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APPROXIMATIONS METHOD FOR FIXED POINTS OF FURTHER 2-GENERALIZED HYBRID MAPPINGS IN FLAT HADAMARD SPACES

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Abstract. In this paper, a hybrid iterative algorithm for approximating fixed points of noncommutative further 2-generalized hybrid mappings is proposed. We prove that the sequence generated by the proposed algorithm converges strongly to a common fixed point of such mappings in flat Hadamard spaces. The result established in this paper generalizes and improves some recently results announced in the literature.

Keywords. Further generalized hybrid mapping; Normally 2-generalized hybrid mapping; Flat Hadamard space; Further 2-generalized hybrid mapping.

1. Introduction

Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $T: C \to H$ be a nonlinear mapping. We denote the set of fixed points of T by $F(T)$, i.e., $F(T) = \{u \in C : Tu = u\}$. A mapping $T: C \to H$ is said to be (α, β)-generalized hybrid [1] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \ \forall\, x, y \in C.$$

Observe that the mapping T is reduced to a nonexpansive mapping if $\alpha = 1$ and $\beta = 0$, i.e., $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$. If $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, then T is said to be hybrid [2, 3], i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall\, x, y \in C.$$

Also, it is called nonspreading [4], i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall\, x, y \in C,$$

if $\alpha = 2$ and $\beta = 1$. Recall that a mapping T is called normally generalized hybrid [5] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

(a) $\alpha + \beta + \gamma + \delta \geq 0$; (b) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0, \ \forall\, x, y \in C.$$

The class of normally generalized hybrid mapping is a subclass of the generalized hybrid mapping. As a generalization of this class of normally generalized hybrid mapping, the classes of

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normally 2-generalized hybrid and further generalized hybrid mappings were introduced. A mapping T is called

(i) normally 2-generalized hybrid [6] if there exist $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3 \in \mathbb{R}$ such that (a) $\sum_{i=1}^{3}(\alpha_i+\beta_i) \geq 0$, (b) $\sum_{i=1}^{3}\alpha_i > 0$ and

$$\begin{aligned}\alpha_1\|T^2x-Ty\|^2+\alpha_2\|Tx-Ty\|^2+\alpha_3\|x-Ty\|^2\\ +\quad \beta_1\|T^2x-y\|^2+\beta_2\|Tx-y\|^2+\beta_3\|x-y\|^2 \leq 0,\ \forall\, x,y\in C.\end{aligned}$$

(ii) further generalized hybrid [7] if there exist $\alpha,\beta,\gamma,\delta,\varepsilon \in \mathbb{R}$ such that (a) $\alpha+\beta+\gamma+\delta \geq 0$, $\varepsilon \geq 0$ (b) $\alpha+\beta > 0$ or $\alpha+\gamma > 0$ and

$$\begin{aligned}\alpha\|Tx-Ty\|^2 \quad &+ \quad \beta\|x-Ty\|^2+\gamma\|Tx-y\|^2\\ &+ \quad \delta\|x-y\|^2+\varepsilon\|x-Tx\|^2 \leq 0,\ \forall\, x,y\in C.\end{aligned}$$

In 2019, Takahashi, Wen and Yao [8] proved strong convergence theorems via hybrid methods for two noncommutative normally 2-generalized hybrid mappings in Hilbert spaces. They proved that the sequence $\{x_n\} \subset C$ defined by

$$\begin{cases} x_1 = x \in C, \\ y_n = a_nx_n + b_n(\gamma_nS+(1-\gamma_n)T)x_n + c_n(\delta_nS^2+(1-\delta_n)T^2)x_n, \\ C_n = \{z\in C : \|y_n-z\| \leq \|x_n-z\|\}, \\ Q_n = \{z\in C : \langle x_n-z, x_1-x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x_1, \quad \forall n\in\mathbb{N}, \end{cases} . \tag{1.1}$$

converges strongly to $z_0 = P_{F(S)\cap F(T)}x$, where $P_{F(S)\cap F(T)}$ is the metric projection of H on $F(S)\cap F(T)$.

Let (X,d) be a metric space and $x,y\in X$. An isometry $c:[0,d(x,y)]\to X$ satisfying $c(0)=x$ and $c(d(x,y))=y$ is called a geodesic path joining x to y. A geodesic segment between x and y is the image of a geodesic path joining x to y, which is denoted by $[x,y]$ provided that it is unique. A geodesic space is a metric space (X,d) in which every two points of X are joined by a geodesic segment. It is said to be a uniquely geodesic space if every two points of X are joined by only one geodesic segment. Let X be a uniquely geodesic space, and let $(1-t)x\oplus ty$ denote the unique point z of the geodesic segment joining x to y for each $x,y\in X$ such that $d(z,x)=td(x,y)$ and $d(z,y)=(1-t)d(x,y)$. Set $[x,y] := \{(1-t)x\oplus ty : t\in[0,1]\}$. Then, a subset $C\subset X$ is said to be convex if $[x,y]\subset C$ for all $x,y\in C$.

A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic space (X,d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of points (the edges of Δ). A comparison triangle for $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\bar{\Delta}(x_1,x_2,x_3)=\Delta(\bar{x_1},\bar{x_2},\bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i},\bar{x_j}) = d(x_i.x_j)$ for all $i,j\in\{1,2,3\}$.

A geodesic space X is called a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom: Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle in $\mathbb{R}^2$. Then the triangle Δ is said to satisfy the $CAT(0)$ in equality if $d(x,y)\leq d_{\mathbb{R}^2}(\bar{x},\bar{y})$ for all $x,y\in\Delta$ and all comparison points $\bar{x},\bar{y}\in\bar{\Delta}$.

Lemma 1.1. [9] *Let X be a uniquely geodesic space. The following assertions are equivalent:*

(i) X is a CAT(0) space;

(ii) X satisfies the CAT(0)(CN) inequality, i.e., if x,y,z are points in X, and q is the midpoint of the segment $[y,z]$, then

$$d^2(x,q) \leq \frac{1}{2}d^2(x,y) + \frac{1}{2}d^2(x,z) - \frac{1}{4}d^2(y,z).$$

A complete CAT(0) space is called a Hadamard space. The examples of Hadamard spaces include Hilbert spaces, the Hilbert ball [10], Euclidean spaces $\mathbb{R}^n$, $\mathbb{R}$-trees, hyperbolic spaces [11] and any complete simply connected Riemanian manifold with non-positive sectional curvatures.

Definition 1.1. [12] A Hadamard space X is called flat if and only if, for each $x,y,z \in X$ and $t \in [0,1]$, $d^2((1-t)x \oplus ty, z) = (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y)$.

It is known [12] that, in a flat Hadamard space X, for each $x,y,z,w \in X$ and $t \in [0,1]$,

$$\langle \overrightarrow{xy}, \overrightarrow{x(tz \oplus (1-t)w)} \rangle = t\langle \overrightarrow{xy}, \overrightarrow{xz} \rangle + (1-t)\langle \overrightarrow{xy}, \overrightarrow{xw} \rangle. \tag{1.2}$$

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X and, for $x \in X$, $r(x,\{x_n\}) := \limsup_{n\to\infty} d(x,x_n)$, the asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}$, and the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}$. In a complete CAT(0) sapce, it is generally known that $A(\{x_n\})$ consists of exactly one point; see [13]. A sequence $\{x_n\}$ is said to be Δ-convergent to a point $x \in X$ if, for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $A(\{x_{n_k}\}) = \{x\}$. In this case, x is called Δ-limit of $\{x_n\}$ and it is written as $\Delta - \lim_{n\to\infty} x_n = x$.

In 2008, Berg and Nicolev [14] introduced the concept of the quasilinearisation in a complete CAT(0) space. They denoted a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and called it a vector. The quasilinearisation is a map $\langle .,. \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d))$$

for every $a,b,c,d \in X$. From the definition, it is easy to see that, for all $a,b,c,d,e \in X$, $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$. The space X is said to satisfy the Cauchy Schwartz inequality if, for all $a,b,c,d \in X$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d)$. It is known (see [14]) that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality.

In 2010, Kakavandi and Amini [15] introduced the concept of dual spaces in a complete CAT(0) space X as follow. Let $C(X,\mathbb{R})$ be the space of all continuous real-valued functions on X. Consider a map $\Theta : \mathbb{R} \times X \times X \to C(X,\mathbb{R})$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \ (t \in \mathbb{R}, a,b,x \in X).$$

The Cauchy-Schwartz inequality implies that $\Theta(t,a,b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t,a,b)) = |t|d(a,b)(t \in \mathbb{R}, a,b \in X)$, where the Lipschitz semi-norm $L(\phi)$ of any function $\phi : X \to \mathbb{R}$ is given by

$$L(\phi) = \sup\{\frac{\phi(x) - \phi(y)}{d(x,y)} : x,y \in X, x \neq y\}.$$

A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d)),$$

for $t,s \in \mathbb{R}$ and $a,b,c,d \in X$. In a complete CAT(0) space, it was shown [15] that $D((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle$, for all $x,y \in X$. Thus D induces an equivalence relation on $\mathbb{R}\times X\times X$ with equivalence class defined by

$$[t\overrightarrow{ab}] := \{s\overrightarrow{cd} : D((t,a,b),(s,c,d)) = 0\}.$$

The pair (X^*,D) is called the dual space of the metric space (X,d), where $X^* = \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R}\times X\times X\}$ and the function D on X^* is a metric.

In 2019, Cuntavepanit and Phuengrattana [16] studied the class of further generalized hybrid mappings [7] in Hadamard spaces. Very recently, Ali and Haruna [17] introduced a new class of mappings called further 2-generalized hybrid mappings in Hadamard spaces. This class of mappings contains further generalized hybrid and normally 2 generalized hybrid mappings as special cases.

Definition 1.2. Let X be a complete CAT(0) space and C be a nonempty subset of X. A mapping $T : C \to C$ is called further 2-generalized hybrid if there exist $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\varepsilon_1,\varepsilon_2 \in \mathbb{R}$ such that (i) $\sum_{i=1}^{3}(\alpha_i+\beta_i) \geq 0, \varepsilon_1,\varepsilon_2 \geq 0$, (ii) $\sum_{i=1}^{3}\alpha_i > 0$ and

$$\begin{aligned}(iii)\ &\alpha_1 d^2(T^2x,Ty)+\alpha_2 d^2(Tx,Ty)+\alpha_3 d^2(x,Ty)+\beta_1 d^2(T^2x,y)\\ +\quad &\beta_2 d^2(Tx,y)+\beta_3 d^2(x,y)+\varepsilon_1 d^2(x,T^2x)+\varepsilon_2 d^2(x,Tx) \leq 0.\end{aligned}$$

for all $x,y \in C$.

Remark 1.1. If $\alpha_1 = \beta_1 = \varepsilon_1 = 0$, then the mapping reduces to further generalized hybrid in the sense of [16]. Also, the mapping reduces to normally 2-generalized hybrid mapping if $\varepsilon_1 = \varepsilon_2 = 0$.

It is our purpose in this paper to construct a hybrid type iterative scheme that converges strongly to a common fixed point of further 2 generalized hybrid mappings in a flat Hadamard space. Our results improved and generalize the results in Takahashi, Wen and Yao [8] in the sense of the mapping and the space considered.

2. Preliminaries

Throughout this paper, the symbols " $\to$ " and " $\rightharpoonup$ " represent the strong and Δ-convergence, respectively. The following results play vital roles in establishing our main result.

Lemma 2.1. [13] *Let X be a CAT(0) space and $x,y \in X$, $t \in [0,1]$. Then*

(i) $d(z,tx\oplus(1-t)y) \leq td(z,x)+(1-t)d(z,y)$;
(ii) $d^2(z,tx\oplus(1-t)y) \leq td^2(z,x)+(1-t)d^2(z,y)-t(1-t)d^2(x,y)$.

The following Lemma can be deduced from Lemma 12 of [18] (see also [17]).

Lemma 2.2. *Let X be a complete CAT(0) space and $x,y \in X$, $t_i \in (0,1)$. Then*

$$d^2(z,\oplus_{i=0}^{n}t_ix_i) \leq \sum_{i=0}^{n} t_id^2(z,x_i) - t_it_jd^2(x_i,x_j),$$

where $i,j \in \{0,1,...,n\}$ and $\sum_{i=0}^{n}t_i = 1$

Lemma 2.3. [19] *Let X be a complete CAT(0) space. Then every bounded sequence in X has a Δ-convergence subsequence.*

Lemma 2.4. [20] *Let X be a complete CAT(0) space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if* $\limsup_{n\to\infty}\langle \overrightarrow{x_n x}, \overrightarrow{yx}\rangle \le 0$, $\forall y \in X$.

3. Main results

In this section, X is considered to be a flat Hadamard space. We then construct a hybrid type iterative scheme that converges strongly to a common fixed point of further 2 generalized hybrid mappings in the space.

Lemma 3.1. *Let C be a nonempty subset of a flat Hadamard space X, and let $T : C \to C$ be a further 2-generalized hybrid mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\Delta - \lim\limits_{n\to\infty} x_n = z$ and $d(x_n, Tx_n) \to 0$, $d(x_n, T^2 x_n) \to 0$ as $n \to \infty$. Then $z \in F(T)$.*

Proof. Let $\{x_n\} \subset C$ be a sequence such that $x_n \rightharpoonup z$, $(x_n - Tx_n) \to 0$ and $(x_n - T^2 x_n) \to 0$ as $n \to \infty$. Note that $T : C \to C$ is a generic 2-generalized Bregman nonspreading mapping. Following the similar techniques as in [17, Lemma 3.1], we get $d(z, Ty) \le d(z, y)$ for all $y \in C$. Setting $y = z$, we obtain $d(z, Tz) = 0$. Thus, $z \in F(T)$. This completes the proof. □

Lemma 3.2. *Let C be a nonempty subset of a flat Hadamard space X, and let $T : C \to C$ be a further 2-generalized hybrid mapping. If $F(T) \neq \emptyset$, then T is quasi-nonexpansive.*

Proof. Let T be a further 2-generalized hybrid mapping with $F(T) \neq \emptyset$. In particular, let $x \in F(T)$ such that $T^2 x = Tx = x$. Now, from the definition of T, we get

$$\begin{aligned} \alpha_1 d^2(x, Ty) \quad &+ \quad \alpha_2 d^2(x, Ty) + \alpha_3 d^2(x, Ty) + \beta_1 d^2(x, y) \\ &+ \quad \beta_2 d^2(x, y) + \beta_3 d^2(x, y) + \varepsilon_1 d^2(x, x) + \varepsilon_2 d^2(x, x) \le 0. \end{aligned}$$

for all $y \in C$. This implies

$$(\alpha_1 + \alpha_2 + \alpha_3) d^2(x, Ty) \le -(\beta_1 + \beta_2 + \beta_3) d^2(x, y),$$

for all $y \in C$. Since $\sum_{i=1}^{3} \alpha_i > 0$, we obtain

$$d^2(x, Ty) \le -\frac{(\beta_1 + \beta_2 + \beta_3)}{(\alpha_1 + \alpha_2 + \alpha_3)} d^2(x, y),$$

for all $y \in C$. From the fact that $\sum_{i=1}^{3} (\alpha_i + \beta_i) \ge 0$, we have

$$d^2(x, Ty) \le d^2(x, y).$$

Thus, $d(x, Ty) \le d(x, y)$ for all $y \in C$. Hence T is quasi-nonexpansive. This completes the prove. □

Theorem 3.1. *Let C be a nonempty, convex subset of a flat Hadamard space X which satisfies the $(\mathbb{S})$ property and the $(\bar{Q}_4)$ condition. Let $S, T : C \to C$ be further 2-generalized hybrid mappings such that $\mathscr{F} = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$*

$$\begin{cases} y_n = \alpha_n x_n \oplus \beta_n(\delta_n S x_n \oplus (1 - \delta_n) T x_n) \oplus \gamma_n(\lambda_n S^2 x_n \oplus (1 - \lambda_n) T^2 x_n)), \\ C_n = \{p \in C : d(p, y_n) \le d(p, x_n)\}, \\ Q_n = \{p \in C : \langle \overrightarrow{x_1 x_n}, \overrightarrow{x_n p}\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1), \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\lambda_n\} \subset [a, b] \subset (0, 1)$ and $\alpha_n + \beta_n + \gamma_n = 1$. Then $\{x_n\}$ converges strongly to $z = P_{\mathscr{F}}(x_1)$.

Proof. Note that $\mathscr{F}$ is closed and convex and hence $z = \mathrm{P}_{\mathscr{F}}(x_1)$ is well defined. From the definition of C_n, $d(p,y_n) \leq d(p,x_n)$ if and only if $-d^2(x_n,y_n) + \langle \overrightarrow{py_n}, \overrightarrow{x_ny_n} \rangle \leq 0$. Thus, it is easy to see that both C_n, Q_n and $C_n \cap Q_n$ are closed and convex. For the closedness, let $p_k \in C_n$ be such that $p_k \to p$ as $k \to \infty$. Then, we show $p \in C_n$. But

$$\begin{aligned} & -d^2(x_n,y_n) + \langle \overrightarrow{py_n}, \overrightarrow{x_ny_n} \rangle \\ &= -d^2(x_n,y_n) + \langle \overrightarrow{\lim_{k\to\infty} p_k y_n}, \overrightarrow{x_ny_n} \rangle \\ &= -d^2(x_n,y_n) + \lim_{k\to\infty} \langle \overrightarrow{p_ky_n}, \overrightarrow{x_ny_n} \rangle \\ &= \lim_{k\to\infty} (-d^2(x_n,y_n) + \langle \overrightarrow{p_ky_n}, \overrightarrow{x_ny_n} \rangle) \leq 0. \end{aligned}$$

Thus, C_n is closed. For the convexity, let $p_1, p_2 \in C_n$. Then, we show that $p = tp_1 \oplus (1-t)p_2 \in C_n$ for $t \in [0,1]$. Using equation (1.2), we get

$$\begin{aligned} & -d^2(x_n,y_n) + \langle \overrightarrow{py_n}, \overrightarrow{x_ny_n} \rangle \\ &= -d^2(x_n,y_n) + \langle \overrightarrow{(tp_1 \oplus (1-t)p_2)y_n}, \overrightarrow{x_ny_n} \rangle \\ &= -d^2(x_n,y_n) + t\langle \overrightarrow{p_1y_n}, \overrightarrow{x_ny_n} \rangle + (1-t)\langle \overrightarrow{p_2y_n}, \overrightarrow{x_ny_n} \rangle \leq 0. \end{aligned}$$

Thus, C_n is convex. Therefore, C_n is closed and convex. Following similar techniques, we see that Q_n is closed and convex. Hence $C_n \cap Q_n$ is closed and convex.

Let $u_n = \delta_n Sx_n \oplus (1-\delta_n)Tx_n$ and $v_n = \lambda_n S^2 x_n \oplus (1-\lambda_n)T^2 x_n$. Let $z \in \mathscr{F}$. Using Lemma 2.1, we get

$$\begin{aligned} d(z,u_n) &= d(z, \delta_n Sx_n \oplus (1-\delta_n)Tx_n) \\ &\leq \delta_n d(z,Sx_n) + (1-\delta_n)d(z,Tx_n) \\ &= d(z,x_n). \end{aligned}$$

Similarly,

$$\begin{aligned} d(z,v_n) &= d(z, \lambda_n S^2 x_n \oplus (1-\lambda_n)T^2 x_n) \\ &\leq \lambda_n d(z,S^2x_n) + (1-\lambda_n)d(z,T^2x_n) \\ &\leq \lambda_n d(z,Sx_n) + (1-\lambda_n)d(z,Tx_n) \\ &= d(z,x_n). \end{aligned}$$

It follows that

$$\begin{aligned} d(z,y_n) &= d(z, \alpha_n x_n \oplus \beta_n u_n \oplus \gamma_n v_n) \\ &\leq \alpha_n d(z,x_n) + \beta_n d(z,u_n) + \gamma_n d(z,v_n) \\ &\leq \alpha_n d(z,x_n) + \beta_n d(z,x_n) + \gamma_n d(z,x_n) \\ &= d(z,x_n). \end{aligned}$$

Therefore, $z \in C_n$ and hence $\mathscr{F} \in C_n$. We can now use induction to show that $\mathscr{F} \in Q_n \cap C_n$, $\forall n \in \mathbb{N}$. For $n = 1$, we see that $\mathscr{F} \subset C_1 \cap Q_1$ since $\mathscr{F} \subset C_1$ and $Q_1 = C$. Suppose, for some $k \geq 1$, $\mathscr{F} \subset C_k \cap Q_k$. Using $x_{k+1} = P_{C_k \cap Q_k} x_1$ and the property of metric projections, we have

$$\langle \overrightarrow{x_{k+1}x_1}, \overrightarrow{px_{k+1}} \rangle \geq 0$$

for all $p \in C_k \cap Q_k$. Since $\mathscr{F} \subset C_k \cap Q_k$, we have

$$\langle \overrightarrow{x_{k+1}x_1}, \overrightarrow{zx_{k+1}} \rangle \geq 0$$

for all $z \in \mathscr{F}$. This implies $\mathscr{F} \subset C_{k+1} \cap Q_{k+1}$ and therefore $\mathscr{F} \subset C_n \cap Q_n$ $\forall n \in \mathbb{N}$. Hence, the sequence $\{x_n\}$ is well defined.

We next show that the sequence $\{x_n\}$ is bounded. We know from the definition of Q_n that $x_n = P_{Q_n}x_1$. Thus, using the property of metric projection, we get

$$\begin{aligned} d(x_n, x_1) &= d(P_{Q_n}x_1, x_1) \\ &\leq d(z, x_1) - d(z, P_{Q_n}x_1) \\ &= d(z, x_1). \end{aligned}$$

Thus, the sequence $\{d(x_n, x_1)\}$ is bounded, so is $\{x_n\}$. In view of $x_n = P_{Q_n}x_1$ and $x_{n+1} \in Q_n$, we have

$$\begin{aligned} 0 &\leq \langle \overrightarrow{x_1x_n}, \overrightarrow{x_nx_{n+1}} \rangle \\ &= d^2(x_1, x_{n+1}) + d^2(x_n, x_n) - d^2(x_1, x_n) - d^2(x_n, x_{n+1}) \\ &\leq d^2(x_1, x_{n+1}) - d^2(x_1, x_n). \end{aligned} \tag{3.2}$$

This implies $d(x_1, x_n) \leq d(x_1, x_{n+1})$. Thus, $\{d(x_1, x_n)\}$ is monotone increasing. Since it is bounded, then $\lim_{n\to\infty} d(x_n, x_1)$ exists. From equation (3.2), we see that

$$d^2(x_n, x_{n+1}) \leq d^2(x_1, x_{n+1}) - d^2(x_1, x_n). \tag{3.3}$$

From $x_{n+1} \in C_n$, we obtain

$$d(x_{n+1}, y_n) \leq d(x_{n+1}, x_n). \tag{3.4}$$

From the property of metric distances, we get that

$$d(x_n, y_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n). \tag{3.5}$$

From the existence of $\lim_{n\to\infty} d(x_n, x_1)$, (3.3), (3.4) and (3.5), we get that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \ \lim_{n\to\infty} d(x_{n+1}, y_n) = 0, \ \lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.6}$$

On the other hand, using Lemma 2.2, we have

$$\begin{aligned} d^2(z, y_n) &= d^2(z, \alpha_n x_n \oplus \beta_n u_n \oplus \gamma_n v_n) \\ &\leq \alpha_n d^2(z, x_n) + \beta_n d^2(z, u_n) + \gamma_n d^2(z, v_n) - \alpha_n \beta_n d^2(x_n, u_n) \\ &\leq \alpha_n d^2(z, x_n) + \beta_n d^2(z, x_n) + \gamma_n d^2(z, x_n) - \alpha_n \beta_n d^2(x_n, u_n) \\ &= d^2(z, x_n) - \alpha_n \beta_n d^2(x_n, u_n). \end{aligned}$$

Thus,

$$d^2(z, y_n) \leq d^2(z, x_n) - \alpha_n \beta_n d^2(x_n, u_n). \tag{3.7}$$

Similarly, one has

$$d^2(z, y_n) \leq d^2(z, x_n) - \alpha_n \gamma_n d^2(x_n, v_n). \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\alpha_n \beta_n d^2(x_n, u_n) \leq d^2(z, x_n) - d^2(z, y_n) \tag{3.9}$$

and

$$\alpha_n\gamma_n d^2(x_n,v_n)\leq d^2(z,x_n)-d^2(z,y_n). \tag{3.10}$$

Using the quasilinearization definition, the Cauchy-Schwartz inequality, (3.6) and (3.9), we conclude that

$$\begin{aligned}\alpha_n\beta_n d^2(x_n,u_n) &\leq d^2(z,x_n)-d^2(z,y_n)\\ &\leq d^2(x_n,x_n)-d^2(x_n,y_n)+2\langle\overrightarrow{zx_n},\overrightarrow{y_nx_n}\rangle\\ &\leq 2\langle\overrightarrow{zx_n},\overrightarrow{y_nx_n}\rangle\\ &\leq 2d(z,x_n)d(y_n,x_n)\to 0 \ as\ n\to\infty.\end{aligned}$$

Similarly, it follows from (3.10) that

$$\alpha_n\gamma_n d^2(x_n,v_n)\leq 2d(z,x_n)d(y_n,x_n)\to 0 \ as\ n\to\infty.$$

Using the fact that $\alpha_n,\beta_n,\gamma_n\subset[a,b]\subset(0,1)$, we conclude that

$$d(x_n,u_n)\to 0,\ d(x_n,v_n)\to 0 \ as\ n\to\infty \tag{3.11}$$

and

$$\begin{aligned}d^2(z,u_n) &= d^2(z,\delta_nSx_n\oplus(1-\delta_n)Tx_n)\\ &\leq \delta_nd^2(z,Sx_n)+(1-\delta_n)d^2(z,Tx_n)-\delta_n(1-\delta_n)d^2(Sx_n,Tx_n)\\ &\leq d^2(z,x_n)-\delta_n(1-\delta_n)d^2(Sx_n,Tx_n).\end{aligned}$$

Thus,

$$\delta_n(1-\delta_n)d^2(Sx_n,Tx_n)\leq d^2(z,x_n)-d^2(z,u_n). \tag{3.12}$$

Similarly, one has

$$\lambda_n(1-\lambda_n)d^2(S^2x_n,T^2x_n)\leq d^2(z,x_n)-d^2(z,v_n). \tag{3.13}$$

Applying the quasilinearization property, the Cauchy-Schwartz inequality, (3.11), (3.12), we get

$$\begin{aligned}\delta_n(1-\delta_n)d^2(Sx_n,Tx_n) &\leq d^2(z,x_n)-d^2(z,u_n)\\ &\leq d^2(x_n,x_n)-d^2(x_n,u_n)+2\langle\overrightarrow{zx_n},\overrightarrow{u_nx_n}\rangle\\ &\leq 2\langle\overrightarrow{zx_n},\overrightarrow{u_nx_n}\rangle\\ &\leq 2d(z,x_n)d(u_n,x_n)\to 0 \ as\ n\to\infty.\end{aligned}$$

Similarly, one has

$$\lambda_n(1-\lambda_n)d^2(S^2x_n,T^2x_n)\leq 2d(z,x_n)d(v_n,x_n)\to 0 \ as\ n\to\infty.$$

Using the fact that $\delta_n,\lambda_n\subset[a,b]\subset(0,1)$, we conclude that

$$d(Sx_n,Tx_n)\to 0,\ d(S^2x_n,T^2x_n)\to 0 \ as\ n\to\infty. \tag{3.14}$$

Now,

$$\begin{aligned}d(x_n,Tx_n) &\leq d(x_n,u_n)+d(u_n,Tx_n)\\ &\leq d(x_n,u_n)+\delta_nd(Sx_n,Tx_n)+(1-\delta_n)d(Tx_n,Tx_n) \qquad (3.15)\end{aligned}$$

and

$$\begin{aligned} d(x_n,T^2x_n) &\leq d(x_n,v_n)+d(v_n,T^2x_n) \\ &\leq d(x_n,v_n)+\lambda_n d(S^2x_n,T^2x_n)+(1-\lambda_n)d(T^2x_n,T^2x_n). \end{aligned} \tag{3.16}$$

Using (3.14), (3.15) and (3.16), we get

$$\lim_{n\to\infty} d(x_n,Tx_n)=0 \textit{ and } \lim_{n\to\infty} d(x_n,T^2x_n)=0. \tag{3.17}$$

But

$$d(x_n,Sx_n)\leq d(x_n,Tx_n)+d(Tx_n,Sx_n) \tag{3.18}$$

and

$$d(x_n,S^2x_n)\leq d(x_n,T^2x_n)+d(T^2x_n,S^2x_n). \tag{3.19}$$

Using (3.14) and (3.17) in (3.18) and (3.19), we get

$$\lim_{n\to\infty} d(x_n,Sx_n)=0 \textit{ and } \lim_{n\to\infty} d(x_n,S^2x_n)=0. \tag{3.20}$$

Since X is a complete CAT(0) space and the sequence $\{x_n\}$ is bounded, we conclude from Lemma 2.3 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta-\lim_{k\to\infty}x_{n_k}=u$. Using Lemma 3.1, (3.17) and (3.20), we conclude that $u\in\mathscr{F}$.

Now, let $v=P_{\mathscr{F}}x_1$. Since $x_{n+1}=P_{C_n\cap Q_n}x_1$ and $v\in C_n\cap Q_n$, then $d(x_{n+1},x_1)\leq d(v,x_1)$. Also, $x_{n_k}\rightharpoonup u$ and $d(.,.)$ is convex and lower semicontinuous (hence, Δ-lower semicontinuous), we get

$$d(u,x_1)\leq \liminf_{k\to\infty} d(x_{n_k},x_1)\leq d(v,x_1).$$

From the definition of v, we can conclude that $u=v$. So, $x_n\rightharpoonup v$. Using the definition of the quasilinearization and Lemma 2.4, we get

$$\begin{aligned} \limsup_{n\to\infty} d^2(x_n,v) &= \limsup_{n\to\infty}(d^2(v,x_1)+d^2(x_n,x_1)-2\langle\overrightarrow{vx_1},\overrightarrow{x_nx_1}\rangle) \\ &\leq \limsup_{n\to\infty}(d^2(v,x_1)+d^2(v,x_1)-2\langle\overrightarrow{vx_1},\overrightarrow{x_nx_1}\rangle) \\ &= 2\limsup_{n\to\infty}(d^2(v,x_1)-\langle\overrightarrow{vx_1},\overrightarrow{x_nx_1}\rangle) \\ &= 2\limsup_{n\to\infty}(\langle\overrightarrow{vx_1},\overrightarrow{vx_1}\rangle+\langle\overrightarrow{vx_1},\overrightarrow{x_1x_n}\rangle) \\ &= 2\limsup_{n\to\infty}\langle\overrightarrow{vx_1},\overrightarrow{vx_n}\rangle\leq 0. \end{aligned}$$

Thus $\lim_{n\to\infty} d^2(x_n,v)=0$. Therefore, $x_n\to v$. This completes the proof. □

In view of Remark 1.1, we have the following results as corollaries of Theorem 3.1.

Corollary 3.1. *Let C be a nonempty, convex subset of a flat Hadamard space X which satisfies the $(\mathbb{S})$ property and the $(\bar{Q}_4)$ condition. Let $S,T:C\to C$ be normally 2-generalized hybrid mappings such that $\mathscr{F}=F(S)\cap F(T)\neq\emptyset$. Let $\{x_n\}$ be a sequence defined by (3.1). Then $\{x_n\}$ converges strongly to $z=P_{\mathscr{F}}(x_1)$.*

Proof. Since further 2-generalized hybrid mapping is reduced to normally 2-generalized hybrid mappings if $\varepsilon_1=\varepsilon_2=0$. It follows from Theorem 3.1 that $\{x_n\}$ converges strongly to $z=\mathrm{P}_{\mathscr{F}}(x_1)$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty, convex subset of a flat Hadamard space X which satisfies the $(\mathbb{S})$ property and the $(\bar{Q}_4)$ condition. Let $S,T : C \to C$ be further generalized hybrid mappings such that $\mathscr{F} = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (3.1). Then $\{x_n\}$ converges strongly to $z = P_{\mathscr{F}}(x_1)$.*

Proof. If $\alpha_1 = \beta_1 = \varepsilon_1 = 0$, i.e., a further 2-generalized hybrid mapping is reduced to a further generalized hybrid mapping, then we see from Theorem 3.1 that $\{x_n\}$ converges strongly to $z = \mathrm{P}_{\mathscr{F}}(x_1)$. This completes the proof. □

In view of the fact that Hilbert spaces are Hadamard spaces, the following result can easily be deduced from Corollary 3.1.

Corollary 3.3. [8, Theorem 3.1] *Let C be a nonempty, closed and convex subset of a Hilbert space X. Let $S,T : C \to C$ be normally 2-generalized hybrid mappings such that $\mathscr{F} = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_n x_n + \beta_n(\delta_n S + (1-\delta_n)T)x_n + \gamma_n(\lambda_n S^2 + (1-\lambda_n)T^2)x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\lambda_n\} \subset [a,b] \subset (0,1)$ and $\alpha_n + \beta_n + \gamma_n = 1$. Then $\{x_n\}$ converges strongly to $z = P_{\mathscr{F}}(x_1)$.

Proof. Since the normally 2-generalized hybrid mapping defined in Hadamard spaces reduces to the one introduced by Kondo and Takahashi [6] in Hilbert spaces, we obtain from Corollary 3.1 the desired conclusion immediately. □

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