \varepsilon\varphi(t)|,\ \forall\, t \in [a,+\infty[, \tag{5.4}$$

and

$$|x'(t) - f(t,x(t))| \leq \varphi(t),\ \forall\, t \in [a,+\infty[. \tag{5.5}$$

As in Section 3, we suppose (see also [1, 21]) that

(C) for each $x_0 \in \mathbb{B}$ the equation (5.1) has a unique solution, denoted by $x(\cdot,x_0) \in C^1([a,\infty[,\mathbb{B})$, such that $x(a) = x_0$.

(PC) the equation (5.2) satisfies condition (C) for each $g \in C([a,+\infty[,\mathbb{B})$.

Let S_0 be the solution set of (5.1), S_ε, $S_{\varepsilon,\varphi}$, S_φ the solution set of (5.3), (5.4) and (5.5), respectively. As in Section 3, we consider the retractions

$$r_\varepsilon : S_\varepsilon \to S_0,\ r_{\varepsilon,\varphi} : S_{\varepsilon,\varphi} \to S_0,\ r_\varphi : S_\varphi \to S_0.$$

For r_ε, we have the estimate

$$|y(t) - r_\varepsilon(y)(t)| \leq (t-a)e^{L(t-a)}\varepsilon,\ \forall\, t \in [a,+\infty[.$$

The problem is that this estimate is not of Ulam-Hyers type.

Moreover, by simple examples, we observe that, in general, the differential equations is not Ulam-Hyers stable ([8, 15]) on $[a,+\infty[$.

Example 5.1. We consider the case $\mathbb{B} := \mathbb{R}$ and $f = 0$.

In this case, we have

$$x'(t) = 0,\ t \in [a,+\infty[\tag{5.6}$$

and

$$|x'(t)| \leq \varepsilon,\ t \in [a,\infty[. \tag{5.7}$$

The function $y(t) = t\varepsilon$, $t \in [a,+\infty[$ is a solution of (5.7) and all solutions of (5.6) are $x = c$, $c \in \mathbb{R}$. We remark that $|y(t) - c| \to \infty$ as $t \to \infty$.

The function $y(t) = \varepsilon \int_a^t \varphi(s)ds$ is a solution of

$$|x'(t)| \leq \varepsilon\varphi(t),\ \text{for } t \in [a,+\infty[. \tag{5.8}$$

In this case,

$$|y(t) - 0| = \left|\varepsilon \int_a^t \varphi(t)ds\right| = \varepsilon \int_a^t \varphi(s)ds.$$

To have an estimate of Ulam-Hyers-Rassias type, the function φ must satisfy a strong restrictive condition (see [8, 15]) of the following form:

(CC) there exists $\lambda > 0$ such that $\int_a^t \varphi(s)ds \leq \lambda\varphi(t)$, $\forall\, t \in [a,+\infty[$.

So, nothing remains but to estimate

$$|y(t) - \rho_{\varepsilon,\varphi}(y)(t)|$$

and

$$|y(t) - \rho_\varphi(y)(t)|.$$

References

[1] R.P. Agarwal, D. O'Regan, Infinite Interval Problems for Differential, Difference and Integral Equations, Kluwer Acad. Publ. 2001.
[2] H. Amann, Ordinary Differential Equations, Degruyter, Berlin, 1990.
[3] D. Bainov, P. Simeonov, Integral Inequalities and Applications, Springer, Netherlands, 1992.
[4] E.A. Coddington, N. Levinson, The Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.
[5] J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960.
[6] P. Găvrutǎ, L. Găvruţă, A new method for the generalized Hyers-Ulam-Rassias stability, Int. J. Nonlinear Anal. Appl. 1 (2010), 11-18.
[7] P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964.

[8] S.-M. Jung, A fixed point approach to the stability of a Volterra integral equation, Fixed Point Theory Appl. 2007 (2007), Article ID 57064.

[9] J.P. La Salle, The Stability of Dynamical Systems, SIAM, Philadelphia, 1976.

[10] A. Petruşel, I.A. Rus, Ulam stability of zero point equations, In: Ulam Type Stability, J. Brzedek, D. Popa, Th.M. Rassias (eds.), pp. 345-364, Springer, 2019,.

[11] A. Petruşel, I.A. Rus, M.A. Şerban, The role of equivalent metrics in fixed point theory, Topological Methods in Nonlinear Anal. 41 (2013), 85-112.

[12] L.C. Piccinini, G. Stampacchia, G. Vidossich, Ordinary Differential Equations in $\mathbb{R}^n$, Springer, New York, 1984.

[13] I.A. Rus, Fixed points, upper and lower fixed points: abstract Gronwall lemmas, Carpathian J. Math. 20 (2004), 125-134.

[14] I.A. Rus, Gronwall lemma approach to the Hyers-Ulam-Rassias stability of an integral equation, In: Nonlinear Analysis and Variational Problems, P. Pardalos, Th.M. Rassias, A.A. Khan, (eds.), pp. 147-152, Springer, 2009.

[15] I.A. Rus, Ulam stability of ordinary differential equations, Studia Univ. Babeş-Bolyai Math. 54 (2009), 125-133.

[16] I.A. Rus, Remarks on Ulam stability of the operatorial equations, Fixed Point Theory, 10 (2009), 305-320.

[17] I.A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, Carpathian J. Math. 26 (2010), 103-107.

[18] I.A. Rus, Results and problems in Ulam stability of operatorial equations and inclusions, In: Handbook of Functional Equations: Stability Theory, Th. M. Rassias (ed.), pp. 323-352, Springer, 2014.

[19] I.A. Rus, A. Petruşel, G. Petruşel, Fixed Point Theory, Cluj Univ. Press, Cluj-Napoca, 2008.

[20] A.M. Samoilenko, Yu.V. Teplinskii, Countable System of Differential Equations, de Gruyter, Berlin, 2003.

[21] C. Vladimirescu, C. Avramescu, Applications of the Fixed Point Method to Ordinary Differential and Integral Equations on Noncompact Intervals, Universitaria Press, Craiova, 2006.