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BYRNE'S EXTENDED CQ-ALGORITHMS IN THE LIGHT OF MOREAU-YOSIDA REGULARIZATION

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Abstract. Given a real $m \times n$ matrix A, a maximal monotone operator $S : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ and a maximal monotone operator $T : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, the split feasibility null-point problem is to find $\bar{x}$ satisfying

$$\bar{x} \in S^{-1}(0) \text{ with } A\bar{x} \in T^{-1}(0).$$

Based on a regularization point of view of a Byrne's idea, by replacing the operators by their Yosida approximates, we propose to consider the problem of finding $\bar{x}$ such that

$$(P_{\alpha^{-1},(1-\alpha)^{-1}}) \qquad 0 = S_{\alpha^{-1}}(\bar{x}) + A^t T_{(1-\alpha)^{-1}}(A\bar{x}),$$

and then to introduce the following extended CQ-Algorithm

$$x_k = \frac{1}{1+\gamma\alpha}(I + \gamma\alpha J^T_{\alpha^{-1}(1+\alpha\gamma)})(I - \gamma(1-\alpha)A^t(I - J^S_{(1-\alpha)^{-1}})A)x_{k-1},$$

where $\alpha \in (0,1)$ and $0 < \gamma < \frac{2}{(1-\alpha)L}$ with $L = \rho(A^tA)$ being the largest eigenvalue of A^tA. In the context of split feasibility problems, this clearly reduces to Byrne's extended CQ-algorithm, namely,

$$x_k = \frac{1}{1+\gamma\alpha}(I + \gamma\alpha P_C)(I - \gamma(1-\alpha)A^t(I - P_Q)A)x_{k-1},$$

where P_C and P_Q are the orthogonal projections onto the nonempty closed convex sets C and Q, respectively.

Keywords. Maximal monotone operators; CQ-algorithms; Moreau-Yosida regularization.

1. INTRODUCTION

The split feasibility problem (SFP) is to find a point $\bar{x}$ with the property that $\bar{x} \in C$ and $A\bar{x} \in Q$, where C and Q are two nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2, respectively, and A is a bounded linear operator from H_1 to H_2. The split feasibility problem (SFP) in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems, which arise from phase retrievals and in medical image reconstruction. It has been found that the (SFP) can also be used to model the intensity-modulated radiation therapy; see, for example, [2, 3] and the references therein. Various algorithms have been designed to solve the (SFP); see, for example, [2, 4] and the references therein. A popular one is the celebrate CQ-algorithm due to Byrne [5], which is found to be a gradient-projection method (GPM) in convex minimization (it is also a special case of the proximal forward-backward splitting

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method). The CQ algorithm only involves the computations of the orthogonal projections P_C and P_Q onto the sets C and Q, respectively, and is therefore implementable in the case where P_C and P_Q have closed-form expressions (e.g., C and Q are the closed balls or half-spaces). It remains however a challenge how to implement the CQ algorithm in the case where the projections P_C and/or P_Q fail to have closed-form expressions though theoretically we can prove the convergence of this algorithm.

In this paper, we are concerned with a generalization of a very recent idea by Byrne. This will be done relying on a regularization technique and developed in the general context of split null-point problems. After introducing our nonlinear operator tools, we look at the CQ-algorithm from different but equivalent ways. Since the extended CQ-algorithm can be derived from an application of the forward-backward algorithm to a regularized version of the initial problem, we develop the idea in the more general context of maximal monotone operators. The intermediate case of split minimization problems can be then obtained by taking subdifferential of proper l.s.c. convex functions. It is worth mentioning that the analysis developed here is still valid in infinite dimensional Hilbert spaces. Having in mind the connection between monotone operators and monotone equilibrium functions, our analysis can be also applied to split equilibrium problems by means of the related maximal monotone operators.

2. Preliminaries: Byrne's Idea

Given a real m by n matrix A, a closed convex set $C \subset \mathbb{R}^n$ and a closed convex set $Q \subset \mathbb{R}^m$, the split feasibility problem is to find $\bar{x}$ verifying

$$(\text{SFP}) \qquad \bar{x} \in C \text{ with } A\bar{x} \in Q,$$

whereas the CQ-algorithm solves the more general problem

$$\min_{x \in C} \|(I - P_Q)(Ax)\|,$$

whenever a solution exists [2].

In a very recent short paper [6], to avoid the hard constraint that $\bar{x} \in C$ and treat the two constraints equally, Byrne [6] proposed to consider the following problem

$$(\text{SFP}_3) \qquad \min\Big(\frac{\alpha}{2}\|(I - P_C)(x)\|^2 + \frac{1-\alpha}{2}\|(I - P_Q)(Ax)\|^2\Big),$$

for some α in the interval $(0,1)$. Using the forward-backward splitting method, this leads to the extended CQ-algorithm, namely,

$$(\text{ECQ}) \quad x_k = \frac{1}{1+\gamma\alpha}(I + \gamma\alpha P_C)(I - \gamma(1-\alpha)A^t(I - P_Q))Ax_{k-1}),$$

with $0 < \gamma < \frac{2}{(1-\alpha)L}$, $L = \rho(A^tA)$ being the largest eigenvalue of A^tA. From Byrne [6], the (SFP) is equivalent to the following minimization problem:

$$(\text{SFP}_1) \qquad \min_{x \in \mathbb{R}^n} F(x); F(x) = f_1(x) + f_2(x) = i_C(x) + i_Q(Ax),$$

where i_C and i_Q are the indicator functions of C and Q, respectively. Whereas the more general problem $\min_{x \in C} \|(I - P_Q)(Ax)\|$ is equivalent to

$$(\text{SFP}_2) \qquad \min_{x \in \mathbb{R}^n} F(x); F(x) = f_1(x) + f_2(x) = i_C(x) + \frac{1}{2}\|(I - P_Q)(Ax)\|^2.$$

The iterative step of the CQ algorithm is given by

$$x_k = P_C(I-\gamma A^t(I-P_Q)A)x_{k-1}.$$

3. MAIN RESULTS: BEYOND SFP BY SMOOTHING

Now, recall that the forward-backward splitting method, which generates a sequence $(x_k)_{k\in\mathbb{N}}$ by

$$\text{(FBSM)} \qquad x_k = prox_{\gamma f_1}(x_{k-1}-\gamma\nabla f_2(x_{k-1})),$$

can be used when we need to minimize $F(x) = f_1(x)+f_2(x)$ with both functions convex, but only f_2 differentiable with a L-Lipschitz continuous gradient. It is well-known, for $0<\gamma<\frac{2}{L}$, that the sequence $(x_k)_{k\in\mathbb{N}}$ converges to a minimizer of the function F when this minimization problem is consistent.

The Moreau proximity function of f_1 is assumed to be computed efficiently. Remember that the Moreau's approximate of a given proper l.s.c convex function f is defined by

$$f_\gamma(x) = \inf_{y\in\mathbb{R}^n}\{f(y)+\frac{1}{2\gamma}\|y-x\|^2\},$$

and enjoys the nice property that it is a continuously differentable function. By noting that $(i_C)_\gamma(x) = \frac{1}{2\gamma}\|(I-P_C)(x)\|^2$, (SFP_2) is nothing but

$$\min_{x\in\mathbb{R}^n} F(x); F(x) = f_1(x)+f_2(x) = i_C(x)+(i_Q)_1(Ax),$$

and (SFP_2) is nothing else than

$$\min_{x\in\mathbb{R}^n} F(x); F(x) = f_1(x)+f_2(x) = (i_C)_{\frac{1}{\alpha}}(x)+(i_Q)_{\frac{1}{1-\alpha}}(Ax).$$

Using the forward-backward splitting method combined with a particular case of a formula in [7], namely,

$$prox_{\gamma f_\lambda}(x) = \frac{\lambda}{\lambda+\gamma}x+\frac{\gamma}{\lambda+\gamma}prox_{(\lambda+\gamma)f}(x),$$

we retrieve exactly the extended CQ-algorithm.

The advantage of this viewpoint is that Byrne's idea can be then extended to split minimization problems and more generally to split null-point problems. Before developing the idea in the context of maximal monotone operators, we would like to emphasize that since in (SFP_2) the two functions are differentiable with gradients $\nabla f_1(x) = \alpha(I-P_C))x$ and $\nabla f_2(x) = (1-\alpha)A^t(I-P_Q)A)x$, we can just use the classical gradient method, namely,

$$x_k = x_{k-1}-\gamma\big(\alpha(I-P_C)x_{k-1}+(1-\alpha)A^t(I-P_Q)Ax_{k-1}\big).$$

To draw a parallel with Byrne's, we first consider the problem of finding a zero of the sum of two maximal monotone operators T and S, namely, finding a solution to

$$(P) \qquad 0\in T(\bar{x})+S(\bar{x}).$$

Throughout this section, we will use some concepts which are of common use in the context of convex and nonlinear analysis (see, for example, Brézis [8]).

By replacing S by its Yosida approximate $S_\lambda = \frac{I-J_\lambda^S}{\lambda}$ with $J_\lambda^S = (I+\lambda S)^{-1}$ being its resolvent mapping, we obtain the partial regularized problem

$$(P_\lambda) \qquad 0\in T(\bar{x})+S_\lambda(\bar{x}).$$

A simple computation shows that this is equivalent to the following fixed-point formulation

$$\bar{x} = J_\lambda^T \circ J_\lambda^S(\bar{x}),$$

which leads to Passty's method [9], namely,

$$x_k = J_\lambda^T \circ J_\lambda^S x_{k-1}.$$

This provides a sequence $(x_k)_{k\in\mathbb{N}}$ that ergodic converges to a solution of problem (P), whenever there is a solution.

If one of the two operators is univoque, say S, then problem (P) can be rewritten as a fixed-point problem and can then be solved by the following forward-backward splitting algorithm

$$x_k = J_\gamma^T(x_{k-1} - \gamma S x_{k-1}).$$

Applied to (P_λ) gives

$$x_k = J_\gamma^T(x_{k-1} - \frac{\gamma}{\lambda}(I - J_\lambda^S)x_{k-1}),$$

and takes, in the context of the split feasibility feasibility null-point problems, the following form

$$x_k = J_\gamma^T(x_{k-1} - \frac{\gamma}{\lambda}A^t(I - J_\lambda^S)Ax_{k-1}),$$

where $\gamma \in]0, \frac{\lambda}{L}[$ with L being the spectral radius of the operator A^tA. An algorithm for the problem was introduced by Byrne, Censor, Gibali and Reich [10] in 2012.

On the other hand, if we regularize the two operators in problem (P), we obtain

$$(P_{\mu,\lambda}) \qquad 0 = T_\mu(\bar{x}) + S_\lambda(\bar{x}).$$

This can be again reformulated as a fixed-point problem, namely,

$$\bar{x} = \frac{\lambda}{\lambda+\mu}J_\mu^T(\bar{x}) + \frac{\mu}{\lambda+\mu}J_\lambda^S(\bar{x}),$$

which leads to the barycentric proximal algorithm introduced by Lehdili and Lemaire [11], namely,

$$x_k = \frac{\lambda}{\lambda+\mu}J_\mu^T(x_{k-1}) + \frac{\mu}{\lambda+\mu}J_\lambda^S(x_{k-1}).$$

The sequence $(x_k)_{k\in\mathbb{N}}$ ergodic converges to a solution of problem (P), whenever a solution exists.

By applying the forward-backward splitting method to $(P_{\mu,\lambda})$ combined with a formula in [7], namely,

$$J_\gamma^{T_\lambda}(x) = \frac{\lambda}{\lambda+\gamma}x + \frac{\gamma}{\lambda+\gamma}J_{\lambda+\gamma}^T(x),$$

we derive

$$x_k = \big(\frac{\mu}{\mu+\gamma}I + \frac{\gamma}{\mu+\gamma}J_{\mu+\gamma}^T\big)\big(I - \frac{\gamma}{\lambda}(I - J_\lambda^S)\big)x_{k-1},$$

which in the context of the split feasibility null-point problem with $\mu = \alpha^{-1}$ and $\lambda = (1-\alpha)^{-1}$, takes the following form

$$\text{(ECQ)} \quad x_k = \frac{1}{1+\gamma\alpha}(I + \gamma\alpha J_{\alpha^{-1}(1+\alpha\gamma)}^T)(I - \gamma(1-\alpha)A^t(I - J_{(1-\alpha)^{-1}}^S)A)x_{k-1}.$$

Observe that we retrieve the extended CQ-algorithm for the feasibility problem with T, S the normal cones to C and Q, respectively. $J_{\alpha^{-1}(1+\alpha\gamma)}^T$ and $J_{(1-\alpha)^{-1}}^S$ reduce in this case to P_C and

P_Q, respectively. Note also that in the intermediate case of split minimization problems, (ECQ) reduces to

$$x_k = \frac{1}{1+\gamma\alpha}(I+\gamma\alpha prox_{\alpha^{-1}(1+\alpha\gamma)f_1})(I-\gamma(1-\alpha)A^t(I-prox_{(1-\alpha)^{-1}f_2})A)x_{k-1}.$$

Now, if A satisfies $AA^t = \nu I$ for $\nu > 0$, following a formula in [12], we have

$$(R) \qquad J_1^{A^tTA}(x) = x - A^t T_\nu(Ax).$$

By interchanging the operators in $(P_{\alpha^{-1},(1-\alpha)^{-1}})$ and applying the (FBSM), we obtain (in this setting) the following extended CQ-algorithm

$$x_k = \big(I-\gamma(1-\alpha)A^t(I-J^T_{(1-\alpha)\nu+1})A\big)\big(I-\gamma\alpha(I-J^S_{\alpha^{-1}})\big)x_{k-1}.$$

In other words,

$$x_k = \big(I-(1-\alpha)(\gamma\nu I+\gamma A^t J^T_{(1-\alpha)\nu+1}A)\big)\big(I-\gamma\alpha(I-J^S_{\alpha^{-1}})\big)x_{k-1}.$$

This reduces, in the context of the convex split feasibility problems, to the following new extended CQ-algorithm

$$x_k = \big(I-(1-\alpha)(\gamma\nu I+\gamma A^t P_Q A)\big)\big(I-\gamma\alpha(I-P_C)\big)x_{k-1},$$

and takes the following form in the intermediate case of split minimization problems

$$x_k = \big(I-(1-\alpha)(\gamma\nu I+\gamma A^t prox_{((1-\alpha)\nu+1)f_1}A)\big)\big(I-\gamma\alpha(I-prox_{\alpha^{-1}f_2})\big)x_{k-1}.$$

Otherwise, some algorithms to compute the resolvent of composite operators were proposed, for instance, in [12].

The assumption $AA^t = \nu I$ is called the *semi-orthogonality* of A, from the viewpoint of the frame theory, such an A corresponds to a synthesis operator of a so-called *tight frame* with the frame constant ν. Note that as a consequence of the assumption, it automatically holds $m \leq n$ due to $m = rank(AA^t) = rank(A) \leq n$. Equation (R) drastically reduces the complexity of computations when the resolvent of the operator T (which reduces to the proximal mapping when T is the subdifferential of a function and to the projection operator when T is the normal cone of a convex set), has a closed-form expression or can be computed efficiently. Since Equation (ECQ) is then solved explicitly when compared with the general case. So, for example, the projection onto the set $\{x; \|Ax-x_0\|_2 \leq r\}$, when A is a synthesis operator of a tight frame, can be evaluate explicitly by

$$P_{\{\bar{x};\|A\bar{x}-x_0\|_2\leq r\}}(x) = x - A^t\big(I-P_{\{\bar{x};\|\bar{x}-x_0\|_2\leq r\}}\big)(Ax),$$

with a simple projection onto the ball $\{\bar{x}; \|\bar{x}-x_0\|_2 \leq r\}$. We refer to [13] for other examples when A is assumed to be surjective and the projection onto affine spaces is expressed by means of the pseudo-inverse operator with applications in audio processing.

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