Appl. Set-Valued Anal. Optim. 2 (2020), No. 3, pp. 317-327
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.2.2020.3.06

STRONG CONVERGENCE OF MODIFIED INERTIAL HALPERN SIMULTANEOUS ALGORITHMS FOR A FINITE FAMILY OF DEMICONTRACTIVE MAPPINGS

JINGFANG XIAO, LU HUANG, YAQIN WANG*

Department of Mathematics, Shaoxing University, Shaoxing 312000, China

Abstract. In this paper, we consider modified inertial Halpern simultaneous algorithms for a finite family of demicontractive mappings. We establish some strong convergence theorems under some suitable conditions in the framework of Hilbert spaces. Our results generalize and improve some known results announced recently.

Keywords. Inertial algorithms; Demicontractive mapping; Fixed point; Strong convergence.

1. INTRODUCTION

There are a number of real problems, such as, quantum physics and image processing, arising in the framework of infinite dimensional spaces. In studying these problems, the strong convergence, the convergence in norm, plays a significant role since we can clearly see the energy of the error between the iteration and the desired solution. In order to obtain the strong convergence, many authors have devised various iterative algorithms; see, e.g., [4, 5, 17, 23]. On the other hand, for the convergence rate of algorithms, it is known that Mann algorithm is slow. Recently, spotlight sheds on various fast algorithms, which are important from the viewpoint of real applications; see, e.g., [2, 6, 16] and the references therein. In 1964, Polyak [15] first proposed an inertial type extrapolation as an acceleration process. In recent years, based on inertial extrapolation techniques, many new iterative algorithms are introduced, such as, inertial forward-backward splitting algorithms [11, 18], inertial Mann algorithms [12] and inertial extragradient algorithms [7, 10].

In 2008, Mainge [12] introduced the following inertial Mann algorithm with the aid of the inertial extrapolation

$$\begin{cases} w_n = x_n + \delta_n(x_n - x_{n-1}), \\ x_{n+1} = \varphi_n w_n + (1-\varphi_n)T(w_n), \quad n \geq 0, \end{cases} \tag{1.1}$$

where T is a nonexpansive mapping, and $\{\delta_n\}$ and $\{\varphi_n\}$ are two real sequences. He proved that the iterative sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T under some mild assumptions in Hilbert spaces.

*Corresponding author.
E-mail addresses: wangyaqin0579@126.com (Y. Wang).
Received October 22, 2020; Accepted December 1, 2020.

Recently, Tan, Zhou and Li [19] proposed a modified inertial Mann algorithm:

$$\begin{cases} w_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = \varphi_n w_n + (1-\varphi_n)T(w_n), \\ x_{n+1} = v_n u + (1-v_n)y_n, \qquad n \geq 0, \end{cases} \tag{1.2}$$

where T is a nonexpansive mapping, and $\{\delta_n\}$ and $\{\varphi_n\}$ are two real sequences. They showed that the iterative sequence $\{x_n\}$ defined by (1.2) converges strongly to $p = P_{Fix(T)}u$ in Hilbert spaces without the aid of compact assumptions.

Recently , the following so-called split common fixed point problem (SCFP) attracts much attention in the community of nonlinear optimization, which consists of

$$\text{finding } \ x^* \in \bigcap_{i=1}^{p} F(U_i) \text{ such that } Ax^* \in \bigcap_{j=1}^{q} F(T_j),$$

where $p, q \geq 1$ are integers, $F(T_i)$ denotes the fixed point set of T_i, $F(U_i)$ denotes the fixed point set of U_i, $A : H_1 \to H_2$ is a bounded linear operator, and $\{U_i\}_{i=1}^{p} : H_1 \to H_1$ and $\{T_j\}_{j=1}^{q} : H_2 \to H_2$ are families of nonlinear operators.

For studying the SCFP, Tang, Peng and Liu [20] introduced the following simultaneous iterative algorithm, which is also called parallel iterative algorithm

$$\begin{cases} u_k = \ x_k + \gamma A^* \Sigma_{j=1}^{r} \eta_j (T_j - I)\, Ax_k, \\ x_{k+1} = \ (1-\alpha_k)u_k + \alpha_k \Sigma_{i=1}^{p} \omega_i U_i(u_k), \end{cases} \tag{1.3}$$

where γ is some positive real number, $\{\alpha_k\} \subset (0,1)$, $\{\omega_i\}_{i=1}^{p} \subset (0,1)$, and $\{\eta_j\}_{j=1}^{q} \subset (0,1)$ with $\Sigma_{i=1}^{p} \omega_i = 1$ and $\Sigma_{j=1}^{q} \eta_j = 1$. They proved the weak convergence of this algorithm and solved the SCFP governed by demicontractive mappings $U_i (1 \leq i \leq p)$ and $T_j (1 \leq i \leq q)$.

In this paper, inspired and motivated by the above results, we first propose a modified inertial Halpern simultaneous algorithm for a finite family of demicontractive mappings. Then, we prove some strong convergence results under some mild conditions. Finally, we provide a numerical example to show the effectiveness of our proposed algorithm.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a closed and convex nonempty subset of H. We denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. From now on, $F(T)$ denotes the fixed-point set of mapping T.

Recall that the metric (or nearest point) projection P_C from H onto C is defined as follows. For any given $x \in H$, there exists a unique vector in C, $P_C x$, such that

$$P_C(x) := \text{argmin}_{y \in C} \|x - y\|.$$

It is well-known [24] that P_C is a nonexpansive mapping and is characterized by the inequality

$$P_C x \ \in \ C, \ \ \langle x - P_C x, \, y - P_C x \rangle \ \leq \ 0, \ \ \forall \, y \in \ C. \tag{2.1}$$

For each $x, y \in H$, we also have the following known facts.

(C_1) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle$;

(C_2) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \quad \forall \, t \in R.$

Definition 2.1. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall\, x, y \in C;$$

(2) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - q\| \leq \|x - q\|, \forall\, x \in C,\ q \in F(T);$$

(3) $\mu-$demicontractive if $F(T) \neq \emptyset$ and there exists a constant $\mu \in (-\infty, 1)$ such that

$$\|Tx - q\|^2 \leq \|x - q\|^2 + \mu\|x - Tx\|^2,\ \ \forall\, x \in\ C,\ q \in\ F(T).$$

Remark 2.1. [21] Note that every 0-demicontractive mapping is exactly quasi-nonexpansive. In particular, one say that it is quasi-strictly pseudo-contractive [13] if $0 \leq \mu < 1$. Moreover, if $\mu \leq 0$, every $\mu-$demicontractive mapping becomes quasi-nonexpansive.

It is known that the following relation holds.

The class of quasi-nonexpansive mappings $\subset$ the class of demicontractive mappings.

Then, is every demicontractive mapping quasi-nonexpansive? The answer is negative, we can see the following example.

Example 2.1. [22] Let $H = l_2$ and $T : l_2 \to l_2$ be defined by $Tx = -kx$, for $\forall\, x \in l_2$, where $k > 1$. Then $F(T) = \{0\}$ and T is a demicontractive mapping which is not quasi-nonexpansive.

In fact, it is obvious that T has unique fixed point $x = 0$. For each $x \in l_2$, we have

$$\|Tx - 0\|^2 = k^2\|x - 0\|^2,$$

which implies that T is not quasi-nonexpansive. Observe that

$$\|x - Tx\|^2 = \|x - (-kx)\|^2 = (1 + k)^2\|x - 0\|^2,$$

which implies that

$$\|x - 0\|^2 = \frac{1}{(1 + k)^2}\|x - Tx\|^2.$$

Thus

$$\|Tx - 0\|^2 = \|x - 0\|^2 + (k^2 - 1)\|x - 0\|^2 = \|x - 0\|^2 + \frac{k^2 - 1}{(k + 1)^2}\|x - Tx\|^2.$$

It follows from $\frac{k^2-1}{(k+1)^2} \in (0, 1)$ that T is a $\frac{k^2-1}{(k+1)^2}$-demicontractive mapping.

Definition 2.2. Let C be a nonempty closed convex subset of a real Hilbert space H. An operator $T : C \to C$ is said to be demiclosed at origin if $\{x_n\}$ converges weakly to x, and $\{Tx_n\}$ converges strongly to 0 for any sequence $\{x_n\}$, then $Tx = 0$.

For a quasi-nonexpansive mapping $T : C \to H$, is $I - T$ still demiclosed on C? The answer is negative even at 0. The following example in [21].

Example 2.2. Let $T : [0, 1] \to [0, 1]$ be a mapping defined by

$$Tx\ =\ \begin{cases} \frac{x}{5},\ \ x\ \in\ [0, \frac{1}{2}], \\ x \sin \pi x,\ \ x\ \in\ (\frac{1}{2}, 1]\ . \end{cases}$$

Then T is quasi-nonexpansive, but $I - T$ is not demiclosed at 0.

In fact, $F(T)=\{0\}$. For any $x \in [0,\frac{1}{2}]$, we have

$$|Tx-0| = \left|\frac{x}{5}-0\right| \leq |x-0|,$$

and for any $x \in (\frac{1}{2},1]$, we have

$$|Tx-0| = |x\sin\pi x-0| \leq |x-0|.$$

Thus T is quasi-nonexpansive. Taking $\{x_n\} \subset (\frac{1}{2},1]$ and $x_n \to \frac{1}{2}$ $(n\to\infty)$, we have

$$|(I-T)x_n| = |x_n[1-\sin\pi x_n]| \to 0 \ (n\to\infty).$$

But $T\frac{1}{2}=\frac{1}{10}\neq\frac{1}{2}$, i.e., $(I-T)\frac{1}{2}\neq 0$. Hence $I-T$ is not demiclosed at 0.

To prove our main results, we need the following lemmas.

Lemma 2.1. [14] *Let T be a $\mu-$demicontractive self-mapping on H with $F(T)\neq\emptyset$ and set $T_\alpha=(1-\alpha)I+\alpha T$ for $\alpha\in[0,1]$. Then, T_α is quasi-nonexpansive provided that $\alpha\in[0,1-\mu]$, and*

$$\|T_\alpha x-q\|^2 \leq \|x-q\|^2-\alpha(1-\mu-\alpha)\|x-Tx\|^2, \ \ x\in H, q\in F(T).$$

Indeed, let C be a nonempty closed convex subset of H, and H is replaced by C in Lemma 2.1, the conlusion of Lemma 2.1 still holds.

Lemma 2.2. [8] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T:C\to H$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C and $x\in H$ such that $x_n \rightharpoonup x$ and $Tx_n-x_n\to 0$ as $n\to+\infty$. Then $x\in Fix(T)$.*

Lemma 2.3. [13] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T:C\to C$ be a self-mapping on C. If T is a μ-demicontractive mapping (which is also called a $\mu-$quasi-strict pseudo-contraction in* [13]*), then the fixed point set $F(T)$ is closed and convex.*

Lemma 2.4. [9] *Assume that $\{S_n\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} S_{n+1} \leq (1-\lambda_n)S_n+\lambda_n\delta_n, \\ S_{n+1} \leq S_n-\eta_n+\mu_n, \end{cases}$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\mu_n\}$ are two real sequences such that

(i) $\Sigma_{n=1}^{\infty}\lambda_n=\infty$;

(ii) $\lim_{n\to\infty}\mu_n=0$;

(iii) $\lim_{l\to\infty}\eta_{n_k}=0$ implies $limsup_{k\to\infty}\delta_{n_k}\leq 0$ for any subsequence $\{n_k\}\subset\{n\}$.

Then $\lim_{n\to\infty}S_n=0$.

3. Modified inertial Halpern simultaneous algorithms

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $U_i:C\to C(1\leq i\leq s)$ be a τ_i-demicontractive mapping. Suppose that $I-U_i(1\leq i\leq s)$ is demiclosed at origin and $\bigcap_{i=1}^{s}Fix(U_i)\neq\emptyset$. Let $\{\alpha_i\}$ be a sequence in $[0,1]$, $(0\leq i\leq s)$ such that $\Sigma_{i=0}^{s}\alpha_n^i=1$. Given a point $u\in C$ and a sequence $\{v_n\}$ in $(0,1)$, the following conditions are assumed to be satisfied:*

(D_1) $\lim_{n\to\infty} v_n = 0$ *and* $\Sigma_{n=0}^{\infty} v_n = \infty$*;*
(D_2) $\lim_{n\to\infty} \frac{\delta_n}{v_n} \|x_n - x_{n-1}\| = 0$*;*
(D_3) $\liminf_{n\to\infty} \alpha_n^0 > \tau$*;*
(D_4) $\liminf_{n\to\infty} \alpha_n^i > 0 (1 \le i \le s)$*,*

where $\tau = max_{1\le i\le s}\tau_i$*. Let* $x_{-1}, x_0 \in C$ *be arbitrary. Define a sequence* $\{x_n\}$ *by the following algorithm:*

$$\begin{cases} u_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = \alpha_n^0 u_n + \Sigma_{i=1}^s \alpha_n^i U_i(u_n), \\ x_{n+1} = v_n u + (1 - v_n) y_n. \end{cases} \tag{3.1}$$

Then the sequence $\{x_n\}$ *defined by (3.1) converges strongly to* $p = P_{\cap_{i=1}^s Fix(U_i)} u$*.*

Proof. From Lemma 2.3, for any $i \in \{1, 2, \cdots n\}$, $Fix(U_i)$ is closed and convex. Then $\bigcap_{i=1}^s Fix(U_i)$ is closed and convex. Now, we show that $\{x_n\}$ is bounded. Indeed, taking $\omega_n^i = \dfrac{\alpha_n^i}{1-\alpha_n^0} (1 \le i \le s)$, we have $\Sigma_{i=1}^s \omega_n^i = 1$, for every $n \ge 0$,

$$\begin{aligned} \alpha_n^0 u_n + \Sigma_{i=1}^s \alpha_n^i U_i(u_n) &= \alpha_n^0 u_n + (1 - \alpha_n^0) \Sigma_i^s \omega_n^i U_i(u_n) \\ &= \Sigma_{i=1}^s \omega_n^i (\alpha_n^0 u_n + (1 - \alpha_n^0) U_i(u_n)). \end{aligned} \tag{3.2}$$

From (3.1), (3.2), the convexity of $\|\cdot\|^2$, Lemma 2.1 and the condition (D_3), we obtain

$$\begin{aligned} \| y_n - p \|^2 \le\ & \Sigma_{i=1}^s \omega_n^i \| \alpha_n^0 u_n + (1 - \alpha_n^0) U_i(u_n) - p \|^2 \\ \le\ & \Sigma_{i=1}^s \omega_n^i (\| u_n - p \|^2 - (1 - \alpha_n^0)(\alpha_n^0 - \tau) \| U_i(u_n) - u_n \|^2) \\ =\ & \| u_n - p \|^2 - (1 - \alpha_n^0)(\alpha_n^0 - \tau) \Sigma_{i=1}^s \omega_n^i \| U_i(u_n) - u_n \|^2 \\ =\ & \| u_n - p \|^2 - (\alpha_n^0 - \tau) \Sigma_{i=1}^s \alpha_n^i \| U_i(u_n) - u_n \|^2 \\ \le\ & \| u_n - p \|^2 \end{aligned} \tag{3.3}$$

for all sufficiently large n, and

$$\| u_n - p \| \le \| x_n - p \| + \delta_n \| x_n - x_{n-1} \| . \tag{3.4}$$

It follows from (3.3) and (3.4), we obtain

$$\begin{aligned} \| x_{n+1} - p \| \le\ & v_n \| u - p \| + (1 - v_n) \| y_n - p \| \\ \le\ & v_n \| u - p \| + (1 - v_n) \| u_n - p \| \\ \le\ & (1 - v_n) \| x_n - p \| + v_n (\| u - p \| + \frac{1 - v_n}{v_n} \delta_n \| x_n - x_{n-1} \|) \end{aligned} \tag{3.5}$$

for all sufficiently large n. By the condition (D_2), we let

$$M := 2 \max \left\{ \| u - p \|, \sup_{n\ge 0} \frac{1 - v_n}{v_n} \delta_n \| x_n - x_{n-1} \| \right\}.$$

It follows from (3.5) that

$$\begin{aligned} \| x_{n+1} - p \| &\leq (1-v_n) \| x_n - p \| + v_n M \\ &\leq \max\{\| x_n - p \|, M\} \\ &\vdots \\ &\leq \max\{\| x_0 - p \|, M\}. \end{aligned}$$

for all sufficiently large n. Hence $\{x_n\}$ is bounded. From conditions (D_1) and (D_2), we have $\lim_{n\to\infty} \delta_n \|x_n - x_{n-1}\| = 0$. This implies that $\{u_n\}$ is bounded. Furthermore, it follows from (3.3) that $\{y_n\}$ is also bounded. By making use of (3.1) and (C_2), we have

$$\begin{aligned} \| y_n - p \|^2 &= \| \alpha_n^0 (u_n - p) + (1-\alpha_n^0) \Sigma_{i=1}^s \omega_n^i (U_i(u_n) - p) \|^2 \\ &= \alpha_n^0 \| u_n - p \|^2 + (1-\alpha_n^0) \Sigma_{i=1}^s \omega_n^i \| U_i(u_n) - p \|^2 \\ &\quad - \alpha_n^0 (1-\alpha_n^0) \Sigma_{i=1}^s \omega_n^i \| U_i(u_n) - u_n \|^2 \\ &\leq \alpha_n^0 \| u_n - p \|^2 + (1-\alpha_n^0) \| u_n - p \|^2 \\ &\quad + (1-\alpha_n^0) \Sigma_{i=1}^s \omega_n^i \tau_i \| U_i(u_n) - u_n \|^2 \\ &\quad - \alpha_n^0 (1-\alpha_n^0) \Sigma_{i=1}^s \omega_n^i \| U_i(u_n) - u_n \|^2 \\ &\leq \| u_n - p \|^2 - (1-\alpha_n^0)(\alpha_n^0 - \tau) \Sigma_{i=1}^s \omega_n^i \| U_i(u_n) - u_n \|^2 \\ &= \| u_n - p \|^2 - (\alpha_n^0 - \tau) \Sigma_{i=1}^s \alpha_n^i \| U_i(u_n) - u_n \|^2 . \end{aligned} \tag{3.6}$$

Therefore, it follows from (3.3), (3.6), (C_1), and condition (D_3) that

$$\begin{aligned} \| x_{n+1} - p \|^2 &\leq (1-v_n)^2 \| y_n - p \|^2 + 2v_n \langle u - p, x_{n+1} - p \rangle. \\ &\leq (1-v_n) \| u_n - p \|^2 + 2v_n \langle u - p, x_{n+1} - p \rangle \\ &\quad - (\alpha_n^0 - \tau)(1-v_n) \Sigma_{i=1}^s \alpha_n^i \| U_i(u_n) - u_n \|^2 \\ &= (1-v_n) \| x_n + \delta_n (x_n - x_{n-1}) - p \|^2 + 2v_n \langle u - p, x_{n+1} - p \rangle \\ &\quad - (\alpha_n^0 - \tau)(1-v_n) \Sigma_{i=1}^s \alpha_n^i \| U_i(u_n) - u_n \|^2 \\ &= (1-v_n) \| x_n - p \|^2 + \delta_n^2 (1-v_n) \| x_n - x_{n-1} \|^2 \\ &\quad + 2\delta_n (1-v_n) \langle x_n - x_{n-1}, x_n - p \rangle + 2v_n \langle u - p, x_{n+1} - p \rangle \\ &\quad - (\alpha_n^0 - \tau)(1-v_n) \Sigma_{i=1}^s \alpha_n^i \| U_i(u_n) - u_n \|^2 \\ &\leq (1-v_n) \| x_n - p \|^2 + \delta_n^2 (1-v_n) \| x_n - x_{n-1} \|^2 \\ &\quad + 2\delta_n (1-v_n) \langle x_n - x_{n-1}, x_n - p \rangle + 2v_n \langle u - p, x_{n+1} - p \rangle, \end{aligned} \tag{3.7}$$

and then

$$\begin{aligned} \| x_{n+1} - p \|^2 &\leq \| x_n - p \|^2 - (\alpha_n^0 - \tau)(1-v_n) \Sigma_{i=1}^s \alpha_n^i \| U_i(u_n) - u_n \|^2 \\ &\quad + \delta_n^2 (1-v_n) \| x_n - x_{n-1} \|^2 + 2\delta_n (1-v_n) \langle x_n - x_{n-1}, x_n - p \rangle \\ &\quad + 2v_n \langle u - p, x_{n+1} - p \rangle \end{aligned} \tag{3.8}$$

for all sufficiently large n. Let $S_n = \| x_n - p \|^2$, $\pi_n = v_n \sigma_n$,

$$\sigma_n = \frac{\delta_n^2 (1-v_n)}{v_n} \| x_n - x_{n-1} \|^2 + \frac{2\delta_n (1-v_n)}{v_n} \langle x_n - x_{n-1}, x_n - p \rangle + 2\langle u - p, x_{n+1} - p \rangle,$$

and

$$\eta_n = (\alpha_n^0 - \tau)(1 - v_n)\Sigma_{i=1}^s \alpha_n^i \parallel U_i(u_n) - u_n \parallel^2 .$$

Using (3.7) and (3.8), we have

$$S_{n+1} \leq (1 - v_n)S_n + v_n\sigma_n, \text{ and } S_{n+1} \leq S_n - \eta_n + \pi_n.$$

From conditions (D_1) and (D_2), we obtain $\Sigma_{n=0}^{\infty} v_n = \infty$ and $\lim_{n\to\infty} \pi_n = 0$.

In order to use Lemma 2.4, it remains to show that $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \sigma_{n_k} \leq 0$ for any subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$. Indeed, let $\{\eta_{n_k}\}$ be a subsequence of $\{\eta_n\}$ such that $\lim_{k\to\infty} \eta_{n_k} = 0$, which implies that

$$\lim_{k\to\infty} \parallel U_i(u_{n_k}) - u_{n_k} \parallel = 0 \ (1 \leq i \leq s) \tag{3.9}$$

due to conditions (D_1), (D_3) and (D_4). From conditions (D_1) and (D_2), we have

$$\parallel u_{n_k} - x_{n_k} \parallel = \delta_{n_k} \parallel x_{n_k} - x_{n_{k-1}} \parallel \to 0 \tag{3.10}$$

as $k \to \infty$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup \bar{x}$ and

$$\limsup_{k\to\infty} \langle u - p, x_{n_k} - p\rangle = \lim_{j\to\infty} \left\langle u - p, x_{n_{k_j}} - p \right\rangle.$$

From (3.10), we have $u_{n_{k_j}} \rightharpoonup \bar{x}$. Since $I - U_i$ $(1 \leq i \leq s)$ is demiclosed at the origin, we get from (3.10) that $\bar{x} \in \bigcap_{i=1}^s Fix(U_i)$. Combining the property of projections and $p = P_{\bigcap_{i=1}^s Fix(U_i)} u$, we obtain

$$\limsup_{k\to\infty} \langle u - p, x_{n_k} - p\rangle = \lim_{j\to\infty} \left\langle u - p, x_{n_{k_j}} - p \right\rangle = \langle u - p, \bar{x} - p\rangle \leq 0. \tag{3.11}$$

On the other hand, by (3.9) we have

$$\begin{aligned} \|y_{n_k} - u_{n_k}\| &= \left\|\alpha_{n_k}^0 u_{n_k} + (1 - \alpha_{n_k}^0)\Sigma_{i=1}^s \omega_{n_k}^i U_i(u_{n_k}) - u_{n_k}\right\| \\ &\leq (1 - \alpha_{n_k}^0)\Sigma_{i=1}^s \omega_{n_k}^i \|U_i(u_{n_k}) - u_{n_k}\| \to 0, \end{aligned}$$

as $k \to \infty$. It follows that

$$\|y_{n_k} - x_{n_k}\| \leq \|y_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0.$$

Further, using condition (D_1), we obtain

$$\begin{aligned} \left\|x_{n_{k+1}} - x_{n_k}\right\| &= \left\|v_{n_k}(u - x_{n_k}) + (1 - v_{n_k})\left(y_{n_k} - x_{n_k}\right)\right\| \\ &\leq v_{n_k} \|u - x_{n_k}\| + (1 - v_{n_k}) \|y_{n_k} - x_{n_k}\| \to 0 \end{aligned} \tag{3.12}$$

as $k \to \infty$. From (3.11) and (3.12), we have

$$\begin{aligned} \limsup_{k\to\infty} \left\langle u - p, x_{n_{k+1}} - p\right\rangle &= \limsup_{k\to\infty} \left\langle u - p, x_{n_{k+1}} - x_{n_k} + x_{n_k} - p\right\rangle \\ &\leq \limsup_{k\to\infty} \|u - p\| \left\|x_{n_{k+1}} - x_{n_k}\right\| \\ &\quad + \limsup_{k\to\infty} \langle u - p, x_{n_k} - p\rangle \\ &\leq 0, \end{aligned}$$

i.e.,

$$\limsup_{k\to\infty} \left\langle u-p, x_{n_{k+1}}-p \right\rangle \leq 0.$$

This together with condition (D_2) and

$$\begin{cases} \limsup_{k\to\infty} \frac{\delta_{n_k}^2(1-v_{n_k})}{v_{n_k}} \left\| x_{n_k}-x_{n_k-1} \right\|^2 = 0, \\ \limsup_{k\to\infty} \frac{2\delta_{n_k}(1-v_{n_k})}{v_{n_k}} \left\langle x_{n_k}-x_{n_k-1},\, x_{n_k}-p \right\rangle = 0, \\ \limsup_{k\to\infty} 2\left\langle u-p,\, x_{n_k+1}-p \right\rangle \leq 0, \end{cases}$$

implies that $\lim_{k\to\infty}\sup\sigma_{n_k} \leq 0$. From Lemma 2.4, we observe that $\lim_{n\to\infty} S_n = 0$. Hence $x_n \to p$ as $n\to\infty$. This completes the proof. □

Let $U_i = U((1\leq i\leq s)$. From Theorem 3.1, we obtain the following result immediately.

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $U: C\to C$ be a $\tau-$demicontractive mapping. Suppose that $I-U$ is demiclosed at the origin and $Fix(U)\neq\emptyset$. Let $\{\alpha_n\}\subset[0,1]$. Given a point $u\in C$ and a sequence $\{v_n\}$ in $(0,1)$, the following conditions are satisfied:*

(D_1) $\lim_{n\to\infty} v_n = 0$ *and* $\Sigma_{n=0}^{\infty} v_n = \infty$*;*
(D_2) $\lim_{n\to\infty} \frac{\delta_n}{v_n}\|x_n - x_{n-1}\| = 0$*;*
(D_3) $\liminf_{n\to\infty} \alpha_n > \tau$*;*
(D_4) $\limsup_{n\to\infty} \alpha_n < 1$,

let $x_{-1}, x_0 \in C$ be arbitrary. Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} u_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = \alpha_n u_n + (1-\alpha_n)U(u_n), \\ x_{n+1} = v_n u + (1-v_n)y_n. \end{cases}$$

Then the sequence $\{x_n\}$ defined above converges strongly to $p = P_{Fix(U)}u$.

Since every quasi-nonexpansive mapping is 0-demicontractive, from Theorem 3.1 and Corollary 3.1, we have the following results if $\tau = 0$.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $U_i: C\to C(1\leq i\leq s)$ be a finite family of quasi-nonexpansive mappings. Suppose that $I-U_i(1\leq i\leq s)$ are demiclosed at the origin and $\bigcap_{i=1}^{s} Fix(U_i)\neq\emptyset$. Let $\{\alpha_i\}$ be a sequence in $[0,1]$, $(0\leq i\leq s)$ with $\Sigma_{i=0}^{s}\alpha_n^i = 1$. Given a point $u\in C$ and a sequence $\{v_n\}$ in $(0,1)$, the following conditions are assumed to be satisfied:*

(D_1) $\lim_{n\to\infty} v_n = 0$ *and* $\Sigma_{n=0}^{\infty} v_n = \infty$*;*
(D_2) $\lim_{n\to\infty} \frac{\delta_n}{v_n}\|x_n - x_{n-1}\| = 0$*;*
(D_3) $\liminf_{n\to\infty} \alpha_n^i > 0$ $(0\leq i\leq s)$.

Let $x_{-1}, x_0 \in C$ be arbitrary. Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} u_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = \alpha_n^0 u_n + \Sigma_{i=1}^{s}\alpha_n^i U_i(u_n), \\ x_{n+1} = v_n u + (1-v_n)y_n. \end{cases}$$

Then the sequence $\{x_n\}$ defined above converges strongly to $p = P_{\cap_{i=1}^{s} Fix(U_i)}u$.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $U : C \to C$ be a quasi-nonexpansive mapping. Suppose that $I - U$ is demiclosed at the origin and $Fix(U) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$. Given a point $u \in C$ and sequence $\{v_n\}$ in $(0,1)$, the following conditions are assumed to be satisfied:*

(D_1) $\lim_{n\to\infty} v_n = 0$ *and* $\Sigma_{n=0}^{\infty} v_n = \infty$;
(D_2) $\lim_{n\to\infty} \frac{\delta_n}{v_n} \|x_n - x_{n-1}\| = 0$;
(D_3) $\liminf_{n\to\infty} \alpha_n > 0$,
(D_4) $\limsup_{n\to\infty} \alpha_n < 1$.

Let $x_{-1}, x_0 \in C$ be arbitrary. Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} u_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = \alpha_n u_n + (1-\alpha_n)U(u_n), \\ x_{n+1} = v_n u + (1-v_n)y_n. \end{cases}$$

Then the sequence $\{x_n\}$ defined above converges strongly to $p = P_{Fix(U)}u$.

Since every nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive, from Corollary 3.2 and 3.3 we have the following results if $\tau = 0$.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $U_i : C \to C (1 \leq i \leq s)$ be a finite family nonexpansive mappings and $\bigcap_{i=1}^{s} Fix(U_i) \neq \emptyset$. Let $\{\alpha_i\} \subset (0,1)(0 \leq i \leq s)$ such that $\Sigma_{i=0}^{s} \alpha_n^i = 1$. Given a point $u \in C$ and a sequence $\{v_n\}$ in $(0,1)$, the following conditions are assumed to be satisfied:*

(D_1) $\lim_{n\to\infty} v_n = 0$ *and* $\Sigma_{n=0}^{\infty} v_n = \infty$;
(D_2) $\lim_{n\to\infty} \frac{\delta_n}{v_n} \|x_n - x_{n-1}\| = 0$;
(D_3) $\liminf_{n\to\infty} \alpha_n^i > 0$ $(0 \leq i \leq s)$.

Let $x_{-1}, x_0 \in C$ be arbitrary. Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} u_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = \alpha_n^0 u_n + \Sigma_{i=1}^{s} \alpha_n^i U_i(u_n), \\ x_{n+1} = v_n u + (1-v_n)y_n. \end{cases}$$

Then the sequence $\{x_n\}$ defined above converges strongly to $p = P_{\cap_{i=1}^{s} Fix(U_i)}u$.

Proof. From Lemma 2.2, we know that, for any $i \in \{1, 2, \cdots, s\}$, $I - U_i$ is demiclosed at the origin. Then we can get from Corollary 3.2 the desired conclusion immediately. □

Corollary 3.5. *Let C be a nonempty close convex subset of a real Hilbert space H and let $U : C \to C$ be a nonexpansive mapping and $Fix(U) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$. Given a point $u \in C$ and a sequence $\{v_n\}$ in $(0,1)$, the following conditions are assumed to be satisfied:*

(D_1) $\lim_{n\to\infty} v_n = 0$ *and* $\Sigma_{n=0}^{\infty} v_n = \infty$;
(D_2) $\lim_{n\to\infty} \frac{\delta_n}{v_n} \|x_n - x_{n-1}\| = 0$;
(D_3) $\liminf_{n\to\infty} \alpha_n > 0$;
(D_4) $\limsup_{n\to\infty} \alpha_n < 1$.

Let $x_{-1}, x_0 \in C$ be arbitrary. Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} u_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = \alpha_n u_n + (1-\alpha_n)U(u_n), \\ x_{n+1} = v_n u + (1-v_n)y_n. \end{cases}$$

Then the sequence $\{x_n\}$ defined above converges strongly to $p = P_{Fix(U)}u$.

Remark 3.1. Theorem 3.1 extends and improves [19, Theorem 1] in the following aspects: (a) one nonexpansive mapping is extended to a finite family of demicontractive mappings; (b) there is a skip in [19] on the condition of $\{\psi_n\}$, where $\{\psi_n\}$ is $\{\alpha_n\}$ in this paper. The condition of $\{\psi_n\}$ is improved as follows:

$$0 < \liminf_{n\to\infty} \psi_n \le \limsup_{n\to\infty} \psi_n < 1.$$

Remark 3.2. For a special choice, the parameter δ_n in (3.1) can be chosen as:

$$0 \le \delta_n \le \bar{\delta}_n, \qquad \bar{\delta}_n = \begin{cases} \min\left\{\frac{\xi_n}{\|x_n - x_{n-1}\|}, \ \frac{n-1}{n+\eta-1}\right\}, & if\ x_n \neq x_{n-1}, \\ \frac{n-1}{n+\eta-1}, & otherwise, \end{cases} \tag{3.13}$$

where $\eta \ge 3$ and $\{\xi_n\}$ is a positive sequence such that $\lim_{n\to\infty} \dfrac{\xi_n}{v_n} = 0$. This idea is from the recent inertial extrapolated step introduced in [1, 3].

Finally, in order to demonstrate the effectiveness, realization and convergence of the algorithm in Theorem 3.1, we consider the following example in $(\mathbb{R}, \|\cdot\|)$.

Example 3.1. Let $C = \mathbb{R}$ and $s = 3$. For any $i \in \{1,2,3\}$, let $U_i : \mathbb{R} \to \mathbb{R}$ be defined by $U_i x = -2ix$. Let the sequence $\{x_n\}$ be generated iteratively by (3.1), where $\tau = \frac{5}{7}$, the value of $\{\delta_n\}$ is chosen by (3.13) with $\xi_n = \frac{1}{n^2}$ and $\eta = 6, \alpha_n^0 = \frac{6}{7}, \alpha_n^1 = \alpha_n^2 = \alpha_n^3 = \frac{1}{21}$, $v_n = \frac{1}{n+1}$ for all $n \ge 0$. Then, the sequence $\{x_n\}$ converges strongly to 0.

It is easy to see that $\bigcap_{i=1}^3 F(U_i) = \{0\} \neq \varnothing$. Also U_i is a τ_i-demicontractive mapping, from Example 2.1, we obtain $\tau_i = \frac{2i-1}{2i+1}$. $I - U_i$ is demiclosed at 0, for each $i = 1,2,3$. Here, we prove the $I - U_i$, $(i = 1,2,3)$ is demiclosed at 0. In fact, since $\forall\ \{x_n\} \subset \mathbb{R}$ and $x_n \rightharpoonup x$ (i.e., $x_n \to x$) and $|(I-U_i)(x_n)| \to 0$ as $n \to \infty$, we have $(I-U_i)(x) = 0$. So $I - U_i$ $(i = 1,2,3)$ is demiclosed at 0. Then iteration (3.1) is reduced to the following: $x_{n+1} = \frac{1}{n+1}u + \frac{n}{n+1}y_n$, where $y_n = \frac{2}{7}x_n + \frac{2}{7}\delta_n(x_n - x_{n-1}), \forall n \ge 0$.

REFERENCES

[1] H. Attouch, J. Peypouquet, The rate of convergence of Nesterov's accelerated forward-backward method is actually faster than $1/k^2$, SIAM J. Optim. 26 (2016), 1824–1834.

[2] N.T. An, N.M. Nam, X. Qin, Solving k-center problems involving sets based on optimization techniques, J. Global Optim. 76 (2020), 189–209.

[3] A. Beck, M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, SIAM J. Imaging Sci. 2 (2009), 183–202.

[4] S.Y. Cho, Implicit extragradient-like method for fixed point problems and variational inclusion problems in a Banach space, Symmetry, 12 (2020), 998.

[5] S.Y. Cho, A convergence theorem for generalized mixed equilibrium problems and multivalued asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 21 (2020), 1017-1026.

[6] T.H. Cuong, J.C. Yao, N.D. Yen, Qualitative properties of the minimum sum-of-squares clustering problem, Optimization, 69 (2020), 2131-2154.
[7] J. Fan, L. Liu, X. Qin, A subgradient extragradient algorithm with inertial effects for solving strongly pseudomonotone variational inequalities, Optimization 69 (2020), 2199-2215.
[8] K. Goebel, W. A. Kirk, Topics In Metric Fixed Point Theory, Cambridge Studies In Advanced Mathematics, Cambridge University Press, Cambridge, 1990.
[9] S. He, C. Yang, Solving the variational inequality problem defined on intersection of finite level sets, Abstr. Appl. Anal. 2013 (2013), 942315.
[10] L. Liu, X. Qin, Strong convergence of an extragradient-like algorithm involving pseudo-monotone mappings, Numerical Algo. 83 (2020), 1577-1590.
[11] D. A. Lorenz, T. Pock, An inertial forward-backward algorithm for monotone inclusions, J. Math. Imaging Vis. 51 (2015), 311–325.
[12] P. E. Mainge, Convergence theorems for inertial KM-type algorithms, J. Comput. Appl. Math. 219 (2008), 223–236.
[13] G. Marino, H.-K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007), 336–346.
[14] A. Moudafi, The split common fixed-point problem for demi-contractive mappings, Inverse Probl. 26 (2010), 587–600.
[15] B. T. Polyak, Some methods of speeding up the convergence of iteration methods, Comput. Math. Math. Phys. 4 (1964), 1–17.
[16] X. Qin, N.T. An, Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets, Comput. Optim. Appl. 74 (2019), 821-850.
[17] X. Qin, A. Petrusel, J.C. Yao, CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces, J. Nonlinear Convex Anal. 19 (2018), 157-165.
[18] B. Tan, Z. Zhou, X. Qin, Accelerated projection-based forward-backward splitting algorithms for monotone inclusion problems, J. Appl. Anal. Comput. 10 (2020), 2184-2197.
[19] B. Tan, Z. Zhou, S. X. Li, Strong convergence of modified inertial Mann algorithms for nonexpansive mappings, Mathematics 8 (2020), 462.
[20] Y. C. Tang, J. G. Peng, L. W. Liu, A cyclic and simultaneous iterative algorithm for the multiple split common fixed point problem of demicontractive mappings, Bull. Korean Math. Soc. 51 (2014), 1527–1538.
[21] Y. Q. Wang, X. L. Fang, Viscosity approximation methods for the multiple-set split equality common fixed-point problems of demicontractive mappings, J. Nonlinear Sci. Appl. 10 (2017), 4254–4268.
[22] Y. Q. Wang, T. H. Kim, Simultaneous iterative algorithm for the split equality fixed-point problem of demicontractive mappings, J. Nonlinear Sci. Appl. 10 (2017), 154–165.
[23] H.K. Xu, Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces, Inverse Probl. 26 (2010), 105018.
[24] H. Zhou, X. Qin, Fixed Points of Nonlinear Operators. Iterative Methods, De Gruyter, Berlin, 2020.