APPROXIMATION OF COMMON SOLUTIONS FOR A FINITE FAMILY OF GENERALIZED DEMIMETRIC MAPPINGS AND MONOTONE INCLUSION PROBLEMS IN CAT(0) SPACES

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Abstract. In this paper, we introduce a modified Halpern-Mann algorithm and study the strong convergence of the algorithm for approximating common solution of a finite family of monotone inclusion problems and a finite family of generalized demimetric mappings in complete CAT(0) spaces. Some applications are also considered.

Keywords. Monotone operator; Fixed point; Zero point; Strong Convergence; Hadamard Space.

1. INTRODUCTION

The inclusion problem (IP) with a set-valued operator $A$ in a Hilbert space $H$ is consists of finding

$$x \in H \text{ such that } 0 \in Ax. \quad (1.1)$$

The solution set of problem (1.1) is denoted by $A^{-1}(0)$. This problem is closely related to many real-world problems, such as signal processing, medical imaging, and machine learning \cite{1,2,3,4,5} and the references therein.

In 1970, Martinet \cite{6} first studied solutions of problem (1.1) in Hilbert spaces. Later, Rockafellar \cite{7} further studied the inclusion problem by introducing the following iterative algorithm in a Hilbert space $H$

$$x_1 \in H, \quad x_n = J_{\lambda_n}(x_{n-1}), \quad \forall n \geq 1, \quad (1.2)$$

where \{\lambda_n\} is a sequence of positive real numbers and $J_{\lambda}$ is the resolvent of $A$ defined by $J_{\lambda} = (I + \lambda A)^{-1}$ for $\lambda > 0$, and $A$ is a maximal monotone operator in $H$. The algorithm is called the Proximal Point Algorithm (PPA). Rockafellar proved that the sequence \{x_n\} generated by (1.2) converges weakly to a solution of (1.1) provided $\lambda_n \geq \lambda > 0$ for each $n \geq 1$. The generalizations and modified versions of the proximal point algorithm in Hilbert were studied by many authors recently; see, e.g., \cite{8,9,10,11,12,13} and the references therein.

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On the other hand, by using the duality mapping theory introduced by Kakavandi and Amini [14], Khatibzadeh and Ranjbar [15] introduced and study solutions of problem (1.1) via the proximal point algorithm in complete CAT(0) space $\mathcal{X}$

$$x_1 \in \mathcal{X}, \quad x_n = J_{\lambda_n}^{A_n} x_{n-1}, \quad \forall n \geq 1,$$

(1.3)

where $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Recently, Ranjbar and Khatibzadeh [16] proposed the following Mann-type and Halpern-type proximal point algorithms in complete CAT(0) spaces for finding a solution of problem (1.1)

$$x_1 \in \mathcal{X}, \quad x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) J_{\lambda_n}^{A_n} x_n, \quad \forall n \geq 1,$$

(1.4)

and

$$u, x_1 \in \mathcal{X}, \quad x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) J_{\lambda_n}^{A_n} x_n, \quad \forall n \geq 1,$$

(1.5)

where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$. They obtained a $\Delta$-convergence result using the Mann-type proximal point algorithm and they also obtained a strong convergence result using the Halpern-type proximal point algorithm.

Let $\mathcal{X}$ be a metric space, and let $\mathcal{C}$ be a nonempty closed and convex subset of $\mathcal{X}$. A point $x \in \mathcal{C}$ is called a fixed point of a mapping $T : \mathcal{C} \to \mathcal{X}$ provided $Tx = x$. We denote by $\mathcal{F}(T)$ the set of fixed points of $T$. Recently, many authors studied fixed points of nonlinear operators in convex metric spaces; see, e.g., [17, 18, 19, 20] and the references therein.

Recently, Aremu et al. [21] and Ugwunnadi et al. [22] used the concept of quasilinearization to define new operators in CAT(0) spaces as follows.

**Definition 1.1.** Let $\mathcal{X}$ be a complete CAT(0) space, and let $\mathcal{C}$ be a nonempty closed and convex subset of $\mathcal{X}$. The mapping $T$ from $\mathcal{C}$ into $\mathcal{X}$ is said to be

(i) $k$-demimetric (see [21]) if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$ such that

$$\langle \overrightarrow{x}, xT \rangle \geq \frac{1-k}{2} d^2(x, Tx), \quad \text{for all} \quad x \in \mathcal{X} \quad \text{and} \quad p \in F(T).$$

(1.6)

(ii) $\theta$-generalized demimetric (see [22]) if $F(T) \neq \emptyset$ and there exists $\theta \in \mathbb{R}$ such that

$$d^2(x, Tx) \leq \theta \langle \overrightarrow{x}, xT \rangle$$

(1.7)

for all $x \in \mathcal{C}$ and $u \in F(T)$.

**Remark 1.1.** It is clear in Definition 1.1 that, for any $k \in (-\infty, 1)$, a $k$-demimetric mapping is $\frac{2}{1-k}$-generalized demimetric. Also, for $\theta > 0$, a $\theta$-generalized demimetric is $\left(1 - \frac{2}{\theta}\right)$-demimetric.

Motivated by the above results, in this paper, we study a modified Halpern-Mann type algorithm for approximating common solution of a finite family of monotone inclusion problems and a finite family of generalized demimetric mappings. We also obtain a strong convergence theorem in Hadamard spaces. Our results unify and compliments many results in the current literature.
2. Preliminaries

A geodesic path joining two elements \( x, y \) in a metric space \( X \) is an isometry \( c : [0, l] \rightarrow X \), where \( d(x, y) = l \) such that \( c(0) = x \) and \( c(l) = y \). The image of a geodesic path is called a geodesic segment. A metric space for which every two points can be joined by a geodesic segment is called a geodesic space. We say that a metric space \( X \) is uniquely geodesic if every two points of \( X \) are joined by only one geodesic segment (i.e., CAT(0) space). They denoted a pair \( (a, b) \) which is called the quasilinearization mapping. It is easy to verify that \( a \) is unique if \( b \) is unique. Let \( x \in X \), \( y \in X \) and \( x \neq y \), which they called a vector and defined a mapping \( \langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \rightarrow \mathbb{R} \) by

\[
\langle ab, cd \rangle = \frac{1}{2} (d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)), \quad (a, b, c, d) \in X, \tag{2.1}
\]

which is called the quasilinearization mapping. It is easy to verify that \( \langle ab, ab \rangle = d^2(a, b) \), \( \langle ba, cd \rangle = -\langle ab, cd \rangle \), \( \langle ab, cd \rangle = \langle ac, bd \rangle + \langle be, cd \rangle \) and \( \langle ab, cd \rangle = \langle cd, ab \rangle \) for all \( a, b, c, d, e \in X \). It has been established that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Swartz inequality (see [27]). Recall that the space \( X \) is said to satisfy the Cauchy-Swartz inequality if \( \langle ab, cd \rangle \leq d(a, b)d(c, d) \) for all \( a, b, c, d \in X \). Let \( X \) be a complete CAT(0) space, and let \( X^* \) be its dual space. A multivalued operator \( A : X \rightarrow 2^X \) with domain \( D(A) := \{ x \in X : Ax \neq \emptyset \} \) is monotone if and only if, for all \( x, y \in D(A) \), \( x^* \in Ax \), \( y^* \in Ay \),

\[
\langle x^* - y^*, xx^* \rangle \geq 0 \quad \text{(see [15])}.
\]

The resolvent of the operator \( A \) of order \( \lambda > 0 \) is the multivalued mapping \( J_\lambda^A : X \rightarrow 2^X \) defined in [15] as

\[
J_\lambda^A(x) := \{ z \in X | \frac{1}{\lambda} z \} \in Ax \}.
\]

The operator \( A \) satisfies the range condition if for every \( \lambda > 0 \), \( D(J_\lambda^A) = X \) (see [15]). For simplicity, we shall write \( J_\lambda \) for the resolvent of a monotone operator \( A \). Since our main contribution in this paper is on Hadamard spaces for monotone inclusion problems, it is worthwhile to provide a detailed proof of example of a monotone mapping in Hadamard spaces.

**Example 2.1.** [28] Let \( X = \mathbb{R}^2 \) be an \( \mathbb{R} \)-tree with the radical metric \( d_r \), where \( d_r(x, y) = d(x, y) \) if \( x \) and \( y \) are situated on the euclidean straight line passing through the origin and

\[
d_r(x, y) = d(x, 0) + d(y, 0) := ||x|| + ||y||,
\]

otherwise let \( r = (1, 0) \) and \( X = B \cup C, \) where

\[
B = \{ (h, 0) : h \in [0, 1] \} \quad \text{and} \quad C = \{ (h, k) : h + k = 1, \ h \in [0, 1] \}.
\]
Then, \((X,d_r)\) is an Hadamard space and \(X^*\), which is a space of element \(\overrightarrow{tab}\) such that

\[
\overrightarrow{tab} = \begin{cases} 
\{\overrightarrow{scd} : c,d \in B, s \in \mathbb{R}, t(||b|| - ||a||) = S(||d|| - ||c||) \} a,b \in B, \\
\{\overrightarrow{scd} : c,d \in C \in \{0\}, s \in \mathbb{R}, t(||b|| - ||a||) = s(||d|| - ||c||) \} a,b \in C \cup \{0\}, \\
\{\overrightarrow{tab}\}
\end{cases}
\]

is the dual space of \(X\) (see [29]). Now, defined \(A : X \to 2^{X^*}\) by

\[
Ax := \begin{cases} 
\{[0p]\}, & x \in B, \\
\{[0p],[0x]\}, & x \in C.
\end{cases}
\]

Then \(A\) is a multivalued monotone operator. To see this we consider the cases:

(I) If \(x,y \in B\), then \(Ax = Ay = \{[0p]\}\) and \(x^* = y^* = [0p]\). So, \(\langle x^* - y^*, \overrightarrow{xy} \rangle = 0 \geq 0\).

(ii) If \(x^* = y^* = [0p]\), then \(Ax = \{[0p]\}\) and \(Ay = \{[0p]\}\).

(iii) If \(x \in B, y \in C\). Then \(Ax = \{[0p]\}\) and \(Ay = \{[0p],[0y]\}\).

Let \(\{x_n\}\) be a bounded sequence in a complete CAT(0) space \(X\). For \(x \in X\), we set

\[
r(x, \{x_n\}) = \limsup_{n \to \infty} d(x,x_n).
\]

The asymptotic radius \(r(\{x_n\})\) of \(\{x_n\}\) is given by

\[
r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},
\]
Remark 2.1. (see [35]) If for all $u \in A$, it is well known that, in a CAT(0) space, $A = \bigtriangleup^-$ is the resolvent of a monotone operator $A : X \to 2^{X^*}$ of order $\lambda > 0$, then $d^2(u, J_A^\lambda x) + d^2(J_A^\lambda x, x) \leq d^2(u, x)$, for all $u \in A^{-1}(0)$ and $x \in D(J_A^\lambda)$.

Proof. Indeed, for any $u \in A^{-1}(0)$, $x \in D(J_A^\lambda)$ and $\lambda > 0$, we obtain from Theorem 2.1 (i) and (ii) that

$$d^2(J_A^\lambda x, u) \leq \langle J_A^\lambda x u, \overrightarrow{xu} \rangle = \frac{1}{2} \left( d^2(J_A^\lambda x, u) + d^2(u, x) - d^2(J_A^\lambda x, x) \right),$$

which implies

$$d^2(u, J_A^\lambda x) + d^2(J_A^\lambda x, x) \leq d^2(u, x).$$

□
Lemma 2.5. [36] Let \( \{x_n\} \) be a sequence in a complete CAT(0) space \( X \), and \( x \in X \). Then \( \{x_n\} \) is \( \Delta - \)convergent to \( x \) if and only if \( \limsup_{n \to \infty} (x_n, x) \leq 0 \) for all \( y \in X \).

Lemma 2.6. [19] Every bounded sequence in a complete CAT(0) spaces always has a convergent subsequence.

Lemma 2.7. [17] Let \( C \) be a nonempty, closed and convex subset of CAT(0) space \( X \). Let \( \{x_i : i = 1, 2, \ldots, N\} \) be in \( C \), and \( \alpha_1, \alpha_2, \ldots, \alpha_N \in (0, 1) \) such that \( \sum_{i=1}^{N} \alpha_i = 1 \). Then the following inequality hold:

(i) \( d(z, \oplus_{i=1}^{N} \alpha_i x_i) \leq \sum_{i=1}^{N} \alpha_i d(z, x_i) \) for all \( z \in C \).

(ii) \( d^2(z, \oplus_{i=1}^{N} \alpha_i x_i) \leq \sum_{i=1}^{N} \alpha_i d^2(z, x_i) - \sum_{i,j=1, i \neq j}^{N} \alpha_i \alpha_j d^2(x_i, x_j) \) for all \( z \in C \).

Lemma 2.8. [37] Let \( C \) be a nonempty, convex subset of CAT(0) space \( X \). Let \( \{u_i : i = 1, 2, \ldots, N\} \subset C \), and \( \alpha_1, \alpha_2, \ldots, \alpha_N \in (0, 1) \) such that \( \sum_{i=1}^{N} \alpha_i = 1 \). Then the following inequalities hold:

\[
\langle \bigoplus_{i=1}^{N} \alpha_i u_i, x \rangle \leq \sum_{i=1}^{N} \alpha_i \langle u_i, x \rangle + \frac{1}{2} \left( \sum_{i=1}^{N} \alpha_i d^2(u_i, x) - d^2(\bigoplus_{i=1}^{N} \alpha_i u_i, x) \right)
\leq \sum_{i=1}^{N} \alpha_i \langle u_i, x \rangle + \frac{1}{2} \sum_{i=1}^{N} \alpha_i d^2(u_i, x).
\] (2.4)

Lemma 2.9. [37] Let \( X \) be a CAT(0) space and let \( C \) a nonempty convex subset of \( X \). Assume that \( \{S_i\}_{i=1}^{N} : C \to X \) is a finite family of \( k_i \)-demimetric mapping with \( k_i \in (-\infty, 1) \) for each \( i \in \{1, 2, \ldots, N\} \) such that \( \bigcap_{i=1}^{N} F(S_i) \neq \emptyset \). Let \( \{\alpha_i\}_{i=1}^{N} \) be a positive sequence with \( \sum_{i=1}^{N} \alpha_i = 1 \). Then \( \bigoplus_{i=1}^{N} \alpha_i S_i : C \to X \) is a \( k \)-demimetric mapping if \( k := \max\{k_i : i = 1, 2, \ldots, N\} \leq 0 \) and \( F(\bigoplus_{i=1}^{N} \alpha_i S_i) = \bigcap_{i=1}^{N} F(S_i) \).

Definition 2.1. Let \( C \) be a nonempty closed and convex subset of a complete CAT(0) space \( X \). The metric projection \( P_C : X \to C \) is defined by

\[ u = P_C(x) \iff d(u, x) = \inf\{d(y, x) : y \in C\}, \text{ for all } x \in X. \]

Lemma 2.10. [27] Let \( C \) be a nonempty closed and convex subset of complete CAT(0) space \( X \). For any \( x \in X \) and \( u \in C \), \( u = P_C x \) if and only if

\[ \langle y - u, \bar{u} \rangle \geq 0. \]

Lemma 2.11. [22] Let \( C \) be a nonempty closed and convex subset of a CAT(0) space \( X \) and let \( T : C \to X \) be a \( \theta \)-generalized demimetric mapping with \( \theta \in \mathbb{R} \). Then, it is closed and convex.

Lemma 2.12. [22] Let \( C \) be a nonempty closed and convex subset of a CAT(0) space \( X \) and let \( T : C \to X \) be a \( \theta \)-generalized demimetric mapping. Then, for any \( \theta \in [0, \infty) \), and \( \alpha \in (0, 1) \), \( (1 - \alpha)I \oplus kT \) is \( \theta \)-generalized demimetric from \( C \) into \( X \).

Lemma 2.13. [21] Let \( X \) be a CAT(0) space, \( T : X \to X \) a \( k \)-demimetric mapping with \( k \in (-\infty, \lambda) \) with \( \lambda \in (0, 1) \) and \( F(T) \neq \emptyset \). Suppose that \( T_{\lambda} x := (1 - \lambda) \oplus \lambda T x \). Then \( T_{\lambda} \) is quasinonexpansive mapping and \( F(T_{\lambda}) = F(T) \).

Lemma 2.14. [34] Let \( X \) be a complete CAT(0) space. Then, for all \( u, x, y \in X \), the following inequality holds:

\[ d^2(x, u) \leq d^2(y, u) + 2 \langle y - x, \bar{x} \rangle. \]
Lemma 2.16. \cite{39} Let $X$ be a complete CAT(0) space. For all $u, x, y \in X$ and $\alpha \in [0, 1]$, let $z_1 = \alpha x \oplus (1 - \alpha)u$ and $z_2 = \alpha y \oplus (1 - \alpha)u$. Then

$$
\langle z_1, z_2 \rangle \leq \alpha \langle x, y \rangle.
$$

Lemma 2.17. \cite{40} If $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the following inequality:

$$
a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, n \geq 0,
$$

where, (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; (n $\geq 0$) and $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$. 

Lemma 2.17. \cite{40} If $\{a_n\}$ is a sequence of real numbers and there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$, then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied: $a_{m_k} \leq a_{m_{k+1}}$ and $a_k \leq a_{m_k+1}$. for all sufficiently large numbers $k \in \mathbb{N}$. In fact, $m_k = \max \{ j \leq k : a_j < a_{j+1} \}$.

3. Main results

Theorem 3.1. Let $X$ be a complete CAT(0) space with dual $X^*$ and let $C$ be a nonempty closed and convex subset of $X$. Let $\{T_i\}_{i=1}^N : C \to X$ be a finite family of $\theta_i$-generalized demimetric mapping and $\Delta$-demiclosed at 0 with $\theta_i \in (0, \infty)$ for each $i \in \{1, 2, ..., N\}$. Let $A_i : X \to 2^{X^*}$ $(i = 1, 2, ..., N)$ be multivalued monotone mappings which satisfy the range condition. Assume that $Y := \cap_{i=1}^N F(T_i) \cap (\cap_{i=1}^N A_i^{-1}(0)) \neq \emptyset$. Let $\{u_n\}$ be a sequence in $X$ such that $u_n \to u \in X$. Assume for $k \in (0, \gamma)$ with $\gamma \in (0, 1)$ and $\theta_k > 0$. For any $x_1 \in X$, let $\{x_n\}$ in $X$ be a sequence generated by

$$
\begin{align*}
y_n &= J^N_\lambda \circ J^{N-1}_\lambda \circ \cdots \circ J^2_\lambda \circ J^1_\lambda x_n, \\
z_n &= (1 - \gamma)y_n \oplus \gamma [\oplus_{i=1}^N \xi_i ((1 - k \oplus kT_i)y_n)], \\
x_{n+1} &= \alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n,
\end{align*}
$$

where $\lambda \in (0, \infty)$, $\{\alpha_n\}$, $\{\sigma\}$, $\{\beta_n\}$ and $\{\xi_i\}_{i=1}^N$ are sequences in $(0, 1)$ satisfying the following conditions

(i) $\lim \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\alpha_n + \beta_n + \sigma_n = 1$.

Then $\{x_n\}$ converges strongly to $x^* \in Y$.

Proof. Let $S_i = (1 - k \oplus kT_i)$ and $W_N = \bigoplus_{i=1}^N \xi_i S_i$. Then we can rewrite algorithm (3.1) as:

$$
\begin{align*}
y_n &= J^N_\lambda \circ J^{N-1}_\lambda \circ \cdots \circ J^2_\lambda \circ J^1_\lambda x_n, \\
z_n &= (1 - \gamma)y_n \oplus \gamma W_N y_n, \\
x_{n+1} &= \alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n,
\end{align*}
$$

since $T_i : C \to X$ is $\theta_i$-generalized demimetric, by Lemma 2.11, we have that $F(T_i)$ is closed and convex for each $i \in \{1, 2, \cdots, N\}$. Also, $J^i_\lambda$ is firmly nonexpansive by Theorem 2.1 and hence nonexpansive for each $i = 1, 2, ..., N$. Therefore $F(J^i_\lambda)$ is closed and convex for each $i = 1, 2, ..., N$. Hence, $\cap_{i=1}^N F(T_i) \cap (\cap_{i=1}^N A_i^{-1}(0))$ is nonempty closed and convex. Therefore, $F(\cap_{i=1}^N F(T_i) \cap (\cap_{i=1}^N A_i^{-1}(0))$ is well defined. Furthermore, $T_i$ is $\theta_i$-generalized demimetric with $\theta_i \in (0, \infty)$ for each $i \in \{1, 2, ..., N\}$. So, for any $k \in (0, \gamma)$, with $\gamma \in (0, 1)$, we find from
Lemma 2.12 that $S_i$ is $\theta k$-generalized demimetric for each $i$. By Remark 1.1, we have that
$S_i$ is $\left(1 - \frac{\theta k}{\lambda} \right)$-demimetric. We obtain from Lemma 2.9 that $W_N = \bigoplus_{i=1}^N S_i$ is demimetric. It follows by Lemma 2.13 that $V_N := (1 - \gamma) \oplus W_N$ is quasi-nonexpansive and $F(V_N) = F(W_N) = \bigcap_{i=1}^N F(S_i) = \bigcap_{i=1}^N F(T_i)$. Let $p \in Y$, $\Psi^N := J^N \circ J_{\lambda}^{N-1} \circ \cdots \circ J^2 \circ J_{\lambda}^1$, where $\Psi_0 = I$. Then by the definition of $(y_n)$ in (3.2) and Remark 2.1, we obtain
\[
\begin{align*}
   d^2(y_n, p) &\leq d^2(\Psi_{\lambda}^{N-1}x_n, p) - d^2(\Psi_{\lambda}^{N-1}x_n, y_n) \\
   &\leq d^2(\Psi_{\lambda}^{N-2}x_n, p) - d^2(\Psi_{\lambda}^{N-2}x_n, \Psi_{\lambda}^{N-1}x_n) - d^2(\Psi_{\lambda}^{N-1}x_n, y_n) \\
   &\leq d^2(\Psi_{\lambda}^{N-3}x_n, p) - d^2(\Psi_{\lambda}^{N-3}x_n, \Psi_{\lambda}^{N-2}x_n) - d^2(\Psi_{\lambda}^{N-2}x_n, \Psi_{\lambda}^{N-1}x_n) - d^2(\Psi_{\lambda}^{N-1}x_n, y_n) \\
   &\leq d^2(x_n, p) - \sum_{i=1}^N d^2(\Psi_{\lambda}^{i-1}x_n, \Psi_{\lambda}^ix_n).
   \end{align*}
\]
Using $(z_n)$ in (3.2), we get
\[
   d(z_n, p) \leq d(V_Ny_n, p) \leq d(y_n, p) \leq d(x_n, p),
\]
which together the definition of $(x_{n+1})$ implies that
\[
   \begin{align*}
   d(x_{n+1}, p) &= \alpha_n d(u_n, p) + \beta_n d(x_n, p) + \sigma_n d(z_n, p) \\
   &\leq \alpha_n d(u_n, p) + (\alpha_n + \sigma_n) d(x_n, p) \\
   &= (1 - \alpha_n) d(x_n, p) + \alpha_n d(u_n, p).
   \end{align*}
\]
Since $\{u_n\}$ is bounded, there exists $M > 0$ such that $\sup d(u_n, p) \leq M$. Letting $M^* = \max \{d(x_1, p), M\}$ for all $n \in \mathbb{N}$ implies that $d(x_1, p) \leq M^*$. Suppose that, for some $t \in \mathbb{N}$, $d(x_t, p) \leq M^*$, then
\[
   \begin{align*}
   d(x_{n+1}, p) &\leq (1 - \alpha_t) d(x_t, p) + \alpha_t d(x_t, p) \\
   &\leq (1 - \alpha_t) M^* + \alpha_t M^* = M^*.
   \end{align*}
\]
By induction, we obtain that $d(x_n, p) \leq M^*$ for all $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded. From Lemma 2.1(ii), we obtain
\[
   \begin{align*}
   d^2(x_{n+1}, p) &= d^2(\alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n, p) \\
   &\leq d^2(1 - \alpha_n) \left( \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n \right) \oplus \sigma_n z_n, p) \\
   &\leq (1 - \alpha_n) d^2 \left( \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n, p \right) + \sigma_n d^2(z_n, p) \\
   &\leq \alpha_n d^2(u_n, p) + \beta_n d^2(x_n, p) - \frac{\alpha_n\beta_n}{1 - \sigma_n} d^2(u_n, z_n) + \sigma_n d^2(y_n, p) \\
   &\leq \alpha_n d^2(u_n, p) + \beta_n d^2(x_n, p) + \sigma_n d^2(x_n, p) - \sigma_n \sum_{i=1}^N d^2(\Psi_{\lambda}^{i-1}x_n, \Psi_{\lambda}^ix_n) \\
   &\leq (1 - \alpha_n) d^2(x_n, p) + \alpha_n d^2(u_n, p) - \sigma_n \sum_{i=1}^N d^2(\Psi_{\lambda}^{i-1}x_n, \Psi_{\lambda}^ix_n).
   \end{align*}
\]
We divide the remaining proof in two cases.
**Case 1.** Assume that \( \{d(x_n, p)\}_{n=1}^{\infty} \) is a non-increasing sequence of a real numbers. Since \( \{d(x_n, p)\}_{n=1}^{\infty} \) is bounded, then its limit exists. With the fact that \( \alpha_n \to 0 \) as \( n \to \infty \) and \( \sigma_n > 0 \), (3.4) gives

\[
\sum_{i=1}^{N} d^2(\Psi_{\lambda}^{-1} x_n, \Psi_{\lambda} x_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p) + \alpha_n d^2(u_n, p),
\]

and

\[
\lim_{n \to \infty} \sum_{i=1}^{N} d^2(\Psi_{\lambda}^{-1} x_n, \Psi_{\lambda} x_n) = 0.
\]

Note that \( d^2(\Psi_{\lambda}^{-1} x_n, \Psi_{\lambda} x_n) \) is nonnegative for each \( i = 1, 2, \cdots, N \). Hence, for each \( i = 1, 2, \cdots, N \), we obtain

\[
\lim_{n \to \infty} d(\Psi_{\lambda}^{-1} x_n, \Psi_{\lambda} x_n) = 0.
\]

(3.5)

Using \((y_n)\) in (3.2) and (3.5), we get

\[
d(y_n, x_n) \leq \sum_{i=1}^{N} d(\Psi_{\lambda}^{-1} x_n, \Psi_{\lambda} x_n) \to 0, \text{ as } n \to \infty.
\]

(3.6)

It follows from (3.2) that

\[
d^2(x_{n+1}, p) = d^2(\alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n, p)
\]

\[
\leq d^2 \left[ (1 - \sigma_n) \left( \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n \right) \oplus \sigma_n z_n, p \right]
\]

\[
\leq (1 - \sigma_n) d^2 \left[ \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n \right] + \sigma_n d^2(z_n, p)
\]

\[
\leq \alpha_n d^2(u_n, p) + \beta_n d^2(x_n, p) - \frac{\alpha_n \beta_n}{1 - \sigma_n} d^2(u_n, x_n) + \sigma_n d^2(z_n, p)
\]

\[
\leq (1 - \alpha_n) d^2(x_n, p) + \alpha_n d^2(u_n, p) - \frac{\alpha_n \beta_n}{1 - \sigma_n} d^2(u_n, x_n),
\]

which implies that

\[
\frac{\alpha_n \beta_n}{1 - \sigma_n} d^2(u_n, x_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p) + \alpha_n d^2(x_n, p).
\]

Hence,

\[
\lim_{n \to \infty} d(u_n, x_n) = 0.
\]

(3.7)
Also, using Lemma 2.1(ii), we can get that
\[
d^2(x_{n+1}, p) \leq d^2((1 - \sigma_n) \left( \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n \right) \oplus \sigma_n z_n, p)
\]
\[
\leq (1 - \sigma_n)d^2 \left( \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n, p \right) + \sigma_n d^2(z_n, p)
\]
\[
- \sigma_n(1 - \sigma_n)d^2 \left( \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n, z_n \right)
\]
\[
\leq \alpha_n d^2(u_n, p) + \beta_n d^2(x_n, p) - \sigma_n(1 - \sigma_n)d^2 \left( \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n, z_n \right).
\]

Therefore,
\[
\sigma_n(1 - \sigma_n)d^2 \left( \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n, z_n \right) \leq d^2(x_n, p) - d^2(x_{n+1}, p) + \alpha_n d^2(u_n, p)
\]
and
\[
\lim_{n \to \infty} d \left( \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n, z_n \right) = 0. \quad (3.8)
\]

On the other hand, we obtain from (3.7) and (3.8) that
\[
d(z_n, x_n) \leq d \left( z_n, \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n \right) + d \left( \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n, x_n \right)
\]
\[
\leq d \left( z_n, \frac{\alpha_n}{1 - \sigma_n} u_n \oplus \frac{\beta_n}{1 - \sigma_n} x_n \right) + \frac{\alpha_n}{1 - \sigma_n} d(u_n, x_n).
\]
Hence,
\[
\lim_{n \to \infty} d(z_n, x_n) = 0. \quad (3.9)
\]
We obtain from (3.6) and (3.9) that
\[
d(y_n, z_n) \leq d(y_n, x_n) + d(x_n, z_n) \to 0, \quad \text{as} \quad n \to \infty. \quad (3.10)
\]
From (3.2), (3.8) and (3.9), we get
\[
d(x_{n+1}, x_n) \leq \alpha_n d(u_n, x_n) + \beta_n d(x_n, x_n) + \sigma_n d(z_n, x_n) \to 0, \quad n \to \infty. \quad (3.11)
\]
Furthermore, since $S_i$ is $k_i$-demimetric mapping for each $i \in \{1, 2, \ldots, N\}$ with $k = \max\{k_i\} \leq 0$, then

$$
\langle \overrightarrow{y_n}, \overrightarrow{y_n} \rangle = -\langle \overrightarrow{z_n}, \overrightarrow{y_n} \rangle \\
= -\langle ((1 - \gamma) y_n \oplus \gamma W_N y_n), y_n \overrightarrow{p} \rangle \\
\geq -\gamma \langle W_N y_n, y_n \overrightarrow{p} \rangle - \gamma \langle W_N y_n, y_n \overrightarrow{p} \rangle \\
\geq -\gamma \langle W_N y_n, y_n \overrightarrow{p} \rangle = -\gamma \left( \bigoplus_{i=1}^{N} \xi_i S_i y_n, y_n \overrightarrow{p} \right) \\
\geq -\gamma \left( \bigoplus_{i=1}^{N} \xi_i S_i y_n, y_n \overrightarrow{p} \right) \\
\geq -\gamma \sum_{i=1}^{N} \frac{1 - k_i}{2} \xi_i d^2(S_i y_n, y_n) \\
\geq \gamma \sum_{i=1}^{N} -\frac{k_i}{2} \xi_i d^2(S_i y_n, y_n) \\
\geq -\frac{k_i}{2} \gamma \sum_{i=1}^{N} \xi_i d^2(S_i y_n, y_n).
$$

Therefore

$$
-\frac{k_i}{2} \gamma \sum_{i=1}^{N} \xi_i d^2(S_i y_n, y_n) \leq \langle \overrightarrow{y_n}, \overrightarrow{y_n} \rangle \\
\leq d(y_n, z_n) d(y_n, p). \tag{3.12}
$$

Since $\{y_n\}$ is bounded, $k \leq 0$, and $\gamma, \xi_i \in (0, 1)$ for all $n \geq 1$ and $i \in \{1, 2, \ldots, N\}$, then we find from (3.10) and (3.12) that

$$
\lim_{n \to \infty} d(S_i y_n, y_n) = 0, \text{ for } i \in \{1, 2, \ldots, N\}. \tag{3.13}
$$

Now, since $\{x_n\}$ is bounded and $X$ is complete CAT(0) spaces, we conclude from Lemma 2.6 that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\Delta\text{-lim} x_{n_j} = v \in X$. By (3.6), we get $\Delta\text{-lim} x_{n_j} = v$. With (3.13) and the fact that $S_i$ is $\Delta$-demiclosed at 0, for each $i \in \{1, 2, \ldots, N\}$, we obtain that $v \in \bigcap_{i=1}^{N} F(S_i) = \bigcap_{i=1}^{N} F(T_i)$. Furthermore, $\Psi^i_\lambda$ is firmly nonexpansive, in particular, it is nonexpansive for each $i = 1, 2, \ldots, N$. Hence by (3.6), we obtain that $v \in \bigcap_{i=1}^{N} A_i^{-1}(0)$. Therefore, $v \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} A_i^{-1}(0) = Y$. Thus, from Lemma 2.5, we get

$$
\lim_{n \to \infty} \sup_{n \to \infty} \langle \overrightarrow{u_n}, \overrightarrow{x_n} \rangle \leq 0. \tag{3.14}
$$

Letting $w_n := \frac{\beta_n}{1 - \alpha_n} x_n \oplus \frac{\sigma_n}{1 - \alpha_n} z_n$, we have

$$
\langle \overrightarrow{u_n}, \overrightarrow{w_n} \rangle = \langle \overrightarrow{u_n}, \overrightarrow{x_n} \rangle + \langle \overrightarrow{u_n}, \overrightarrow{z_n} \rangle \\
\leq d(u_n, v) d(w_n, x_n) + d(u_n, w_n) + d(u_n, v) + d(u_n, v) + d(u_n, v) + \langle \overrightarrow{u_n}, \overrightarrow{z_n} \rangle.
$$
Therefore, in view of the fact that \( u_n \to u \) as \( n \to \infty \) with (3.9) and (3.14), we obtain
\[
\limsup_{n \to \infty} \langle \overrightarrow{u_n v}, \overrightarrow{w_n v} \rangle \leq 0. \tag{3.15}
\]
Also,
\[
d(w_n, v) = d \left( \frac{\beta_n}{1 - \alpha_n} x_n + \frac{\sigma_n}{1 - \alpha_n} z_n, v \right)
\leq \frac{\beta_n}{1 - \alpha_n} d(x_n, v) + \frac{\sigma_n}{1 - \alpha_n} d(z_n, v)
\leq \frac{\beta_n}{1 - \alpha_n} d(x_n, v) + \frac{\sigma_n}{1 - \alpha_n} d(x_n, v)
= d(x_n, v).
\]

Finally, we show that \( x_n \to v \). Using (3.2), and letting \( \bar{\partial}_n := \alpha_n v \oplus (1 - \alpha_n) z_n \) and \( x_{n+1} = \alpha_n u_n \oplus (1 - \alpha_n) w_n \), we conclude from Lemma 2.14 and Lemma 2.15 that
\[
d^2(x_{n+1}, v) \leq d^2(\bar{\partial}_n, v) + 2 \langle \overrightarrow{x_{n+1} \bar{\partial}_n}, \overrightarrow{x_{n+1} v} \rangle
\leq (1 - \alpha_n) d^2(w_n, v) + 2 \langle \overrightarrow{\bar{\partial}_n x_{n+1} v}, \overrightarrow{x_{n+1} v} \rangle
\leq (1 - \alpha_n) d^2(x_n, v) + 2 \alpha_n \langle \overrightarrow{u_n v}, \overrightarrow{w_n v} \rangle.
\]
Therefore
\[
d^2(x_{n+1}, v) \leq (1 - \alpha_n) d^2(x_n, v) + 2 \alpha_n \langle \overrightarrow{u_n v}, \overrightarrow{w_n v} \rangle. \tag{3.16}
\]
From (3.15), (3.16) and Lemma 2.16, we obtain \( d(x_n, v) \to 0 \) as \( n \to \infty \), that is, \( x_n \to v \) as \( n \to \infty \).

**Case 2.** Suppose that \( \{d(x_n, p)\}_{n=1}^{\infty} \) is a not monotone decreasing real sequence. Set \( \Upsilon_n := d(x_n, x^* ) \) for all \( n \geq 1 \). Then, there exists a subsequence \( \Upsilon_{n_k} \) of \( \Upsilon_n \) such that \( \Upsilon_{n_k} < \Upsilon_{n_{k+1}} \) for all \( k \geq 1 \). Now, define \( \tau : \mathbb{N} \to \mathbb{N} \) by
\[
\tau(n) = \max \{ k \leq n : \Upsilon_k < \Upsilon_{k+1} \}.
\]
It follows from Lemma 2.17 that \( \Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1} \). Using (3.4), we get
\[
\sigma_{\tau(n)} \sum_{i=1}^{N} d^2(\Psi_{\lambda}^{i-1} x_{\tau(n)}, \Psi_{\lambda}^{i} x_{\tau(n)}) \leq d^2(x_{\tau(n)}, v) - d^2(x_{\tau(n)+1}, v) + \alpha_{\tau(n)} d^2(u_{\tau(n)}, v).
\]
Now, \( \alpha_{\tau(n)} \to 0 \) as \( n \to \infty \) gives
\[
\lim_{n \to \infty} \sum_{i=1}^{N} d^2(\Psi_{\lambda}^{i-1} x_{\tau(n)}, \Psi_{\lambda}^{i} x_{\tau(n)}) = 0.
\]
Following an argument similar to the one in Case 1, we obtain
\[
\lim_{n \to \infty} d(y_{\tau(n)}, x_{\tau(n)}) = 0, \quad \lim_{n \to \infty} d(y_{\tau(n)}, z_{\tau(n)}) = 0, \quad \lim_{n \to \infty} d(u_{\tau(n)}, x_{\tau(n)}) = 0
\]
and
\[
\lim_{n \to \infty} d(x_{\tau(n)+1}, x_{\tau(n)}) = 0 = \lim_{n \to \infty} d(S y_{\tau(n)}, y_{\tau(n)}).
\tag{3.17}
\]
Following the similar argument of proof in Case 1, we get
\[
\langle \overrightarrow{u_{\tau n} p}, \overrightarrow{w_{\tau(n)} p} \rangle \leq 0.
\]
From (3.16), we have
\[ d^2(x_{\tau(n)+1}, p) \leq (1 - \alpha_{\tau(n)})d^2(x_{\tau(n)}, p) + 2\alpha_{\tau(n)}\langle \overrightarrow{u_{\tau(n)}p}, \overrightarrow{w_{\tau(n)}p} \rangle. \]

Since \(d^2(x_{\tau(n)}, p) < d^2(x_{\tau(n)+1}, p)\), then
\[ \alpha_{\tau(n)}d^2(x_{\tau(n)}, p) \leq d^2(x_{\tau(n)}, p) - d^2(x_{\tau(n)+1}, p) + 2\alpha_{\tau(n)}\langle \overrightarrow{u_{\tau(n)}p}, \overrightarrow{w_{\tau(n)}p} \rangle < 2\alpha_{\tau(n)}\langle \overrightarrow{u_{\tau(n)}p}, \overrightarrow{w_{\tau(n)}p} \rangle. \]

Using the fact that \(\alpha_{\tau(n)} > 0\), we obtain
\[ d^2(x_{\tau(n)}, p) < 2\langle \overrightarrow{u_{\tau(n)}p}, \overrightarrow{w_{\tau(n)}p} \rangle. \]

Since \(\limsup_{n \to \infty} \langle \overrightarrow{u_{\tau(n)}p}, \overrightarrow{w_{\tau(n)}p} \rangle \leq 0\), then
\[ \limsup_{n \to \infty} d^2(x_{\tau(n)}, p) \leq 0. \]

Hence, \(\lim_{n \to \infty} d(x_{\tau(n)}, p) = 0\). Since \(\lim_{n \to \infty} d(x_{\tau(n)+1}, x_{\tau(n)}) = 0\), then
\[ \lim_{n \to \infty} d(x_{\tau(n)}, p) = \lim_{n \to \infty} d(x_{\tau(n)+1}, p) = 0. \]

Therefore, by Lemma 2.17, we obtain \(d(x_n, p) \leq d(x_{\tau(n)+1}, p) \to 0\) as \(n \to \infty\). Hence \(x_n \to p\) as \(n \to \infty\).

Let \(X\) be complete CAT(0) space and let \(X^*\) be its dual space. Let \(f: X \to (-\infty, \infty]\) be a proper lower semicontinuous and convex function with domain \(\mathbb{D}(f) := \{x \in X : f(x) < +\infty\}\). Then the subdifferential of \(f\) is a set-valued function \(\partial f: X \to 2^{X^*}\) is defined by
\[ \partial f(x) = \begin{cases} \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, xz \rangle, \ (z \in X)\}, & \text{if } x \in \mathbb{D}(f), \\ \emptyset, & \text{otherwise} \end{cases} \]

It has been shown in [14] that

(1) \(\partial f\) is a monotone operator;
(2) \(\partial f\) satisfies the range condition. That is, \(\mathbb{D}(\lambda \partial f) = X\) for all \(\lambda > 0\);
(3) \(f\) attains its minimum at \(x \in X\) if and only if \(0 \in \partial f(x)\).

Now, we consider the following Minimization Problem (MP), which consists of finding \(x \in X\) such that
\[ f(x) = \min_{y \in X} f(y). \tag{3.18} \]

From Theorem 3.1, we obtain the following result.

**Corollary 3.1.** Let \(X\) be a complete CAT(0) space with dual \(X^*\) and let \(C\) be a nonempty closed and convex subset of \(X\). Let \(\{T_i\}_{i=1}^N : C \to X\) be a finite family of \(\theta_i\)-generalized demimetric mappings and \(\Delta\)-demiclosed at 0 with \(\theta_i \in (0, \infty)\) for each \(i \in \{1, 2, ..., N\}\). Let \(f_i: X \to (-\infty, \infty]\) \((i = 1, 2, ..., N)\) be a finite family of proper, lower semicontinuous and convex functions. Assume that \(Y := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N (\bigcap_{i=1}^N \partial f_i^{-1}(0)) \neq \emptyset\). Let \(\{u_n\}\) be a sequence in \(X\) such that \(u_n \to u \in X\).
Assume $k \in (0, \gamma)$ with $\gamma \in (0, 1)$ and $\theta_k > 0$. For any $x_1 \in X$, let \( \{x_n\} \) in $X$ be a sequence generated by

\[
\begin{align*}
  &y_n = J^\lambda_f \circ J^\lambda_f x_n - 1 \circ \cdots \circ J^\lambda_f x_n, \\
  &z_n = (1 - \gamma)y_n \oplus \gamma \left[ \bigoplus_{i=1}^N \xi_i ((1 - k) \oplus kT_i)y_n \right], \\
  &x_{n+1} = \alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n,
\end{align*}
\]

(3.19)

where $\lambda \in (0, \infty)$, $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\beta_n\}$ and $\{\xi_i\}_{i=1}^N$ are sequences in $(0, 1)$ satisfying the following conditions

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\alpha_n + \beta_n + \sigma_n = 1$.

Then $\{x_n\}$ converges strongly to $x^* \in Y$.

4. APPLICATIONS

In this section, using Theorem 3.1, we obtain new strong convergence theorems in complete CAT(0) space.

**Definition 4.1.** [41] Let $C$ be a nonempty subset of a CAT(0) space $X$. A mapping $T : C \to X$ is called a strict pseudo-contraction if there exists a constant $0 \leq \delta < 1$ such that

\[
d^2(Tx, Ty) \leq d^2(x, y) + 4\delta \left( \frac{1}{2} x \oplus \frac{1}{2} Ty, \frac{1}{2} Tx \oplus \frac{1}{2} y \right)
\]

(4.1)

for all $x, y \in C$. If (4.1) holds, we also say that $T$ is a $\delta$-strict pseudo-contraction.

The definition of pseudo-contractions finds its origin in Hilbert spaces. Note that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings. That is, $T$ is nonexpansive if and only if $T$ is a $0$-strict pseudo-contraction.

**Lemma 4.1.** [41] Let $C$ be a nonempty, closed and convex subset of a Hadamard space $X$ and let $T : C \to X$ be $\delta$-strict pseudocontraction. Define $T_\delta : C \to X$ by $T_\delta x = \delta x \oplus (1 - \delta)Tx$. Then $T_\delta$ is nonexpansive mapping and $F(T) = F(T_\delta)$.

**Lemma 4.2.** Let $C$ be a nonempty closed and convex subset of a Hadamard space $X$. Let $S : C \to C$ be a nonexpansive mapping and let $T : C \to C$ be a $\delta$-strict pseudo-contraction such that $F(S) \cap F(T) \neq \emptyset$. Let $W_\alpha = ((1 - \alpha)I \oplus \alpha T)Sx$, for any $x \in C$, where $0 < \alpha < \frac{1}{1 + \delta}$ and $\delta \in (0, 1)$. Then $F(W_\alpha) = F(S) \cap F(T)$. Furthermore, if $F(W_\alpha) \neq \emptyset$, then $W_\alpha$ is 2-generalized demimetric.

**Proof.** First, we show that $F(W_\alpha) = F(S) \cap F(T)$. It is easy to prove that $F(S) \cap F(T) \subseteq F(W_\alpha)$. Next, we show that $F(W_\alpha) \subseteq F(S) \cap F(T)$ for any $x \in F(W_\alpha)$ and $y \in F(S) \cap F(T)$

\[
\begin{align*}
  d^2(x, y) &= d^2((1 - \alpha)I \oplus \alpha T)Sx, y) \\
  &= (1 - \alpha)d^2(Sx, y) + \alpha d^2(TSx, y) - \alpha (1 - \alpha)d^2(Sx, TSx) \\
  &= d^2(Sx, y) - \alpha d^2(Sx, y) \\
  &\quad + \alpha d^2(TSx, y) - \alpha (1 - \alpha)d^2(Sx, TSx).
\end{align*}
\]

(4.2)
By the virtue of 0 and then
\[ d^2(TSx, y) \leq d^2(Sx, y) + 4\delta d^2\left(\frac{1}{2}Sx \oplus \frac{1}{2}y, \frac{1}{2}TSx \oplus \frac{1}{2}y\right) \]
\[ \leq d^2(Sx, y) + \delta[d^2(Sx, y) + d^2(TSx, y) + d^2(Sx, TSx) + d^2(y, TSx)] \]
\[ = d^2(Sx, y) + \delta d^2(Sx, y) + \delta d^2(TSx, y) + \delta d^2(Sx, TSx) - \delta d^2(Sx, y) - \delta d^2(y, TSx) \]
\[ = d^2(Sx, y) + \delta d^2(Sx, TSx). \]  

(4.3)

By (4.2) and (4.3), we obtain
\[ d(x, y) \leq d^2(Sx, y) + \alpha \delta d^2(Sx, TSx) - \alpha(1 - \alpha)d^2(Sx, TSx) \]
\[ \leq d^2(x, y) - \alpha(1 - \alpha(1 + \delta))d^2(Sx, TSx). \]

Hence,
\[ \alpha(1 - \alpha(1 + \delta))d^2(Sx, TSx) \leq 0. \]

By the virtue of 0 < \alpha < \frac{1}{1+\delta}, we have \alpha(1 - \alpha(1 + \delta)) > 0 and then
\[ TSx = Sx. \]  

(4.4)

By (4.4), we have
\[ d(x, Sx) = d(((1 - \alpha)I \oplus \alpha T)Sx, Sx) \]
\[ \leq (1 - \alpha)d(Sx, Sx) + \alpha d(TSx, Sx) \]
\[ \leq \alpha d(x, Sx), \]

which implies that
\[ (1 - \alpha)d(x, Sx) \leq 0. \]

Since \alpha \in (0, 1), we have
\[ d(x, Sx) = 0, \quad \implies \quad x = Sx. \]  

(4.5)

Therefore, we obtain that \( x \in F(S) \). From (4.4) and (4.5), we get \( x = Sx = TSx = Tx \), so \( x \in F(T) \). Hence \( x \in F(S) \cap F(T) \). So, \( F(W_\alpha) \subset F(S) \cap F(T) \) hold.

Next, we show that \( W_\alpha \) is 2-generalized demimetric. Let \( p \in F(W_\alpha) \). Then, \( p \in F(S) \cap F(T) \).

From Lemma 4.1, we obtain
\[ d^2(W_\alpha x, p) \leq d^2([(1 - \alpha)I \oplus \alpha T]Sx, p) \]
\[ \leq d^2(Sx, p) \]
\[ \leq d^2(x, p). \]  

(4.6)

Then, it follows (2.1) and (4.6) that
\[ 2\langle \overrightarrow{xW_\alpha x}, \overrightarrow{p}\rangle + d^2(x, p) + d^2(W_\alpha x, x) \leq d^2(x, p), \]
thus,
\[ d^2(W_\alpha x, x) \leq 2\langle \overrightarrow{xW_\alpha x}, \overrightarrow{p}\rangle. \]

Hence, \( W_\alpha \) is 2-generalized demimetric mapping. This complete the proof. \( \square \)
Lemma 4.3. [19] Let C be a closed and convex subset of a complete CAT(0) space X and let \( T : C \rightarrow X \) be a nonexpansive mapping. Let \( \{x_n\} \) be a bounded sequence in C such that \( \lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0 \) and \( \Delta - \lim_{n \rightarrow \infty} x_n = p \). Then \( p = Tp \).

With the help of Lemma 4.2, we obtain the following result.

Theorem 4.1. Let \( X \) be a complete CAT(0) space with dual \( X^* \) and let C be a nonempty closed and convex subset of X. Let \( S : C \rightarrow C \) be a nonexpansive mapping and let \( T : C \rightarrow C \) be a \( \delta \)-strict pseudo-contraction such that \( F(S) \cap F(T) \neq \emptyset \). For \( 0 < \alpha < \frac{1}{1+\delta} \), let \( W_\alpha := ((1-\alpha)I \oplus \alpha T)S \). Let \( A_i : X \rightarrow 2^{X^*} \ (i = 1, 2, \ldots, N) \) be multivalued monotone mappings, which satisfy the range condition. Assume that \( \mathcal{Y} := F(T) \cap F(S) \cap (\bigcap_{i=1}^N A_i^{-1}(0)) \neq \emptyset \). Let \( \{u_n\} \) be a sequence in X such that \( u_n \rightarrow u \in X \). For any \( x_1 \in X \), let \( \{x_n\} \) in X be a sequence generated by

\[
\begin{align*}
&\left\{ \begin{align*}
y_n &= J^{N}_\lambda \circ J^{N-1}_\lambda \circ \cdots \circ J^2_\lambda \circ J^1_\lambda x_n, \\
z_n &= (1-\gamma)y_n \oplus \gamma W_\alpha y_n, \\
x_{n+1} &= \alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n,
\end{align*} \right.
\end{align*}
\]

where \( \lambda \in (0, \infty) \), \( \gamma \in (0, 1) \) and \( \{\alpha_n\}, \{\beta_n\}, \{\sigma_n\} \) are sequences in \( (0, 1) \) satisfying the following conditions

(i) \( \lim_{n \rightarrow \infty} \alpha_n = 0 \) and \( \sum_{i=1}^{\infty} \alpha_n = \infty \);

(ii) \( \alpha_n + \beta_n + \sigma_n = 1 \).

Then \( \{x_n\} \) converges strongly to \( x^* \in \mathcal{Y} \).

Proof. Since \( T \) is \( \delta \)-strictly pseudo-contractive and S is nonexpansive. From Lemma 4.1, we have that \( T_\alpha := (1-\alpha)I \oplus \alpha T \) is nonexpansive. From Lemma 4.3, \( (1-\alpha)I \oplus \alpha T \) and S are \( \Delta \)-demicycloid at zero. If \( \lim_{n \rightarrow \infty} d(T_\alpha x_n, x_n) = 0 = \lim_{n \rightarrow \infty} d(Sx_n, x_n) \), then

\[
d(W_\alpha x_n, x_n) = d(((1-\alpha)I \oplus \alpha T)Sx_n, x_n) \\
\leq d(T_\alpha Sx_n, T_\alpha x_n) + d(T_\alpha x_n, x_n) \\
\leq d(Sx_n, x_n) + d(T_\alpha x_n, x_n).
\]

Hence \( \lim_{n \rightarrow \infty} d(W_\alpha x_n, x_n) = 0 \). Since \( F(T) \cap F(S) \neq \emptyset \), we find from Lemma 4.2 that \( W_\alpha \) is 2-generalized demimetric mapping. We obtain the desired Theorem 3.1.

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REFERENCES


