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VARIANTS OF THE NONCONVEX MINIMIZATION THEOREM AND THE CARISTI'S FIXED POINT THEOREM

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Abstract. The aim of this paper is to obtain variants of the nonconvex minimization theorem and the Caristi's fixed point theorem in quasi-metric spaces. We also prove a generalized Ekeland's variational principle.
Keywords. Caristi's fixed point theorem; Ekeland's variational principle; Nonconvex minimization theorem; Quasi-metric space.

1. INTRODUCTION

In 1996, Kada, Suzuki and Takahashi [9] introduced the notion of the w-distance metric space and proved extended versions of Takahashi's minimization theorem (see [16]), Caristi's fixed point theorem and Ekeland's variational principle.

A nonnegative real-valued function p on a metric space (X,d) is called a w-distance if the following conditions are satisfied:

(i): for each $x,y,z \in X$, $p(x,z) \leq p(x,y) + p(y,z)$;
(ii): for any $x \in X$, a mapping $p(x,.) : X \longrightarrow [0,\infty)$ is lower semicontinuous;
(iii): for any $\delta > 0$, there exists $\mu > 0$ such that $p(z,x) \leq \mu$ and $p(z,y) \leq \mu$ imply $d(x,y) \leq \delta$.

The metric d is a classical example of the w-distance. For more details on the properties and examples of w-distance, we refer to [9].

In 1997, Suzuki [13] proved several fixed point theorems via w-distance and gave the characterization of the metric completeness. Since then, various results appeared as the generalizations of his results in many directions; see, e.g., [12, 14, 17]. One of those generalizations of the w-distance on a metric space was introduced again by Suzuki [15] and it is called τ-distance. Park [10] extended this concept to quasi-metric spaces. He proved equivalent formulations to the Ekeland's Variational Principle (see Theorems 1, 1' and 2 in [15]). Inspired by Park's work, Al-Homidan, Ansari and Yao [1] suggested a generalization of the w-distance on a quasi-metric space, which is called the Q-function. In 2010, Ume [18] introduced the u-distance, which is

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a generalization of the w-distance, the τ-distance and the Tataru's distance, and gave a minimization theorem, the Ekeland's variational principle, the Caristi's fixed point theorem using the u-distance on a metric space.

A nonnegative real-valued function ρ on a nonempty set X is called *quasi-metric* (see [5, 6]) if

(i): $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$;
(ii): $\rho(x,y) = \rho(y,x) = 0 \Leftrightarrow x = y$, for all $x, y, z \in X$.

The pair (X, ρ) is called a *quasi-metric space*. The *conjugate* of a quasi-metric ρ is a quasi-metric $\bar{\rho}$ defined by $\bar{\rho}(x,y) = \rho(y,x), x, y \in X$. If ρ is a quasi-metric on X, a mapping $\rho^s : X \times X \longrightarrow \mathbb{R}^+$ defined by $\rho^s(x,y) = \max\{\rho(x,y), \bar{\rho}(x,y)\}$ is a metric on X.

Let (X, ρ) be a quasi-metric space. Then, the open balls and closed balls are defined as follows: for $x \in X$ and $r > 0$,

$$B_\rho(x,r) = \{y \in X : \rho(x,y) < r\} \text{ (open ball)},$$
$$B_\rho[x,r] = \{y \in X : \rho(x,y) \leq r\} \text{ (closed ball)}.$$

The topology τ_ρ generated by a quasi-metric ρ on X is T_0 and it is T_1 if and only if $\rho(x,y) > 0$ for any two distinct points x and y in X.

Since the quasi-metric space X is equipped with two topologies τ_ρ and $\tau_{\bar{\rho}}$, the quasi-metric space $(X, \tau_\rho, \tau_{\bar{\rho}})$ is considered as a bitopological space with respect to Kelly [7]. For more details on bitopological spaces, the reader can refer to [6, 7].

A sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, ρ) is ρ-*convergent* (resp. $\overline{\rho}$-*convergent*) to some $x \in X$ if $\rho(x, x_n) \longrightarrow 0$ (resp. $\overline{\rho}(x, x_n) \longrightarrow 0 \Longleftrightarrow \rho(x_n, x) \longrightarrow 0$).

Lack of the symmetry condition in the definition of quasi-metric spaces makes various notions of Cauchy sequences, completeness and compactness. For instance, we refer to [8, 11]. Our basic structure can be found in [6, 11].

A sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, ρ) is called

(i): *left (right)* ρ-*Cauchy* if, for every $\varepsilon > 0$, there exists $x \in X$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\rho(x, x_n) < \varepsilon, \text{ (resp. } \rho(x_n, x) < \varepsilon).$$

(ii): ρ^s-*Cauchy* if, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$,

$$\rho^s(x_m, x_n) < \varepsilon,$$

that is, a sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy in a metric space (X, ρ^s).

The quasi-metric space (X, ρ) is said to be ρ-*sequentially* (resp. $\overline{\rho}$-*sequentially*) *complete* if every ρ^s-Cauchy sequence in X is ρ-convergent (resp. $\overline{\rho}$-convergent) to $x \in X$.

Note that the notion of the left ρ-Cauchy sequence is equivalent to the notion of the right $\overline{\rho}$-Cauchy sequence, but for the completeness, the two notions are distinct. By the right $\overline{\rho}$-completeness, we mean that every left ρ-Cauchy sequence in X is $\overline{\rho}$-convergent while by the left ρ-completeness, we mean the convergence of such sequence with respect to τ_ρ. For more details and examples, see [11].

In this paper, by using the notion of the w-distance, we aim to obtain variants of the nonconvex minimization theorem and the Caristi's fixed point theorem in quasi-metric spaces. Furthermore, we prove a general version of the Ekeland's variational principle.

2. PRELIMINARIES

In a quasi-metric space (X,ρ), a nonnegative real-valued function p on X is called a w-distance if the following conditions are satisfied [1, 10]:

(w_1) $p(x,z) \leq p(x,y) + p(y,z)$ for all $x,y,z \in X$;

(w_2) a mapping $p(x,.) : X \longrightarrow \mathbb{R}^+$ is $\overline{\rho}$-lower semicontinuous for any $x \in X$;

(w_3) for any $\delta > 0$, there exists $\mu > 0$ such that $p(z,x) \leq \mu$ and $p(z,y) \leq \mu$ imply $\rho(x,y) \leq \delta$.

Clearly, a quasi-metric ρ is not necessary to be a w-distance on (X,ρ). The following example illustrates this case [8].

Let $X = \mathbb{R}$ and ρ be a quasi-metric defined by

$$\rho(x,y) = \begin{cases} y-x, & \text{if} \quad x \leq y, \\ 1, & \text{if} \quad x > y. \end{cases}$$

Then, ρ satisfies (w_1) and (w_2), however, (w_3) does not hold.

The following two lemmas play an important role in our theorems. Note that the following lemma is a general form of Lemma 1 in [13] in the case of the quasi-metric.

Lemma 2.1. *Let (X,ρ) be a quasi-metric space. Let $(x_n),(y_n)$ and (z_n) be sequences in X and let $x,y,z \in X$. Assume that p is a $w-$distance on X. Then the following hold:*

(i): If $p(y_n,x_n) \longrightarrow 0$ and $p(y_n,z_n) \longrightarrow 0$, then $\rho(x_n,z_n) \longrightarrow 0$.

(ii): If $p(y_n,x_n) \longrightarrow 0$ and $p(y_n,z) \longrightarrow 0$, then (x_n) is $\overline{\rho}$-convergent to z.

(iii): If $p(y_n,x) \longrightarrow 0$ and $p(y_n,z) \longrightarrow 0$, then $x = z$. In particular, if $p(y,x) = 0$ and $p(y,z) = 0$, then $x = z$.

Lemma 2.2. [2, Lemma 1] *Let (X,ρ) be a quasi-metric space and p a $w-$distance on X. Then, for any $\delta > 0$, there exists $\mu > 0$ such that $p(z,x) \leq \mu$ and $p(z,y) \leq \mu$ imply $\rho^s(x,y) \leq \delta$.*

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^+ := [0,\infty)$ denotes the set of nonnegative real numbers.

3. THE MINIMIZATION THEOREM

In this section, we prove a nonconvex minimization theorem on a sequentially complete quasi-metric space (X,ρ). Our results generalize the results of Kada et.al [1] and Park [8]. We set the following hypotheses.

Hypothesis:

1. Let $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be subadditive, i.e. $\gamma(u+v) \leq \gamma(u) + \gamma(v), \forall u,v \in \mathbb{R}^+$, amenable, i.e., $\gamma^{-1}(\{0\}) = \{0\}$ and an increasing continuous map. For example, $\gamma(u) = t^{\alpha}, (0 < \alpha \leq 1)$, for $u \in \mathbb{R}^+$. Let Γ be the family of all such functions γ.

2. Let $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a map for which there exist $\varepsilon > 0$ and $\gamma \in \Gamma$ such that if $\eta(u) \leq \varepsilon \Longrightarrow \eta(u) \geq \gamma(u)$. The family of all such functions η is denoted by $\mathscr{A}$.

3. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $F(0) = 0$, $F^{-1}[0,\infty) \subset [0,\infty)$ and for $u \in \mathbb{R}^+$, F is increasing upper semicontinuous. Also, we suppose that F is superadditive, i.e., $F(u) + F(v) \leq F(u+v)$ for $u,v \in \mathbb{R}^+$. The family of all such functions F is denoted by $\mathscr{F}$. For more details, see [19].

Theorem 3.1. *Let (X,ρ) be a $\overline{\rho}$-sequentially complete quasi-metric space and $\varphi : X \longrightarrow \mathbb{R}$ a $\overline{\rho}$-lower semicontinuous and bounded below function. Assume that there exists a w-distance p on X such that for any $x \in X$ with*

$$\varphi(x) > \inf\{\varphi(t) : t \in X\},$$

there exists $y \in X$ with $y \neq x$ and

$$\eta(p(x,y)) \leq F(\varphi(x) - \varphi(y)),$$

where $\eta \in \mathscr{A}$ and $F \in \mathscr{F}$. Then there exists $z \in X$ such that

$$\varphi(z) = \inf_{t \in X} \varphi(t).$$

Proof. Set $\varphi_0 = \inf_{t \in X} \varphi(t)$. Assume, to the contrary, that $\varphi_0 < \varphi(y)$ for all $y \in X$. By use of the upper semicontinuity of F on $\mathbb{R}^+$, we have

$$\limsup_{u \to 0^+} F(u) \leq F(0) = 0.$$

Then, for $\varepsilon > 0$, there exists $\varepsilon_0 > 0$ such that $F(u) < \varepsilon$ for $0 \leq u \leq \varepsilon_0$. Let

$$X_0 = \{x \in X : \varphi(x) \leq \varphi_0 + \varepsilon_0\}.$$

Clearly, $X_0 \neq \varnothing$. We show that (X_0,ρ) is $\overline{\rho}$-sequentially complete. To show this, let (x_n) be a ρ^s-Cauchy sequence in X_0. Then $(x_n) \subset X$ and

$$\varphi(x_n) \leq \varphi_0 + \varepsilon_0. \tag{3.1}$$

By use of the $\overline{\rho}$-sequential completeness of X, there exists $z \in X$ such that $x_n \stackrel{\overline{\rho}}{\longrightarrow} z \in X$, that is, for any $\varepsilon > 0$, there exists $n_\varepsilon > 0$ such that

$$\overline{\rho}(z,x_n) < \varepsilon, \tag{3.2}$$

for all $n \geq n_\varepsilon$. Since φ is $\overline{\rho}$-lower semicontinuous, it follows from (3.1) that

$$\varphi(z) \leq \liminf_{i \to \infty} \varphi(x_i) \leq \varphi_0 + \varepsilon_0,$$

which implies that $z \in X_0$. It then follows from (3.2) that $x_n \stackrel{\overline{\rho}}{\longrightarrow} z \in X_0$. Hence (X_0,ρ) is $\overline{\rho}$-sequentially complete. For $x,y \in X_0$ with $0 \leq \eta(p(x,y)) \leq F(\varphi(x) - \varphi(y))$, we find from $F^{-1}([0,\infty)) \subset [0,\infty)$ that $\varphi(x) - \varphi(y) \geq 0$. Also, for any $x,y \in X_0$, we have

$$\begin{aligned} \varphi_0 \leq \varphi(x) \leq \varphi_0 + \varepsilon_0, \\ \varphi_0 \leq \varphi(y) \leq \varphi_0 + \varepsilon_0. \end{aligned} \tag{3.3}$$

Thus

$$0 \leq \varphi(x) - \varphi(y) \leq \varepsilon_0,$$

and

$$\eta(p(x,y)) \leq F(\varphi(x) - \varphi(y)) \leq \varepsilon.$$

Since $\eta \in \mathscr{A}$, we have

$$\gamma(p(x,y)) \leq \eta(p(x,y)) \leq F(\varphi(x) - \varphi(y)).$$

For each $x \in X_0$, we define

$$S(x) = \{y \in X_0 : \gamma(p(x,y)) \leq F(\varphi(x) - \varphi(y))\}. \tag{3.4}$$

By the assumptions, we have, for each $x \in X_0$, there exists $s \in X_0$ with $s \neq x$ such that $s \in S(x)$ and $S(y) \subseteq S(x)$ for each $y \in S(x)$. Now, for each $x \in X_0$, let

$$I(x) = \inf\{\varphi(y) : y \in S(x)\}. \tag{3.5}$$

Choose $x \in X_0$. Then define a sequence (x_n) in $S(x)$ when $x_1 = x$, $x_i, i = 2,3,\ldots,n$, have been chosen, and choose $x_{n+1} \in S(x_n)$ such that

$$\varphi(x_{n+1}) < I(x_n) + \frac{1}{n} \tag{3.6}$$

for $n \in \mathbb{N}$. Thus we have constructed a sequence (x_n) such that

$$\gamma(p(x_n, x_{n+1})) \leq F(\varphi(x_n) - \varphi(x_{n+1})), \tag{3.7}$$

$$\varphi(x_{n+1}) - \frac{1}{n} < I(x_n) \leq \varphi(x_{n+1}). \tag{3.8}$$

Since $F^{-1}([0,\infty)) \subset [0,\infty)$, we conclude from (3.7) that $\varphi(x_{n+1}) \leq \varphi(x_n)$ for all $n \in \mathbb{N}$. Thus $(\varphi(x_n))$ is a decreasing sequence of reals and bounded below. Therefore, it converges, that is, there is $\alpha \in \mathbb{R}$ such that

$$\alpha = \lim_{n\to\infty} \varphi(x_n) = \inf_{n\in\mathbb{N}} \varphi(x_n).$$

In view of (3.8), we have

$$\alpha = \lim_{n\to\infty} I(x_n) = \lim_{n\to\infty} \varphi(x_n) = \inf_{n\in\mathbb{N}} \varphi(x_n). \tag{3.9}$$

Then (x_n) is a ρ^s-Cauchy sequence. Indeed, if $m > n$, then it follows from (w_1) and the subadditivity of γ that

$$\begin{aligned}\gamma(p(x_n, x_m)) &\leq \gamma\left(\sum_{i=n}^{m-1} p(x_i, x_{i+1})\right) \leq \sum_{i=n}^{m-1} \gamma(p(x_i, x_{i+1})) \\ &\leq \sum_{i=n}^{m-1} (F(\varphi(x_i) - \varphi(x_{i+1}))).\end{aligned}$$

Since $F(u) + F(v) \leq F(u+v)$, we obtain

$$\begin{aligned}\gamma(p(x_n, x_m)) &\leq F\left(\sum_{i=n}^{m-1} (\varphi(x_i) - \varphi(x_{i+1}))\right) \\ &= F(\varphi(x_n) - \varphi(x_m)).\end{aligned} \tag{3.10}$$

It then follows from the upper semicontinuity of F on $\mathbb{R}^+$that

$$\begin{aligned}0 &\leq \limsup_{n,m\to\infty} \gamma(p(x_n, x_m)) \\ &\leq \limsup_{n,m\to\infty} F(\varphi(x_n) - \varphi(x_m)) \\ &\leq F(0) = 0\end{aligned}$$

and so $\lim_{n,m\to\infty} \gamma(p(x_n, x_m)) = 0$, which implies that

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Thus for any $\delta > 0$, there exists $\mu > 0$ and $n_\mu > 0$ such that $p(x_n, x_m) < \mu$. whenever $m > n \geq n_\mu$. In particular, $p\left(x_{n_\mu}, x_n\right) \leq \mu$ and $p\left(x_{n_\mu}, x_m\right) \leq \mu$ whenever $m, n \geq n_\mu$. Hence, by use of Lemma 2.2, we have $\rho^s(x_n, x_m) \leq \delta$. whenever $m, n \geq n_\mu$. It follows that (x_n) is ρ^s-Cauchy sequence in X_0. By use of the $\overline{\rho}$-sequential completeness of X_0, there exists $z \in X_0$ such that

$$x_n \xrightarrow{\overline{\rho}} z. \tag{3.11}$$

Since φ is $\overline{\rho}$-lower semicontinuous, we have

$$\varphi(z) \leq \liminf_{n\to\infty} \varphi(x_n) = \alpha. \tag{3.12}$$

By using (w_2), we have

$$p(x_n, z) \leq \liminf_{m\to\infty} p(x_n, x_m). \tag{3.13}$$

Since F is increasing, upper semicontinuous on $\mathbb{R}^+$ and γ is continuous, we find from (3.10), (3.12) and (3.13) that

$$\begin{aligned} \gamma(p(x_n, z)) &\leq \limsup_{m\to\infty} \gamma(p(x_n, x_m)) \\ &\leq \limsup_{m\to\infty} F(\varphi(x_n) - \varphi(x_m)) \\ &\leq F(\varphi(x_n) - \alpha) \\ &\leq F(\varphi(x_n) - \varphi(z)). \end{aligned} \tag{3.14}$$

It follows from (3.4), (3.5) and (3.14) that $z \in S(x_n)$ and therefore

$$I(x_n) \leq \varphi(z) \text{ for all } n \in \mathbb{N}. \tag{3.15}$$

Taking the limit as $n \to \infty$ in (3.15), we get

$$\lim_{n\to\infty} I(x_n) \leq \varphi(z). \tag{3.16}$$

This together with (3.9) and (3.12) implies that

$$\varphi(z) = \alpha. \tag{3.17}$$

Since $z \in S(x_n)$ and $x_n \in S(x)$, we have $z \in S(x)$. Suppose that $s_1 \in S(z)$ and $s_1 \neq z$. Then $\varphi(s_1) < \varphi(z) = \alpha$. Again, since $s_1 \in S(z)$, $z \in S(x_n)$ and $x_n \in S(x)$, we have that $S(z) \subseteq S(x_n) \subseteq S(x)$. So $s_1 \in S(x_n)$ and $s_1 \in S(x)$, which gives $I(x_n) \leq \varphi(s_1)$ for all $n \in \mathbb{N}$. It further implies that

$$\alpha = \lim_{n\to\infty} I(x_n) \leq \varphi(s_1).$$

This contradicts the fact that $\varphi(s_1) < \alpha$. Thus $S(z) = \{z\}$. However, we can find $y \in X$ with $y \neq z$ and $y \in S(z)$, which is a contradiction. Hence, there exists $z \in X$ such that $\varphi(z) = \inf_{x\in X} \varphi(x)$. □

4. The Caristi's Fixed Point Theorem

In this section, we use the notion of the w-distance to generalize the well-known Caristi's fixed point theorem [3, 9].

Theorem 4.1. *Let (X, ρ) be a $\overline{\rho}$-sequentially complete quasi-metric space and $\varphi : X \longrightarrow \mathbb{R}$ a $\overline{\rho}$-lower semicontinuous and bounded below function. Let $T : X \longrightarrow X$ be a mapping. Assume that there exists a w-distance p on X such that*

$$\eta(p(x, Tx)) \leq F(\varphi(x) - \varphi(Tx)) \tag{4.1}$$

for all $x \in X$*, where* $\eta \in \mathscr{A}$ *and* $F \in \mathscr{F}$*. Then there is* $z_0 \in X$ *such that* $z_0 = Tz_0$ *and* $p(z_0, z_0) = 0$*.*

Proof. Choose $x \in X$ and let

$$Y = \{y \in X : \varphi(y) \le \varphi(x)\}.$$

Then $Y \ne \varnothing$. We show that Y is $\overline{\rho}$-sequentially complete. Let $(y_n) \subset Y$ be a ρ^s-Cauchy sequence such that $y_n \xrightarrow{\overline{\rho}} y$. We show that $y \in Y$. Since $y_n \in Y$, then $\varphi(y_n) \le \varphi(x)$. By use of the lower semicontinuity of φ with respect to $\overline{\rho}$, we have

$$\varphi(y) \le \liminf_{i \to \infty} \varphi(y_i) \le \varphi(x),$$

which shows that $y \in Y$. Now, let $y \in Y$. Since $F^{-1}([0,\infty)) \subset [0,\infty)$, we have from (4.1) that

$$\varphi(Ty) \le \varphi(y) \le \varphi(x),$$

which implies that $Ty \in Y$ and so Y is invariant under T. Suppose that $Ty \ne y$ for all $y \in Y$. Then, by Theorem 3.1, there exists $z \in Y$ such that $\varphi(z) = \inf_{y \in Y} \varphi(y)$. Since $\eta(p(z,Tz)) \le F(\varphi(z) - \varphi(Tz))$, we have from $F^{-1}([0,\infty)) \subset [0,\infty)$ that

$$\varphi(Tz) = \varphi(z) \text{ and } \eta(p(z,Tz)) = 0. \tag{4.2}$$

So, there exists $\varepsilon_0 > 0$ such that

$$\eta(p(z,Tz)) \le F(\varphi(z) - \varphi(Tz)) \le \varepsilon_0$$

and thus,

$$\gamma(p(z,Tz)) \le \eta(p(z,Tz)) = 0. \tag{4.3}$$

From (4.1), we have

$$\eta\left(p\left(Tz, T^2z\right)\right) \le F\left(\varphi(Tz) - \varphi\left(T^2z\right)\right).$$

This together with (4.2) and the condition $F^{-1}([0,\infty)) \subset [0,\infty)$ implies that

$$\varphi(Tz) = \varphi\left(T^2z\right) \text{ and so } \gamma\left(p\left(Tz, T^2z\right)\right) = 0. \tag{4.4}$$

It follows from (w_1) and the subadditivity of γ that

$$\begin{aligned} \gamma\left(p\left(z, T^2z\right)\right) &\le \gamma\left(p(z,Tz) + p\left(Tz, T^2z\right)\right) \\ &\le \gamma(p(z,Tz)) + \gamma\left(p\left(Tz, T^2z\right)\right) = 0. \end{aligned} \tag{4.5}$$

Since (4.3), (4.5) and γ is amenable, we have

$$p(z,Tz) = 0 \text{ and } p\left(z, T^2z\right) = 0. \tag{4.6}$$

Thus, from Lemma 2.1 (iii), it follows that

$$Tz = T^2z. \tag{4.7}$$

which is a contradiction. Hence, there exists $z_0 \in Y \subset X$ such that $Tz_0 = z_0$. Now,

$$\begin{aligned} \eta(p(z_0,z_0)) &= \eta(p(z_0,Tz_0)) \le F(\varphi(z_0) - \varphi(Tz_0)) = F(\varphi(z_0) - \varphi(z_0)) \\ &= F(0) = 0. \end{aligned}$$

So $\gamma(p(z_0,z_0)) = 0$. Since γ is amenable, we obtain $p(z_0,z_0) = 0$. This completes the proof. □

We conclude this section by the following examples which show the applicability of Theorem 4.1.

Example 4.1. Let $X = \omega$ and ρ be a quasi-metric on X defined by

$$\begin{array}{ll} \rho(x,x) = 0, & \text{for all } x \in X, \\ \rho(n,0) = \frac{1}{n}, & \text{for all } n \in \mathbb{N}, \\ \rho(0,n) = 1, & \text{for all } n \in \mathbb{N}, \\ \rho(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|, & \text{for all } n,m \in \mathbb{N}. \end{array}$$

Then (X,ρ) is $\overline{\rho}$-sequentially complete quasi-metric space (see [2]). Define $\varphi : X \longrightarrow \mathbb{R}$ by $\varphi(x) = x^2$, $\forall x \in X$. Then, φ is a $\overline{\rho}$-lower semicontinuous and bounded below function. Let $T : X \longrightarrow X$ be a mapping on X defined by

$$T(x) = \begin{cases} \frac{x}{2}, & \text{if } \; x \text{ is even,} \\ x, & \text{if } \; x \text{ is odd} \end{cases}$$

for all $x \in X$. Now, define $p : X \times X \longrightarrow \mathbb{R}^+$ by $p(x,y) = |x-y|$ for all $x,y \in \mathbb{N}$ and $p(x,y) = 0$ for $x = 0$ or $y = 0$ is a w-distance on X . Take $\eta(x) = x^\alpha$ for $0 < \alpha \leq 1$ and $F(x) = x^\beta$ for $\beta \geq 1$ and $x \geq 0$. Therefore, we have the following cases:

Case1: If $x \in \mathbb{N}$ is odd, then

$$\eta(p(x,T(x))) = \eta(p(x,x)) = \eta(|x-x|) = \eta(0) = 0 = F(\varphi(x) - \varphi(T(x))).$$

Case 2: If $x \in \mathbb{N}$ is even, then

$$\eta(p(x,T(x))) = \eta\left(p\left(x,\frac{x}{2}\right)\right) = \eta\left(\left|x - \frac{x}{2}\right|\right) = \eta\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^\alpha,$$

and

$$F(\varphi(x) - \varphi(T(x))) = F\left(\varphi(x) - \varphi\left(\frac{x}{2}\right)\right) = F\left(x^2 - \frac{x^2}{4}\right) = F\left(\frac{3}{4}x^2\right) = \left(\frac{3}{4}x^2\right)^\beta.$$

Since $x \leq x^2$, it follows that $\frac{1}{2}x \leq \frac{1}{2}x^2 \leq \frac{3}{4}x^2$ and then $\left(\frac{1}{2}x\right)^\alpha \leq \left(\frac{3}{4}x^2\right)^\alpha \leq \left(\frac{3}{4}x^2\right)^\beta$, where $\alpha \leq 1 \leq \beta$. It follows that

$$\eta(p(x,T(x))) \leq F(\varphi(x) - \varphi(T(x))).$$

Case 3: If $x = 0$, then

$$\eta(p(0,T(0))) = \eta(p(0,0)) = \eta(0) = 0 = F(\varphi(0) - \varphi(T(0))).$$

Hence, in each cases, we have $\eta(p(x,T(x))) \leq F(\varphi(x) - \varphi(T(x)))$ for all $x \in X$, that is, T satisfies the Caristi's condition and we can apply Theorem 4.1. Note that T has 0 and all odd points as fixed points.

Example 4.2. Let $X = \mathbb{R}^+$ and ρ be a quasi-metric on X given by

$$\rho(x,y) = \max\{y-x,0\}, \text{ for all } x,y \in X.$$

Then (X,ρ) is $\overline{\rho}$-sequentially complete quasi-metric space (see [2]). In fact, since $\rho(x,0) = \max\{-x,0\} = 0$, we have $\overline{\rho}(0,x) = 0$ for all $x \in X$. Thus, every sequence in X is $\overline{\rho}$-convergent to 0. Let $T : X \longrightarrow X$ be a mapping on X defined by $Tx = x+1$ for all $x \in X$. Let $\varphi : X \longrightarrow \mathbb{R}$ be defined by $\varphi(x) = x-1$ for every $x \in X$. Then φ is $\overline{\rho}$-lower semicontinuous and bounded below function. Take $p(x,y) = y$ for all $x,y \in X$. Then p is a $w-$distance on X .

Let $\eta(x) = x^\alpha$ for $0 < \alpha \leq 1$ and $F(x) = x^\beta$ for $\beta \geq 1$ and $x \in X$. Then

$$\eta(p(x,Tx)) = (x+1)^\alpha \quad \text{and} \quad F(\varphi(x) - \varphi(Tx)) = (-1)^\beta.$$

So,

$$\eta\left(p\left(x,Tx\right)\right)\geq F\left(\varphi\left(x\right)-\varphi\left(Tx\right)\right)\quad \text{if}\quad \beta\ \text{ even}$$

and

$$\eta\left(p\left(x,Tx\right)\right)>F\left(\varphi\left(x\right)-\varphi\left(Tx\right)\right)\quad \text{if}\quad \beta\ \text{ odd.}$$

Note that T has no fixed points in X and T does not satisfy the Caristi's condition.

5. Ekeland's Variational Principle

In this section, we prove a general form of the Ekeland's variational principle in quasi-metric spaces. Our result generalizes various known results of some authors, including Kada, Suzuki and Takahashi [9], Zhang and Jiang [19] and Cobzas [4] in the setting of quasi-metric spaces via w-distance p.

Theorem 5.1. *Let (X,ρ) be a $\overline{\rho}$-sequentially complete quasi-metric space and $\varphi : X \longrightarrow \mathbb{R}$ be a $\overline{\rho}$-lower semicontinuous and bounded below function. Let p be a w-distance on X . Then the following statements hold:*

(1) For any $x\in X$, there exists $y\in X$ such that $\varphi(y)\leq\varphi(x)$ and

$$\gamma(p(y,s))>F(\varphi(y)-\varphi(s)),$$

for all $s\in X\setminus\{y\}$.

(2) For any $\varepsilon>0,\lambda>0$ and $x\in X$ with $p(x,x)=0$ and

$$\varphi(x)\leq\inf_{t\in X}\varphi(t)+\varepsilon,$$

where $\gamma\in\Gamma$, $\eta\in A$ and $F\in\mathscr{F}$, there exists $y\in X$ such that

(i): $\varphi(y)\leq\varphi(x)$;
(ii): $\gamma(p(x,y))\leq\frac{\lambda}{\varepsilon}F(\varepsilon)$;
(iii): $F(\varphi(y)-\varphi(s))<\frac{\varepsilon}{\lambda}\gamma(p(y,s))$, *for all $s\in X\setminus\{y\}$.*

Proof. (1) Let $x\in X$ and define

$$Z=\{y\in X:\varphi(y)\leq\varphi(x)\}.$$

Clearly, $Z\neq\varnothing$ as $\varphi(x)\leq\varphi(x)$. We show that Z is a $\overline{\rho}$-sequentially complete. Let (y_n) be a ρ^s-Cauchy sequence in Z such that $y_n\xrightarrow{\overline{\rho}}y$. Since $y_n\in Z$, then $\varphi(y_n)\leq\varphi(x)$, By use of the lower semicontinuity of φ with respect to $\overline{\rho}$, we have

$$\varphi(y)\leq\liminf_{i\to\infty}\varphi(y_i)\leq\varphi(x),$$

which follows that $y\in Z$. Now, we show that there exists $y\in Z$ such that

$$\gamma(p(y,s))>F(\varphi(y)-\varphi(s)),$$

for all $s\in X$ with $y\neq s$. If not, then, for each $y\in Z$, there is $a\in X$ such that $a\neq y$ and

$$\gamma(p(y,a))\leq F(\varphi(y)-\varphi(a)).$$

Since $\varphi(a)\leq\varphi(y)\leq\varphi(x)$, $a\in Z$, we find from the minimization theorem (Theorem 3.1 with $\eta=\gamma$) that there is $z\in Z$ such that $\varphi(z)=\inf_{t\in Z}\varphi(t)$. Again, for $z\in Z$, there exists $b\in Z$ such that

$$b\neq z \text{ and } \gamma(p(z,b))\leq F(\varphi(z)-\varphi(b)). \tag{5.1}$$

Using $F^{-1}([0,\infty)) \subset [0,\infty)$, we get $\varphi(b) \leq \varphi(z)$, From the minimality of $\varphi(z)$, we have $\varphi(z) = \varphi(b)$. In view of (5.1), we have $\gamma(p(z,b)) = 0$. Since γ is amenable, we have

$$p(z,b) = 0. \tag{5.2}$$

Similarly, for $b \in Z$, there exists $c \in Z$ such that

$$c \neq b \text{ and } \gamma(p(b,c)) \leq F(\varphi(b) - \varphi(c)). \tag{5.3}$$

So, $\varphi(b) = \varphi(c)$ and by (5.3) we obtain $\gamma(p(b,c)) = 0$. Since γ is amenable, we have

$$p(b,c) = 0. \tag{5.4}$$

Then, we immediately get from (w_1), (5.2) and (5.4) that

$$p(z,c) = 0. \tag{5.5}$$

In view of (5.2), (5.5) and using Lemma 2.1 (iii), we have $b = c$, which contradicts to (5.3).

(2). Define

$$Y = \left\{ y \in X : F(\varphi(x) - \varphi(y)) \geq \frac{\varepsilon}{\lambda}\gamma(p(x,y)) \right\}.$$

Clearly, $x \in Y$ and then $Y \neq \varnothing$. To prove that Y is a $\overline{\rho}$-sequentially complete, let (y_n) be a ρ^s-Cauchy sequence in Y such that (y_n) is $\overline{\rho}$-convergent to some $y \in X$. Since $y_n \in Y$, we obtain

$$\frac{\varepsilon}{\lambda}\gamma(p(x,y_n)) \leq F(\varphi(x) - \varphi(y_n)). \tag{5.6}$$

By use of the lower semicontinuity of φ with respect to $\overline{\rho}$, we have

$$\varphi(y) \leq \liminf_{n \longrightarrow \infty} \varphi(y_n).$$

Setting $\beta = \liminf_{n \longrightarrow \infty} \varphi(y_n)$, we find that there is a subsequence $\varphi(y_{n_k})$ such that $\varphi(y_{n_k}) \longrightarrow \beta$. It follows from $F^{-1}([0,\infty)) \subset [0,\infty)$ and (5.6) that $\varphi(x) - \varphi(y_n) \geq 0$. Since γ is continuous and F is increasing, upper semicontinuous on $\mathbb{R}^+$, we have from (5.6) that

$$\begin{aligned} \frac{\varepsilon}{\lambda}\gamma(p(x,y)) &= \frac{\varepsilon}{\lambda}\limsup_{k \longrightarrow \infty} \gamma(p(x,y_{n_k})) \\ &\leq \limsup_{k \longrightarrow \infty} F(\varphi(x) - \varphi(y_{n_k})) \\ &\leq F(\varphi(x) - \beta) \leq F(\varphi(x) - \varphi(y)), \end{aligned}$$

which implies that $y \in Y$. The same manner of proof of statement (1) of this theorem yields $y \in Y$ such that

$$\frac{\varepsilon}{\lambda}\gamma(p(y,s)) > F(\varphi(y) - \varphi(s)),$$

for each $s \in X$ with $s \neq y$. On the other side, since $y \in Y$, we obtain

$$\frac{\varepsilon}{\lambda}\gamma(p(x,y)) \leq F(\varphi(x) - \varphi(y)). \tag{5.7}$$

Using $F^{-1}([0,\infty)) \subset [0,\infty)$, we get

$$\varphi(y) \leq \varphi(x). \tag{5.8}$$

Also, by using (5.8) and the monotonicity of F on $\mathbb{R}^+$, we have

$$\begin{aligned}\frac{\varepsilon}{\lambda}\gamma(p(x,y)) &\leq F(\varphi(x)-\varphi(y)) \\ &\leq F\left(\varphi(x)-\inf_{t\in X}\varphi(t)\right) \\ &\leq F(\varepsilon).\end{aligned}$$

This completes the proof. □

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