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TWO METHODS FOR SOLVING SPLIT COMMON FIXED POINT PROBLEMS OF STRICT PSEUDO-CONTRACTVE MAPPINGS IN HILBERT SPACES WITH APPLICATIONS

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Abstract. In this paper, we construct Halpern-viscosity and hybrid-projection algorithms. The sequences generated by these algorithms are proved to converge to a common fixed point of two finite families of strictly pseudo-contractive mappings, respectively. Applications of the main convergence theorems are considered. Furthermore, numerical examples are given to show the efficiency of these algorithms.

Keywords. Halpern algorithm; Hybrid algorithm; Haugazeau's approach; Strict pseudo-contraction; Viscosity algorithm.

1. INTRODUCTION

Let C and Q be nonempty, closed, convex, subsets of real Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* (SFP) is given by the following:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q,$$

where $A : H_1 \to H_2$ is a bounded linear map with the adjoint operator $A^* : H_2 \to H_1$. This problem was introduced by Censor and Elfving [1] for the modeling of inverse problems stemming from phase retrievals, image processing and intensity modulated radiation therapy (IMRT) (see, e.g., [2, 3, 4]). The SFP has also been successfully applied in other areas such as immaterial science, computerized tomography, antenna design, sensor networks, data denoising and data compression (see, e.g., [5, 6, 7]). Several iterative algorithms have been introduced for solving SFP, the most notable among them is the CQ method introduced by Byrne [3, 4]. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two mappings with fixed point sets $F(T)$ and $F(S)$, respectively. Then, the *split common fixed point problem* (SCFPP) is to:

$$\text{find } x^* \in F(T) \text{ such that } Ax^* \in F(S), \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear map. This problem originated from Censor and Segal [8] and was studied by employing the following algorithm:

$$x_{n+1} = T(x_n - \tau_n A^*(I-S)Ax_n), \quad n \geq 1,$$

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where S and T are firmly nonexpansive mappings, and $\tau_n \in \left(0, \frac{2}{||A||^2}\right)$ for all $n \geq 1$. The generated sequence was proven to converge weakly to a solution of (1.1). Motivated by the result of Censor and Segal [8], Wang [9] proved a weak convergence theorem for a solution of SCFPP (1.1) given by the following algorithm:

$$x_{n+1} = x_n - \tau_n \left[(I-T)x_n + A^*(I-S)Ax_n\right], \quad n \geq 1, \tag{1.2}$$

where S and T are firmly nonexpansive mappings and $\{\tau_n\} \subset (0,\infty)$ is such that

$$\sum_{n=1}^{\infty} \tau_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \tau_n^2 < \infty.$$

We observe that the step size τ in algorithm (1.2) is independent of the norm of the bounded linear operator A. Cui and Ceng [10] extended Wang's results to a class of strict pseudo contractions and proved a weak convergence result for approximating solutions of SCFFP.

An important generalization of the SCFPP is the *multiple-set split feasibility problem* (MSSFP) which is to

$$\text{find } x^* \in \cap_{i=1}^{\mathrm{N}} C_i \text{ such that } Ax^* \in \cap_{j=1}^{\mathrm{M}} Q_j,$$

where $\{C_i\}_{i=1}^{\mathrm{N}}$ and $\{Q_j\}_{j=1}^{\mathrm{M}}$ are nonempty, closed and convex subsets of H_1 and H_2, respectively. This problem has been studied by several authors; see, e.g., [11, 12, 13, 14, 15].

Recently, Reich, Tuyen and Trang [16] studied the SCFPP in a more general setting. They formulated the problem as follows:

$$\text{find } x^* \in H_1 \text{ such that } x^* \in \Omega := \cap_{i=1}^{\mathrm{N}} \mathrm{Fix}(T_i) \cap A^{-1}\left(\cap_{j=1}^{\mathrm{M}} \mathrm{Fix}(S_j)\right),$$

where $\{T_i\}_{i=1}^{\mathrm{N}} : H_1 \to H_1$ and $\{S_j\}_{j=1}^{\mathrm{M}} : H_2 \to H_2$ are two finite families of nonexpansive mappings. In fact, they proved the following theorem.

Theorem 1.1. *Let H_1 and H_2 be two real Hilbert spaces. Let $T_i : H_1 \to H_1$, $i = 1,2,3,\ldots,N$ and $S_j : H_2 \to H_2$, $j = 1,2,3,\ldots,M$, be nonexpansive mappings on H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator from H_1 to H_2. Suppose that*

$$\Omega := \cap_{i=1}^{\mathrm{N}} \mathrm{Fix}(T_i) \cap A^{-1}\left(\cap_{j=1}^{\mathrm{M}} \mathrm{Fix}(S_j)\right) \neq \varnothing.$$

For any $u, x_0 \in H_1$, let $\{x_n\}$ be the sequence generated by

$$\begin{aligned}
y_n &= \sum_{i=1}^{M} a_{i,n} \tilde{T}_i x_n, \\
z_n &= \sum_{j=1}^{N} b_{j,n} S_j(Ay_n), \\
t_n &= y_n + \delta A^*(z_n - Ay_n), \\
x_{n+1} &= \alpha_n u + (1-\alpha_n)t_n, \ n \geq 0,
\end{aligned}$$

where $\{a_{i,n}\}$, $i = 1,\ 2,\ldots,N$, $\{b_{j,n}\}$, $j = 1,\ 2,\ldots,M$, and $\{\alpha_n\}$ are positive sequences, $\bar{T} = b_{j,n}I + (1-b_{j,n})T$ and $\{b_{i,n}\} \subset (0,1)$ for each $i = 1,\ 2,\ldots,N$ satisfying appropriate conditions. Then, $\{x_n\}$ converges strongly to $x = P_{\Omega}^{H_1} u$, where $P_{\Omega}^{H_1}$ is the metric projection of H_1 unto Ω.

In this paper, we present a strong convergence theorem of solutions of the MSSFP involving strict pseudo-contractive mappings based Halpern-viscosity and hybrid-projection algorithms. Numerical examples are given to show the efficiency of the two algorithms. Our results improve and complement some recent results in the literature.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space H. We denote '$x_n \rightharpoonup x^*$' to mean that the sequence $\{x_n\}$ converges weakly to x^*. We present some definitions and lemmas that will be used in the next section.

Let $P_C : H \to C$ be the metric projection from H onto C, that is,

$$||x - P_C(x)|| \leq ||x - z||, \quad \forall z \in C.$$

Given any $x \in H$ and $z \in C$, the following facts are known:

(i) $z = P_C(x) \iff \langle x - z, y - z \rangle \leq 0, \quad \forall y \in C;$
(ii) $||P_C x - y||^2 + ||x - P_C x||^2 \leq ||x - y||^2, \quad \forall y \in C.$

For a space E, we say that E has the Kadec Klee property [17] if, for any sequence $\{x_n\}$ in E with $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, then $||x_n - x|| \to 0$ as $n \to \infty$. It is known that Hilbert spaces have the Kadec Klee property.

Recall that a mapping $T : C \to C$ is said to be a strict pseudo-contraction if there exists a $\kappa \in [0,1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + \kappa ||(I - T)x - (I - T)y||^2, \quad \forall x, y \in C. \tag{2.1}$$

If $\kappa = 0$ in (2.1), T is said to be nonexpansive. Furthermore, if there exists an $\eta \in (0,1)$ such that

$$||Tx - Ty|| \leq \eta ||x - y||, \quad \forall x, y \in C,$$

then T is called a contraction. The following examples show that the class of strict pseudo-contractions properly contains the class of nonexpansive mappings.

Example 2.1. Let $T : \mathbb{R} \to \mathbb{R}$ be a mapping defined by

$$Tx = \begin{cases} x, & -\infty < x < 0, \\ -\dfrac{3}{2}x, & 0 \leq x < \infty. \end{cases}$$

Then, T is $1/5-$strictly pseudo-contractive but not nonexpansive.

Example 2.2. Let $T : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R}) : x \mapsto -\sqrt{2}x$. Then, T is $1/4-$strictly pseudo-contractive but not nonexpansive.

Lemma 2.1. [18] *Let H be a real Hilbert space. Then, we have*

(i) $||x + y||^2 = ||x||^2 + 2\langle x, y \rangle + ||y||^2, \quad \forall x, y \in H;$
(ii) $||x - y||^2 \leq ||y - u||^2 + 2\langle y - x, u - x \rangle, \quad \forall u, x, y \in H;$
(iii) $||\alpha x + (1-\alpha)y||^2 = \alpha ||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x - y||^2, \quad \forall x, y \in H,\ \alpha \in [0,1];$
(iv) $\left\|\sum_{i=1}^{N} \beta_i x_i\right\|^2 = \sum_{i=1}^{N} \beta_i ||x_i||^2 - \dfrac{1}{2}\sum_{i,j=1}^{N} \beta_i \beta_j ||x_i - x_j||^2, \forall \{x_i\}_i^N \subset H, \{\beta_i\}_i^N \subset [0,1], N \geq 2$, *where* $\sum_{i=1}^{N} \beta_i = 1$.

Lemma 2.2. [19] *Let $T : C \to C$ be a strict pseudo-contraction.*

i) *Then, T is Lipschitz, that is,*

$$\|Tx-Ty\| \leq \frac{1+\kappa}{1-\kappa}\|x-y\|, \quad \forall x,\ y\in C.$$

ii) *$I-T$ is demiclosed at 0, that is, for any sequence $\{x_n\}$ in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-T)x_n \to 0$, then $(I-T)\tilde{x}=0$.*

Lemma 2.3. [20] *Let H be a Hilbert space and C a closed, convex, and nonempty subset of H. Let $T: C\to C$ be an η-strict pseudo-contraction. Define $Sx=\zeta x+(1-\zeta)Tx$ for all $x\in C$, where $\eta<\zeta<1$. Then $F(S)=F(T)$.*

Lemma 2.4. [21] *Let $\{s_n\}$ be a real sequence which does not decrease at infinity in the sense that there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k}\leq s_{n_k+1}$, $\forall k\geq 0$. Define an integer sequence $\{\tau(n)\}$, where $n>n_0$, by $\tau(n):=\max\{n_0\leq k\leq n: s_k<s_{k+1}\}$. Then, $\tau(n)\to\infty$ as $n\to\infty$, and for all $n>n_0$, we have $\max\{s_{\tau(n)},s_n\}\leq s_{\tau(n)+1}$.*

Lemma 2.5. [22] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the inequality: $s_{n+1}\leq(1-\gamma_n)s_n+\gamma_n\delta_n$, $\forall n\geq 0$, where $\{\gamma_n\}_{n\in\mathbb{N}}\subset(0,1)$ and $\{\delta_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ such that (i) $\sum_{n=0}^{\infty}\gamma_n=\infty$, (ii) $\limsup_{n\to\infty}\delta_n\leq 0$.* Then, $s_n\to 0$.

3. MAIN RESULTS

Theorem 3.1. *Let H_1 and H_2 be real Hilbert spaces. Let $\{T_i\}_{i=1}^{\mathrm{N}}: H_1\to H_1$ and $\{S_j\}_{j=1}^{\mathrm{M}}: H_2\to H_2$ be $\{\kappa_i\}_{i=1}^{\mathrm{N}}$ and $\{\lambda_j\}_{j=1}^{\mathrm{M}}$ strict pseudo-contractions, respectively. Suppose that $F: H_1\to H_1$ F is an $\eta-$contraction and $\Omega:=\cap_{i=1}^{\mathrm{N}}\mathrm{Fix}(T_i)\cap A^{-1}\left(\cap_{j=1}^{\mathrm{M}}\mathrm{Fix}(S_j)\right)\neq\varnothing$, where $A: H_1\to H_2$ is a bounded linear operator, $A\neq 0$, and $A^*: H_2\to H_1$ is its adjoint operator. For any $x_0\in H_1$, define the sequence $\{x_n\}$ iteratively by*

$$\begin{cases} y_n=\sum\limits_{i=1}^{\mathrm{N}} a_{i,n}\bar{T}_{i,n}x_n, \\ z_n=\sum\limits_{j=1}^{\mathrm{M}} b_{j,n}\bar{S}_{j,n}(Ay_n), \\ t_n=y_n+\delta A^*(z_n-Ay_n), \\ x_{n+1}=\alpha_nF(x_n)+(1-\alpha_n)t_n, \quad n\geq 0, \end{cases} \tag{3.1}$$

where $\kappa=\max\{\kappa_i\}$ and $\lambda=\max\{\lambda_j\}$ with $\kappa<\beta_{i,n}<d<1$ and $\lambda<\gamma_{j,n}<d<1$ for all $i=1,2,\dots,N$, $j=1,2,\dots,M$, and $n\geq 0$. Suppose that $\bar{T}_{i,n}=\beta_{i,n}I+(1-\beta_{i,n})T_i$, $\bar{S}_{j,n}=\gamma_{j,n}I+(1-\gamma_{j,n})S_j$. Assume that the following conditions hold:

C1) $\delta\in\left(0,\frac{1-\lambda}{||A||^2}\right)$, $\varliminf_{n\to\infty}(\beta_{i,n}-\kappa)>0$ *for all* $i=1,2,\dots,N$;

C2) $\{\alpha_n\}\subset(0,1)$, $\lim\limits_{n\to\infty}\alpha_n=0$, $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$, $\sum\limits_{i=1}^{\mathrm{N}}a_{i,n}=\sum\limits_{j=1}^{\mathrm{M}}b_{j,n}=1$;

C3) $\{a_{i,n}\}_{i=1}^{\mathrm{N}},\{b_{j,n}\}_{j=1}^{\mathrm{M}}\subset[a,b]\subset(0,1)$.

Then, the sequence $\{x_n\}$ converges strongly to $x^\dagger=P_S^{H_1}F(x^\dagger)$.

Proof. Let $x,y\in\Omega$ be arbitrarily fixed. Using the fact that F is a contraction and P_S is nonexpansive, one has

$$||P_SF(x)-P_SF(y)||\leq||F(x)-F(y)||\leq\eta||x-y||.$$

Thus, $P_S F$ is an η-contraction on S. By using the Banach contraction principle, there exists a unique $x^\dagger \in S$ such that $x^\dagger = P_S F(x^\dagger)$. Note that

$$||F(x_n) - x^\dagger|| \le ||F(x_n) - F(x^\dagger)|| + ||F(x^\dagger) - x^\dagger|| \le \eta ||t_n - x^\dagger|| + ||F(x^\dagger) - x^\dagger||.$$

From Lemma 2.1 (iv) and the definition of strict pseudo-contractions, one has

$$\begin{aligned}
\|y_n - x^\dagger\|^2 &= \sum_{i=1}^{N} a_{i,n}\|T_i x_n - T_i x^\dagger\|^2 - \frac{1}{2}\sum_{i,k=1}^{N} a_{i,n}a_{k,n}\|T_i x_n - \bar{T}_k x_n\|^2 \\
&\le \sum_{i=1}^{N} a_{i,n}\|T_i x_n - T_i x^\dagger\|^2 \\
&= \sum_{i=1}^{N} a_{i,n}\big[\beta_{i,n}\|x_n - x^\dagger\|^2 + (1-\beta_{i,n})\|T_i x_n - T_i x^\dagger\|^2 - \beta_{i,n}(1-\beta_{i,n})\|x_n - T_i x_n\|^2\big] \\
&\le \sum_{i=1}^{N} a_{i,n}\big[\beta_{i,n}\|x_n - x^\dagger\|^2 + (1-\beta_{i,n})\|x_n - x^\dagger\|^2 + (1-\beta_{i,n})\kappa\|x_n - T_i x_n\|^2 \\
&\quad - \beta_{i,n}(1-\beta_{i,n})\|x_n - T_i x_n\|^2\big] \\
&= \|x_n - x^\dagger\|^2 - \sum_{i=1}^{N} a_{i,n}(\beta_{i,n} - \kappa)(1-\beta_{i,n})\|x_n - T_i x_n\|^2.
\end{aligned} \tag{3.2}$$

Similarly, with $Ax^\dagger = S_j Ax^\dagger$, for all $j = 1, 2, \ldots, M$, we have

$$\begin{aligned}
\|z_n - Ax^\dagger\|^2 \le \sum_{j=1}^{M} b_{j,n}\big[&\gamma_{j,n}\|Ay_n - Ax^\dagger\|^2 + (1-\gamma_{j,n})\|Ay_n - Ax^\dagger\|^2 \\
&+ (1-\gamma_{j,n})\lambda\|Ay_n - S_j(Ay_n)\|^2 - \gamma_{j,n}(1-\gamma_{j,n})\|Ay_n - S_j(Ay_n)\|^2\big] \\
\le \|A\|^2\|&y_n - x^\dagger\|^2 - \sum_{j=1}^{M} b_{j,n}(\gamma_{j,n} - \lambda)(1-\gamma_{j,n})\|Ay_n - S_j(Ay_n)\|^2.
\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
\|t_n - x^\dagger\|^2 &= \|y_n - x^\dagger + \delta A^*(z_n - Ay_n)\|^2 \\
&\le ||y_n - x^\dagger||^2 + 2\delta\langle y_n - x^\dagger, A^*(z_n - Ay_n)\rangle + \delta^2\|A\|^2||z_n - Ay_n||^2 \\
&\le ||y_n - x^\dagger||^2 + 2\delta\langle Ay_n - Ax^\dagger, z_n - Ay_n\rangle + \delta^2||A||^2||z_n - Ay_n||^2.
\end{aligned} \tag{3.3}$$

Again, by using Lemma 2.1 (i), we find

$$\begin{aligned}
\langle z_n - Ay_n, Ay_n - Ax^\dagger\rangle &= \sum_{j=1}^{M} b_{j,n}\left\langle S_j(Ay_n) - Ay_n, Ay_n - Ax^\dagger\right\rangle \\
&= \frac{1}{2}\sum_{j=1}^{M} b_{j,n}\Big[||S_j(Ay_n) - S_j(Ax^\dagger)||^2 - ||S_j(Ay_n) - Ay_n||^2 \\
&\quad - ||Ay_n - Ax^\dagger||^2\Big].
\end{aligned} \tag{3.4}$$

Observe that

$$\begin{aligned}||S_j(Ay_n)-S_j(Ax^\dagger)||^2 &= \left\|\gamma_{j,n}(Ay_n-Ax^\dagger)+(1-\gamma_{j,n})\left(S_j(Ay_n)-Ax^\dagger\right)\right\|^2\\ &= \gamma_{j,n}\left\|Ay_n-Ax^\dagger\right\|^2+(1-\gamma_{j,n})\left\|S_j(Ay_n)-Ax^\dagger\right\|^2\\ &\quad -\gamma_{j,n}(1-\gamma_{j,n})\left\|Ay_n-S_j(Ay_n)\right\|^2\end{aligned}$$

and

$$||S_j(Ay_n)-Ay_n||^2=(1-\gamma_{j,n})^2||S_j(Ay_n)-Ay_n||^2.$$

Substituting these identities in (3.4) and applying the definition of strict pseudo-contractions, we obtain

$$\langle z_n-Ay_n, Ay_n-Ax^\dagger\rangle \leq -\frac{1}{2}\sum_{j=1}^{M}(1-\lambda)b_{j,n}(1-\gamma_{j,n})||S_j(Ay_n)-Ay_n||^2, \tag{3.5}$$

and

$$\begin{aligned}||z_n-Ay_n||^2 &= ||\sum_{j=1}^{M}b_{j,n}S_j(Ay_n)-Ay_n||^2 \\ &\leq \sum_{j=1}^{M}b_{j,n}\left\|S_j(Ay_n)-Ay_n\right\|^2\\ &= \sum_{j=1}^{M}b_{j,n}(1-\gamma_{j,n})^2\left\|S_j(Ay_n)-Ay_n\right\|^2.\end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.3), we arrive at

$$\left\|t_n-x^\dagger\right\|^2 \leq ||y_n-x^\dagger||^2-\delta\left[(1-\lambda)-\delta||A||^2\right]\sum_{j=1}^{M}b_{j,n}(1-\gamma_{j,n})\left\|S_j(Ay_n)-Ay_n\right\|^2. \tag{3.7}$$

Using (3.2) and (3.7), we get that

$$\begin{aligned}||x_{n+1}-x^\dagger|| &\leq \alpha_n||F(x_n)-x^\dagger||+(1-\alpha_n)||t_n-x^\dagger||\\ &\leq \alpha_n||F(x_n)-F(x^\dagger)||+\alpha_n||F(x^\dagger)-x^\dagger||+(1-\alpha_n)||x_n-p||\\ &\leq [1-\alpha_n(1-\eta)]\,||x_n-x^\dagger||+\alpha_n||F(x^\dagger)-x^\dagger||\\ &\leq \max\left\{||x_n-x^\dagger||, \frac{||F(x^\dagger)-x^\dagger||}{1-\eta}\right\}\\ &\ \ \vdots\\ &\leq \max\left\{||x_0-x^\dagger||, \frac{||F(x^\dagger)-x^\dagger||}{1-\eta}\right\}.\end{aligned}$$

Thus, sequence $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{z_n\}$, $\{t_n\}$, and $\{F(x_n)\}$ are also bounded, respectively. Note that $x^\dagger = P_S^{H_1}F(x^\dagger)$. From Lemma 2.1 (iii), Condition C2), (3.2), and (3.7), we see that

$$\begin{aligned}
&||x_{n+1}-x^{\dagger}||^2\\
&=\alpha_n||F(x_n)-x^{\dagger}||^2+(1-\alpha_n)||t_n-x^{\dagger}||^2-\alpha_n(1-\alpha_n)||F(x_n)-t_n||^2\\
&\le \alpha_n||F(x_n)-x^{\dagger}||^2+||t_n-x^{\dagger}||^2\\
&\le \alpha_n||F(x_n)-x^{\dagger}||^2+||y_n-x^{\dagger}||^2-\delta\left(1-\lambda-\delta||A||^2\right)\sum_{j=1}^{M} b_{j,n}(1-\gamma_{j,n})||S_j(Ay_n)-Ay_n||^2\\
&\le \alpha_n||F(x_n)-x^{\dagger}||^2+\left\|x_n-x^{\dagger}\right\|^2-\sum_{i=1}^{N} a_{i,n}(\beta_{i,n}-\kappa)(1-\beta_{i,n})\left\|x_n-T_ix_n\right\|^2\\
&\quad-\delta\left(1-\lambda-\delta||A||^2\right)\sum_{j=1}^{M} b_{j,n}(1-\gamma_{j,n})||S_j(Ay_n)-Ay_n||^2.
\end{aligned} \tag{3.8}$$

Consequently, we have that

$$\begin{aligned}
&\sum_{i=1}^{N} a_{i,n}(\beta_{i,n}-\kappa)(1-\beta_{i,n})\left\|x_n-T_ix_n\right\|^2+\delta\left(1-\lambda-\delta||A||^2\right)\sum_{j=1}^{M} b_{j,n}(1-\gamma_{j,n})||S_j(Ay_n)-Ay_n||^2\\
&\le \alpha_n||F(x_n)-x^{\dagger}||^2+\left(\left\|x_n-x^{\dagger}\right\|^2-||x_{n+1}-x^{\dagger}||^2\right).
\end{aligned} \tag{3.9}$$

By Condition C1), we see that there exists $N_0\in\mathbb{N}$ and a positive number τ such that $(\beta_{i,n}-\kappa)>\tau$, for all $i=1,2,\ldots,N$ and $n\ge 0$. In view of conditions C2) and C3), we obtain

$$\begin{aligned}
&a\tau(1-d)\sum_{i=1}^{N}\left\|x_n-T_ix_n\right\|^2+a\delta\left(1-\lambda-\delta||A||^2\right)(1-d)\sum_{j=1}^{M}||S_j(Ay_n)-Ay_n||^2\\
&<\alpha_n||F(x_n)-x^{\dagger}||^2+\left(\left\|x_n-x^{\dagger}\right\|^2-||x_{n+1}-x^{\dagger}||^2\right).
\end{aligned} \tag{3.10}$$

Next, we split the proof into two cases.

Case 1. There exists an $n_0\in\mathbb{N}$ such that $\{||x_n-x^{\dagger}||\}_{n\ge n_0}$ is nonincreasing. Then, the sequence $\{||x_n-x^{\dagger}||\}_{n\in\mathbb{N}}$ is convergent. Applying C2) and (3.10), we get

$$\lim_{n\to\infty}\left\|x_n-T_ix_n\right\|=0 \quad \text{and} \quad \lim_{n\to\infty}||S_j(Ay_n)-Ay_n||=0, \tag{3.11}$$

for all $i=1,2,\ldots,N$ and $j=1,2,\ldots,M$. From (3.6) and (3.11), we get that $\|z_n-Ay_n\|\to 0$ and $\|t_n-y_n\|\le\delta\|A\|\|z_n-Ay_n\|\to 0$ as $n\to\infty$. Using the definition of each T_i and condition C3), we have that

$$\|y_n-x_n\|^2=\left\|\sum_{i=1}^{N} a_{i,n}(T_ix_n-x_n)\right\|^2\le\sum_{i=1}^{N} a_{i,n}(1-\beta_{i,n})^2\left\|T_ix_n-x_n\right\|^2\to 0 \tag{3.12}$$

as $n\to\infty$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle F(x^{\dagger})-x^{\dagger},x_n-x^{\dagger}\right\rangle=\lim_{k\to\infty}\left\langle F(x^{\dagger})-x^{\dagger},x_{n_k}-x^{\dagger}\right\rangle. \tag{3.13}$$

In Lemma 2.1(ii), setting $y = x_{n+1}$, $x = x^\dagger$, and $u = \alpha_n F(x^\dagger) + (1-\alpha_n)x^\dagger$; we conclude from the convexity of $||\cdot||^2$ that

$$\begin{aligned}||x_{n+1} - x^\dagger||^2 &\le ||\alpha_n(F(x_n) - F(x^\dagger)) + (1-\alpha_n)(t_n - x^\dagger)||^2 + 2\alpha_n\langle x_{n+1} - x^\dagger, F(x^\dagger) - x^\dagger\rangle \\ &\le [1-\alpha_n(1-\eta)]\,||x_n - x^\dagger||^2 + 2\alpha_n(1-\eta)\frac{\langle x_{n+1} - x^\dagger, F(x^\dagger) - x^\dagger\rangle}{1-\eta}.\end{aligned}$$

Without loss of generality, we assume that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Using Lemma 2.2$ii)$ and (3.11), we easily see that $x^* \in \bigcap_{i=1}^{\mathrm{N}} \mathrm{Fix}(T_i)$. From (3.12), it follows that, for any $y \in H_1$,

$$\langle y_{n_k}, y\rangle = \langle y_{n_k} - x_{n_k}, y\rangle + \langle x_{n_k}, y\rangle \to \langle x^*, y\rangle \quad \text{as } k \to \infty.$$

This implies that $y_{n_k} \rightharpoonup x^*$. Since A is bounded linear map and $y_{n_k} \rightharpoonup x^*$, then, for all $z \in H_2$,

$$\langle Ay_{n_k}, z\rangle = \langle y_{n_k}, A^*z\rangle \to \langle x^*, A^*z\rangle = \langle Ax^*, z\rangle \quad \text{as } k \to \infty.$$

Therefore, $Ay_{n_k} \rightharpoonup Ax^*$ as $k \to \infty$. By Lemma 2.2$ii)$ and (3.11), we obtain $x^* \in A^{-1}(\cap_{j=1}^{\mathrm{M}} \mathrm{Fix}(S_j))$. So, $x^* \in \Omega$. In view of Lemma (2), we have

$$\lim_{k\to\infty} \left\langle F(x^\dagger) - x^\dagger, x_{n_k} - x^\dagger \right\rangle = \left\langle F(x^\dagger) - x^\dagger, x^* - x^\dagger \right\rangle \le 0. \tag{3.14}$$

Employing Lemma 2.5 implies that $\{x_n\}$ converges to $x^\dagger = P_S F(x^\dagger)$, which completes the proof.

Case 2. Suppose that Case 1 does not hold. Then, there exists a subsequence $\{x_{n_\ell}\}$ of $\{x_n\}$ such that $||x_{n_\ell} - x^\dagger|| < ||x_{n_\ell+1} - x^\dagger||$ for all $l \in \{0\} \cup \mathbb{N}$. Define an integer sequence $\{\tau(n)\}$, where $n > n_0$, by

$$\tau(n) := \max\left\{n_0 \le \ell \le n : ||x_l - x^\dagger|| < ||x_{l+1} - x^\dagger||\right\},$$

for some n_0 sufficiently large. Usig Lemma 2.4, we have $\tau(n) \to \infty$ as $n \to \infty$, and, for all $n > n_0$,

$$||x_{\tau(n)} - x^\dagger|| < ||x_{\tau(n)+1} - x^\dagger||. \tag{3.15}$$

Using similar arguments used in (3.2) and (3.7) as well as the convexity of $||\cdot||^2$, we arrive at

$$\begin{aligned}0 &\le ||x_{\tau(n)+1} - x^\dagger||^2 - ||x_{\tau(n)} - x^\dagger||^2 \\ &\le \alpha_{\tau(n)}||F(x_{\tau(n)}) - x^\dagger||^2 + (1-\alpha_{\tau(n)})||t_{\tau(n)} - x^\dagger||^2 - ||x_{\tau(n)} - x^\dagger||^2 \\ &\le \alpha_{\tau(n)}||F(x_{\tau(n)}) - x^\dagger||^2 + (1-\alpha_{\tau(n)})||x_{\tau(n)} - x^\dagger||^2 - ||x_{\tau(n)} - x^\dagger||^2 \\ &\le \alpha_{\tau(n)}||F(x_{\tau(n)}) - x^\dagger||^2.\end{aligned}$$

By the boundedness of $\{F(x_{\tau(n)})\}$, there exists an $M > 0$ such that

$$0 \le ||x_{\tau(n)+1} - x^\dagger||^2 - ||x_{\tau(n)} - x^\dagger||^2 \le \alpha_{\tau(n)} M.$$

Using Condition C2) and letting $n \to \infty$, we have

$$||x_{\tau(n)+1} - x^\dagger||^2 - ||x_{\tau(n)} - x^\dagger||^2 \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

Following similar arguments in (3.2) and (3.11), we conclude from (3.16) and (3.10) that

$$\lim_{n\to\infty} \left\|x_{\tau(n)} - T_i x_{\tau(n)}\right\| = 0 \quad \text{and} \quad \lim_{n\to\infty} ||S_j(Ay_{\tau(n)}) - Ay_{\tau(n)}|| = 0, \tag{3.17}$$

for all $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$. Analogous arguments to those in Case 1 yield

$$||x_{\tau(n)+1} - x^\dagger||^2 \leq \left[1 - \alpha_{\tau(n)}(1-\eta)\right] ||x_{\tau(n)} - x^\dagger||^2 + 2\alpha_{\tau(n)}(1-\eta) \frac{\langle x_{\tau(n)+1} - x^\dagger, F(x^\dagger) - x^\dagger \rangle}{1-\eta} \tag{3.18}$$

and

$$\limsup_{n\to\infty} \left\langle F(x^\dagger) - x^\dagger, x_{\tau(n)} - x^\dagger \right\rangle \leq 0. \tag{3.19}$$

From Lemma 2.5, we obtain the desired conclusion immediately.

□

Corollary 3.1. *Let H_1 and H_2 be real Hilbert spaces. Let $\{T_i\}_{i=1}^{N} : H_1 \to H_1$ and $\{S_j\}_{j=1}^{M} : H_2 \to H_2$ be $\{\kappa_i\}_{i=1}^{N}$ and $\{\lambda_j\}_{j=1}^{M}$ strict pseudo-contractions, respectively. Suppose that $\Omega := \cap_{i=1}^{N} \mathrm{Fix}(T_i) \cap A^{-1}\left(\cap_{j=1}^{M} \mathrm{Fix}(S_j)\right) \neq \varnothing$, where $A : H_1 \to H_2$ is a bounded linear operator, $A \neq 0$, and $A^* : H_2 \to H_1$ is its adjoint operator. For any $x_0, u \in H_1$, define the sequence $\{x_n\}$ iteratively by*

$$\begin{cases} y_n = \sum_{i=1}^{N} a_{i,n}\bar{T}_{i,n}x_n, \\ z_n = \sum_{j=1}^{M} b_{j,n}\bar{S}_{j,n}(Ay_n), \\ t_n = y_n + \delta A^*(z_n - Ay_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)t_n, \quad n \geq 0, \end{cases} \tag{3.20}$$

where $\kappa = \max\{\kappa_i\}$ and $\lambda = \max\{\lambda_j\}$ with $\kappa < \beta_{i,n} < d < 1$ and $\lambda < \gamma_{j,n} < d < 1$ for all $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$, and $n \geq 0$. Suppose that $\bar{T}_{i,n} = \beta_{i,n}I + (1-\beta_{i,n})T_i$, $\bar{S}_{j,n} = \gamma_{j,n}I + (1-\gamma_{j,n})S_j$. Assume that the following conditions hold:

C1) $\delta \in \left(0, \frac{1-\lambda}{||A||^2}\right)$, $\underline{\lim}_{n\to\infty}(\beta_{i,n} - \kappa) > 0$ *for all* $i = 1, 2, \ldots, N$;

C2) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{i=1}^{N} a_{i,n} = \sum_{j=1}^{M} b_{j,n} = 1$;

C3) $\{a_{i,n}\}_{i=1}^{N}, \{b_{j,n}\}_{j=1}^{M} \subset [a,b] \subset (0,1)$.

Then, sequence $\{x_n\}$ converges strongly to $x^\dagger = P_S^{H_1}(u)$.

Proof. The proof follows by employing similar arguments to those in the proof of Theorem 3.1 when $F(x_n) = u$, for all $n \geq 0$. □

Theorem 3.2. *Let H_1 and H_2 be real Hilbert spaces. Let $\{T_i\}_{i=1}^{N} : H_1 \to H_1$ and $\{S_j\}_{j=1}^{M} : H_2 \to H_2$ be $\{\kappa_i\}_{i=1}^{N}$ and $\{\lambda_j\}_{j=1}^{M}$ strict pseudo-contractions, respectively. Suppose that $\Omega := \cap_{i=1}^{N} \mathrm{Fix}(T_i) \cap A^{-1}\left(\cap_{j=1}^{M} \mathrm{Fix}(S_j)\right) \neq \varnothing$, where $A : H_1 \to H_2$ is a bounded linear operator, $A \neq 0$, and $A^* : H_2 \to H_1$ is its adjoint operator. For any $x_0, u \in H_1$, define the sequence $\{x_n\}$ iteratively* *by*

$$\begin{cases} y_n = \sum_{i=1}^{N} a_{i,n}\bar{T}_{i,n}x_n, \\ z_n = \sum_{j=1}^{M} b_{j,n}\bar{S}_{j,n}(Ay_n), \\ t_n = y_n + \delta A^*(z_n - Ay_n), \\ w_n = \alpha_n x_n + (1-\alpha_n)t_n, \\ C_n = \{z \in H_1 : ||w_n - z|| \leq ||x_n - z||\}, \\ Q_n = \{z \in H_1 : \langle x_0 - x_n, z - x_n\rangle \leq 0\}, \\ x_{n+1} = P^{H_1}_{C_n \cap Q_n}(x_0), \quad n \geq 0, \end{cases} \tag{3.21}$$

where $C_1 = Q_1 = H_1$ and the conditions on $\{\alpha_n\}$ in Theorem 3.1 are replaced by

D1) $\{\alpha_n\} \subset [\alpha, \beta] \subset (0,1)$.

Suppose that condions C1), C2), and C3) of Theorem 3.1 are satisfied. Then, t $\{x_n\}$ converges strongly to $x^\dagger = P^{H_1}_{\Omega}x_0$.

Proof. We first show that the algorithm is well defined. Indeed, for each $n \geq 0$, we can rewrite C_n as

$$C_n = \left\{ z \in H_1 : \langle x_n - w_n, z\rangle \leq \frac{1}{2}\left(||x_n||^2 - ||w_n||^2\right) \right\}.$$

Clearly, C_n and Q_n are closed and convex subsets of H_1. Fixing $p \in \Omega$, we conclude from Lemma 2.3 that $p = \bar{T}_{i,n}p$ and $Ap = \bar{S}_{j,n}$. By Condition C1), (3.2), (3.7) and the convexity of $||\cdot||$, we have that

$$||w_n - p|| \leq \alpha_n||x_n - p|| + (1-\alpha_n)||t_n - p|| \leq ||x_n - p||.$$

Hence, $p \in C_n$. Consequently, $\Omega \subset C_n$ for all $n \geq 0$. Next, by induction, we show that, for all $n \geq 0$, $\Omega \subset C_n \cap Q_n$. For $n = 0$, we have $\Omega \subset C_0 \cap Q_0 = H_1$. Now, suppose $\Omega \subset C_n \cap Q_n$ for some $n \geq 1$. Thus, by our assumption, $\Omega \neq \varnothing$. Thus, $C_n \cap Q_n$ is non-empty. Since $x_{n+1} = P^{H_1}_{C_n\cap Q_n}(x_0)$, we have $\langle x_0 - x_{n+1}, z - x_{n+1}\rangle \leq 0$ for all $z \in C_n \cap Q_n$. Since $\Omega \subset C_n \cap Q_n$ and $p \in \Omega$, we obtain $\langle x_0 - x_{n+1}, p - x_{n+1}\rangle \leq 0$. This implies that $p \in Q_{n+1}$ and $\Omega \subset C_{n+1} \cap Q_{n+1}$. Consequently, $\Omega \subset C_n \cap Q_n$ for all $n \geq 0$.

We now show that $\{x_n\}$ is bounded. Let $x^\dagger = P^{H_1}_S(x_0)$. Since $x^\dagger \in S \subset Q_n$ for all $n \geq 0$, and $x_n = P^{H_1}_{Q_n}(x_0)$, we have that $||x_n - x_0|| \leq ||x^\dagger - x_0||$ for all $n \geq 0$. Hence, $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{t_n\}$ are bounded.

We now show that $\{x_n\}$ converges to $x^\dagger$. We have, by definition, $x_{n+1} \in Q_n$. Thus,$|x_n - x_0|| \leq ||x_{n+1} - x_0||$ This implies that the sequence $\{||x_n - x_0||\}$ is non-increasing. Therefore, the limit $\lim_{n\to\infty} ||x_n - x_0||$ exists. Moreover, we have

$$||x_n - x_{n+1}||^2 \leq ||x_0 - x_{n+1}||^2 - ||x_0 - x_n||^2.$$

As $n \to \infty$, we obtain that $||x_n - x_{n+1}|| \to 0$ and then $||w_n - x_{n+1}|| \leq ||x_n - x_{n+1}|| \to 0$. They show that

$$||x_n - w_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - w_n|| \to 0 \text{ as } n \to \infty. \tag{3.22}$$

Following similar arguments to those in (3.8) and (3.9) as well as using the convexity of $\|\cdot\|^2$ yield

$$\begin{aligned}
||w_n - x^\dagger||^2 &\le \alpha_n ||x_n - x^\dagger||^2 + (1-\alpha_n)||t_n - x^\dagger||^2 \\
&\le \alpha_n ||x_n - x^\dagger||^2 + (1-\alpha_n)||y_n - x^\dagger||^2 \\
&\quad - \delta\left(1-\lambda-\delta||A||^2\right)(1-\alpha_n)\sum_{j=1}^{M} b_{j,n}(1-\gamma_{j,n})||S_j(Ay_n) - Ay_n||^2 \\
&\le \alpha_n ||x_n - x^\dagger||^2 + (1-\alpha_n)\left\|x_n - x^\dagger\right\|^2 \\
&\quad - (1-\alpha_n)\sum_{i=1}^{N} a_{i,n}(\beta_{i,n}-\kappa)(1-\beta_{i,n})\|x_n - T_i x_n\|^2 \\
&\quad - \delta\left(1-\lambda-\delta||A||^2\right)(1-\alpha_n)\sum_{j=1}^{M} b_{j,n}(1-\gamma_{j,n})||S_j(Ay_n) - Ay_n||^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{N} a_{i,n}(\beta_{i,n}-\kappa)(1-\beta_{i,n})\|x_n - T_i x_n\|^2 + \delta\left(1-\lambda-\delta||A||^2\right)\sum_{j=1}^{M} b_{j,n}(1-\gamma_{j,n})||S_j(Ay_n) - Ay_n||^2 \\
&\le \left(\frac{\left\|x_n - x^\dagger\right\|^2 - ||w_n - x^\dagger||^2}{1-\alpha_n}\right) \\
&\le \left(\frac{\left\|x_n - x^\dagger\right\| + ||w_n - x^\dagger||}{1-\alpha_n}\right)||x_n - w_n||.
\end{aligned}$$

From the boundedness of $\{x_n\}$ and $\{w_n\}$, D1), and (3.22), we find that

$$\|x_n - T_i x_n\| \to 0 \quad \text{and} \quad ||S_j(Ay_n) - Ay_n|| \to 0 \ \text{ as } n \to \infty, \tag{3.23}$$

for all $i = 1,2,\ldots,N$ and $j = 1,2,\ldots,M$. From (3.6) and (3.23), we see that $\|z_n - Ay_n\|^2 \to 0$ as $n \to \infty$, which together with (3.21) yields that

$$\|t_n - y_n\|^2 \le \delta^2||A||^2||z_n - Ay_n||^2 \to 0 \ \text{ as } n \to \infty. \tag{3.24}$$

From Lemma 2.1, we obtain that

$$\begin{aligned}
||w_n - x^\dagger||^2 &\le \alpha_n ||x_n - x^\dagger||^2 + (1-\alpha_n)||y_n - x^\dagger||^2 - \alpha_n(1-\alpha_n)||x_n - t_n||^2 \\
&\quad - \delta\left(1-\lambda-\delta||A||^2\right)(1-\alpha_n)\sum_{j=1}^{M} b_{j,n}(1-\gamma_{j,n})||S_j(Ay_n) - Ay_n||^2 \\
&\le ||x_n - x^\dagger||^2 - \alpha_n(1-\alpha_n)||x_n - t_n||^2 \\
&\quad - (1-\alpha_n)\sum_{i=1}^{N} a_{i,n}(\beta_{i,n}-\kappa)(1-\beta_{i,n})\|x_n - T_i x_n\|^2 \\
&\quad - \delta\left(1-\lambda-\delta||A||^2\right)(1-\alpha_n)\sum_{j=1}^{M} b_{j,n}(1-\gamma_{j,n})||S_j(Ay_n) - Ay_n||^2 \\
&\le ||x_n - x^\dagger||^2 - \alpha_n(1-\alpha_n)||x_n - t_n||^2.
\end{aligned}$$

From Condition D1) and (3.22) as well as the boundedness of $\{x_n\}$ and $\{w_n\}$, we get that

$$||x_n - t_n||^2 \leq \left[\frac{||x_n - x^\dagger||^2 - ||w_n - x^\dagger||^2}{\alpha_n(1-\alpha_n)}\right]$$
$$\leq \left[\frac{||x_n - x^\dagger|| + ||w_n - x^\dagger||}{\alpha_n(1-\alpha_n)}\right] ||x_n - w_n|| \to 0 \text{ as } n \to \infty.$$

Therefore, we have

$$||x_n - y_n|| \leq ||x_n - t_n|| + ||t_n - y_n|| \to 0 \text{ as } n \to \infty. \tag{3.25}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. Using Lemma 2.2*ii)* and (3.23), we get that $x^* \in \cap_{i=1}^{\mathrm{N}} F(T_i)$. From (3.25), it follows that $y_{n_k} \rightharpoonup x^*$. Since A is bounded linear map, then $Ay_{n_k} \rightharpoonup Ax^*$ as $k \to \infty$. Using Lemma 2.2*ii)* and (3.23), we see that $x^* \in A^{-1}\left(\cap_{j=1}^{\mathrm{M}} \mathrm{Fix}(S_j)\right)$. Therefore, $x^* \in \Omega$. Hence, $x_{n_k} - x_0 \rightharpoonup x^* - x_0$. In view of $x^\dagger = P_\Omega^{H_1}(x_0)$ and $x^* \in \Omega$, we conclude from the weak lower semi-continuity of the norm that

$$||x^\dagger - x_0|| \leq ||x^* - x_0|| \leq \underline{\lim}\, ||x_{n_k} - x_0|| \leq \overline{\lim}\, ||x_{n_k} - x_0|| \leq ||x^\dagger - x_0||.$$

This implies that $||x_{n_k} - x_0|| \to ||x^* - x_0||$ as $k \to \infty$. Employing this and $x_{n_k} - x_0 \rightharpoonup x^* - x_0$, we have, by the Kadec-Klee property of H, that $||x_{n_k} - x^*|| \to 0$ as $k \to \infty$. Notice that $x_{n_k} = P_{Q_{n_k}}(x_0)$ and $x^\dagger \in Q_{n_k}$. Thus,

$$-||x_{n_k} - x^\dagger||^2 = -\left\langle x^\dagger - x_0, x^\dagger - x_{n_k}\right\rangle + \left\langle x_0 - x_{n_k}, x_{n_k} - x^\dagger\right\rangle$$
$$\geq \left\langle x_0 - x^\dagger, x^\dagger - x_{n_k}\right\rangle. \tag{3.26}$$

On the other hand, one has $x_{n_k} - x^\dagger \rightharpoonup x^* - x^\dagger$. Using this and the weak lower semi-continuity of $||\cdot||^2$, we get that

$$||x^\dagger - x^*||^2 \leq \underline{\lim}\, ||x^\dagger - x_{n_k}||^2$$
$$\leq \overline{\lim}\, ||x^\dagger - x_{n_k}||^2$$
$$\leq \overline{\lim}\left(||x^\dagger - x^*|| + ||x^* - x_{n_k}||\right)^2$$
$$= ||x^\dagger - x^*||^2.$$

This shows that

$$||x^\dagger - x_{n_k}||^2 \to ||x^\dagger - x^*||^2 \to 0 \quad \text{as } k \to \infty. \tag{3.27}$$

From (3.26) and (3.27), we see that

$$-||x^\dagger - x^*||^2 \geq \left\langle x_0 - x^\dagger, x^\dagger - x^*\right\rangle. \tag{3.28}$$

Since $x^* \in \Omega$ and $x^\dagger = P_\Omega^{H_1}(x_0)$, then

$$\left\langle x_0 - x^\dagger, x^\dagger - x^*\right\rangle \geq 0. \tag{3.29}$$

Plugging (3.29) into (3.28), we have that

$$-||x^\dagger - x^*||^2 \geq 0. \tag{3.30}$$

Therefore, $x^\dagger = x^*$ and, $x_{n_k} \to x^*$ as $k \to \infty$. Thus, $x_n \to x^\dagger = P_S^{H_1}(x_0)$. The proof is complete.

□

4. Applications

4.1. **Split minimum point problems.** A multivalued mapping $A : \mathscr{D}(A) \subset H \to 2^H$ with domain $\mathscr{D}(A) = \{x \in H : Ax \neq \varnothing\}$ and range $\mathscr{R}(A) = \cup\{Ax : x \in H\}$ is monotone if, for every $x, y \in H$,

$$\langle u - v, x - y \rangle \geq 0 \quad \text{for all} \quad u \in Ax, v \in Ay.$$

A monotone map A is maximal monotone if its graph $\mathrm{Gr}(A) = \{(x,y) \in E : x \in \mathscr{D}(A), y \in Ax\}$ is not properly contained in the graph of any other monotone map. It is known that if A is maximal monotone, then the range $\mathscr{R}(I + \lambda A) = H$, for $\lambda > 0$. Consequently, for $\lambda > 0$, the resolvent operator is given by $J_\lambda = (I + \lambda A)^{-1}$, where I is the identity map on H. The operator J_λ is always single-valued (see, for instance, [23]).

Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator, $A \neq 0$, with the adjoint operator $A^* : H_2 \to H_1$. Let $\{\mathscr{A}_i\}_{i=1}^{\mathrm{N}} : H_1 \to 2^{H_1}$ and $\{\mathscr{B}_j\}_{j=1}^{\mathrm{M}} : H_2 \to 2^{H_2}$ be maximal monotone operators such that $\Omega := \cap_{i=1}^{\mathrm{N}} \mathscr{A}_{i=1}^{-1}(0) \cap A^{-1}\left(\cap_{j=1}^{\mathrm{M}} \mathscr{B}_{j=1}^{-1}(0)\right) \neq \varnothing$. The problem under consideration is to find an element $x^\dagger \in \Omega$. Several authors have studied this problems, we refer to [12, 24, 25, 26]. In Theorems 3.1 and 3.2, for $r > 0$, set $T_i = J_r^{\mathscr{A}_i}$ and $S_j = J_r^{\mathscr{B}_j}$, where $J_r^{\mathscr{A}_i} = (I + r\mathscr{A}_i)^{-1}$ and $J_r^{\mathscr{B}_j} = (I + r\mathscr{B}_j)^{-1}$ are the resolvent of the operators $\{\mathscr{A}_i\}_{i=1}^{\mathrm{N}}$ and $\{\mathscr{B}_j\}_{j=1}^{\mathrm{M}}$.

It is well known that the resolvents operators are firmly nonexpansive mappings and thus, strictly pseudo-contractive. Hence, under the assumptions of Theorems 3.1 and 3.2, the sequences generated by algorithms 3.1, 3.20 and 3.21 converge to $x^\dagger \in \Omega$.

4.2. **Split variational inequality problems.** Let C and Q be closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $\mathscr{A}, \mathscr{B} : H_1 \to H_1$ and $\mathscr{B} : H_2 \to H_2$ be two monotone and hemi-continuous operators and $A : H_1 \to H_2$ be a bounded linear map. Then, the split variational inequality problem is to find

$$x^* \in C \text{ such that } \langle v - x^*, Gx^* \rangle \geq 0 \text{ and } \langle q - Ax^*, G(Ax^*) \rangle \geq 0, \forall\, v \in C,\, q \in Q. \tag{4.1}$$

We denote by $\mathrm{SOL}(\mathscr{C}, \mathscr{A})$ and $\mathrm{SOL}(\mathscr{Q}, \mathscr{B})$ the solutions of the variational inequalities with respect to $\mathscr{A}$ and $\mathscr{B}$.

We now prove the following theorem.

Theorem 4.1. *Let H_1 and H_2 be real Hilbert spaces, and let $\mathscr{C}$ and $\mathscr{Q}$ be closed, convex subsets of H_1, H_2, respectively. Assume that $F : H_1 \to H_1$, where F is an $\eta-$contraction. Let $\{\mathscr{A}_i\}_{i=1}^{\mathrm{N}} : \mathscr{C} \to H_1$ and $\{\mathscr{B}_j\}_{j=1}^{\mathrm{M}} : \mathscr{Q} \to H_2$ be two monotone and hemi-continuous operators. Suppose that $\Omega := \cap_{i=1}^{\mathrm{N}} \mathrm{SOL}(C, \mathscr{A}_i) \cap A^{-1}\left(\cap_{j=1}^{\mathrm{M}} \mathrm{SOL}(Q, \mathscr{B}_j)\right) \neq \varnothing$, where $A : H_1 \to H_2$ is a bounded linear operator, $A \neq 0$, and $A^* : H_2 \to H_1$ is its adjoint operator. For any $\alpha, \beta > 0$ and $x_0 \in H_1$,* *define the sequence $\{x_n\}$ iteratively by*

$$\begin{cases} y_n = \sum_{i=1}^{N} a_{i,n}\bar{\mathscr{A}}_{i,n}, \\ z_n = \sum_{j=1}^{M} b_{j,n}\bar{\mathscr{B}}_{j,n}, \\ t_n = y_n + \delta A^*(z_n - Ay_n), \\ x_{n+1} = \alpha_n F(x_n) + (1-\alpha_n)t_n, \quad n \geq 0, \end{cases} \tag{4.2}$$

Suppose that $\bar{\mathscr{A}}_{i,n} = \beta_{i,n}I + (1-\beta_{i,n})\mathrm{SOL}(\mathscr{C}, \alpha\mathscr{A}_i + I^{H_1} - x_n)$, $\bar{\mathscr{B}}_{j,n} = \gamma_{j,n}I + (1-\gamma_{j,n})\mathrm{SOL}(Q, \beta\mathscr{B}_j + I^{H_2} - Ay_n)$. *Assume that the following conditions hold:*

C1) $\delta \in \left(0, \frac{1}{||A||^2}\right)$, $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{i=1}^{N} a_{i,n} = \sum_{j=1}^{M} b_{j,n} = 1$;

C2) $\{a_{i,n}\}_{i=1}^{N}, \{b_{j,n}\}_{j=1}^{M} \subset [a,b] \subset (0,1)$, $\{\beta_{i,n}\}_{i=1}^{N}, \{\gamma_{j,n}\}_{j=1}^{M} \subset [c,d] \subset (0,1)$.

Then, $\{x_n\}$ *converges strongly to* $x^\dagger = P_\Omega^{H_1} F(x^\dagger)$, *where* I^{H_1} *and* I^{H_2} *are the identity operators on* H_1 *and* H_2, *respectively.*

Proof. Define the following mappings $A_{\mathscr{A}_i} : H \to H \times H$ by

$$A_{\mathscr{A}_i}x = \begin{cases} \mathscr{A}_i x + N_{\mathscr{C}}(x), & x \in \mathscr{C}, \\ \varnothing, & x \notin \mathscr{C}, \end{cases} \tag{4.3}$$

and $B_{\mathscr{B}_j} : H \to H \times H$ by

$$B_{\mathscr{B}_j}x = \begin{cases} \mathscr{B}_j x + N_{\mathscr{Q}}(x), & x \in \mathscr{Q}, \\ \varnothing, & x \notin \mathscr{Q}, \end{cases} \tag{4.4}$$

where $N_{\mathscr{C}}(x) = \{y \in H_1 : \langle z - x, y\rangle \text{ for all } z \in \mathscr{C}\}$ and $N_{\mathscr{Q}}(x) = \{u \in H_2 : \langle v - x, u\rangle \text{ for all } v \in \mathscr{Q}\}$. According to Rockafellar [27], $A_{\mathscr{A}_i}$, $i = 1,2,\dots,N$ and $B_{\mathscr{B}_j}$, $j = 1,2,\dots,M$, are maximal monotone. Observe that

$$\chi_n = \mathrm{SOL}(C, \alpha\mathscr{A}_i + I^{H_1} - x_n) \quad \text{if and only if} \, \langle z - \chi_n, \alpha\mathscr{A}_i\chi_n + \chi_n - x_n\rangle \geq 0 \tag{4.5}$$

for all $i = 1,2,\dots,N$ and $z \in H_1$. This implies that $-\alpha\,\mathscr{A}_i\chi_n - \chi_n + x_n \in \alpha N_{\mathscr{Q}}(\chi_n)$. Consequently, $\chi_n = J_\alpha^{\mathscr{A}_i}(x_n)$. Similarly, if we define $\xi_n = \mathrm{SOL}(C, \alpha\mathscr{A}_i + I^{H_2} - x_n)$, then $\xi_n = J_\beta^{\mathscr{B}_j}(Ay_n)$. Since $J_\alpha^{\mathscr{A}_i}$ and $J_\beta^{\mathscr{B}_j}$, $i = 1,2,\dots,N$, $j = 1,2,\dots,M$ are non-expansive mappings, the proof follows that of Theorem 3.1. □

5. Numerical Examples

In this section, we present some numerical examples to show the effectiveness of the proposed algorithms. Numerical experiments were carried out on a MATLAB environment on a system with a processor Pentium(R) Dual − Core CPU T4200 @ 2.00GHz, 3.00 GB RAM, and a 64−bit Operating System.

Example 5.1. In this example, we approximate a common fixed point of some strict pseudo-contraction mappings. We set $H_1 = \mathbb{R}^p, p \geq 1$, and $H_2 = \ell_2(\mathbb{R})$ with their usual norms. For fixed $N, M > 1$, set $i = 1,2,\dots,N$ and $j = 1,2,\dots,M$, and $n \in \mathbb{N}$. Define a map $T_i : \mathbb{R}^p \to \mathbb{R}^p$ by $x \mapsto -\frac{3}{2}x$. It is easy to see that T_i is a $\kappa := \frac{1}{5}-$strict pseudo-contraction map for each

$i = 1,2,\dots,\mathrm{N}$. Choosing $\beta_{i,n} = \frac{1}{2}$ implies that $\bar{T}_{i,n}x = -\frac{1}{4}x$ for all $i = 1,2,\dots,\mathrm{N}$, $n \geq 0$, and $x \in \mathbb{R}^p$. Setting $a_{i,n} = \frac{2i}{\mathrm{N}(\mathrm{N}+1)}$, we have

$$\sum_{i=1}^{\mathrm{N}} a_{i,n} = 1 \;\text{ and }\; \sum_{i=1}^{\mathrm{N}} a_{i,n}\bar{T}_{i,n}x = -\frac{x}{4} \;\text{ for all }\; x \in \mathbb{R}^p.$$

Moreover, it is easy to see that $\cap_{i=1}^{\mathrm{N}}\mathrm{Fix}(T_i) = \cap_{i=1}^{\mathrm{N}}\mathrm{Fix}(\bar{T}_{i,n}) = \{0\}$. Now, define a map $A : \mathbb{R}^p \to \ell_2(\mathbb{R})$ such that

$$x = \{x_k\}_{k=1}^{p} \mapsto Ax = \{A_kx\}_{k=1}^{\infty} : A_kx = \begin{cases} 2x_k, & 1 \leq k \leq p, \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

Obviously, the map A is bounded and linear with $||A|| = 2$. Define also the following map: $S_j : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ by $S_jy = -\frac{5}{2}y$ for all $y \in \ell_2(\mathbb{R})$. It is clear that S_j is a $\lambda := \frac{29}{49}-$strict pseudo-contraction map. Setting $\gamma_{j,n} = \frac{2}{3}$ implies that $\bar{S}_{j,n}y = -\frac{1}{6}y$ for all $j = 1,2,\dots,\mathrm{M}$, $n \geq 0$, $y \in \ell_2(\mathbb{R})$. Taking $b_{j,n} = \frac{2j}{\mathrm{M}(\mathrm{M}+1)}$, we obtain that

$$\sum_{j=1}^{\mathrm{M}} b_{j,n} = 1 \;\text{ and }\; \sum_{j=1}^{\mathrm{M}} b_{j,n}\bar{S}_{j,n}y = -\frac{y}{6} \;\text{ for all }\; y \in \ell_2(\mathbb{R}).$$

Let $x \in \mathbb{R}^p$ and $y \in \ell_2(\mathbb{R})$. Then, from the definition of A in (5.1), we have that

$$\begin{aligned} \langle Ax, y\rangle_{\ell_2(\mathbb{R})} &= 2\left\langle (x^1,x^2,\cdots,x^p,0,0,\cdots),(y^1,y^2,\cdots,y^p,\cdots)\right\rangle_{\ell_2(\mathbb{R})} \\ &= \left\langle (x^1,x^2,\cdots,x^p), 2(y^1,y^2,\cdots,y^p)\right\rangle_{\mathbb{R}^p}. \end{aligned}$$

Therefore, the adjoint operator of A is the map $A^* : \ell_2(\mathbb{R}) \to \mathbb{R}^p$ by $A^*(y) = 2(y^1,y^2,\cdots,y^p)$ for all $y = \{y_k\}_{k=1}^{\infty} \in \ell_2(\mathbb{R})$.

It is clear that $\Omega := \cap_{i=1}^{\mathrm{N}}\mathrm{Fix}(T_i) \cap A^{-1}\left(\cap_{j=1}^{\mathrm{M}}\mathrm{Fix}(S_j)\right) \neq \varnothing$ since $0 \in S$. For computational purposes, we choose $\delta = \frac{1}{2||A||^2}(1-\lambda)$ and $x_1 = (1,1,\cdots,1) \in \mathbb{R}^p$. In Algorithm (3.20), we set $u = (1,1,\cdots,1)$ and define $F : \mathbb{R}^p \to \mathbb{R}^p$ by $F(x) = \frac{1}{3}x$ for all $x \in \mathbb{R}^p$. Then, F is a $\frac{1}{3}-$contraction. In Algorithms (3.20) and (3.1), $\alpha_n = \frac{1}{n+1}$, while $\zeta_n = \frac{3n}{4n+1}$ in Algorithm (3.21).

Tables 1–4 contain the test results and time for this experiment. The symbol N represents the number of iterations, while $||x_{\mathrm{N}}||_2$ represents the corresponding 2–norm. It can be deduced from Tables 1 and 2 that, as the number of iterations increase, the three algorithms converge to the common fixed point. However, from Tables 3 and 4, it can be inferred that these algorithms reduce in efficiency as dimensions increase. Also from the aforementioned tables and Figure 1, it is noticeable that the viscosity algorithm (3.1) performs best among its contemporaries. However, the Halpern scheme (3.20) takes the least computational time.

TABLE 1. Computational results for Example 5.1 in $\mathbb{R}$.

N	Algorithm (3.20) $\|\|x_N\|\|_2$	Algorithm (3.20) Time (secs)	Algorithm (3.21) $\|\|x_N\|\|_2$	Algorithm (3.21) Time (secs)
10	8.0000×10^{-2}	0.000026	1.5573×10^{-1}	0.002370
50	1.6000×10^{-2}	0.000066	1.5403×10^{-4}	0.003078
100	8.0000×10^{-3}	0.000124	3.2480×10^{-8}	0.006308
150	5.3333×10^{-3}	0.000164	7.1255×10^{-12}	0.009161
200	4.0000×10^{-3}	0.000212	1.5015×10^{-15}	0.012050

TABLE 2. Computational results for Example 5.1 in $\mathbb{R}$.

N	Algorithm (3.1) $\|\|u_N\|\|_2$	Algorithm (3.1) Time (secs)
10	2.4900×10^{-9}	0.000017
50	2.1155×10^{-36}	0.000062
100	8.9778×10^{-69}	0.000121
150	7.7584×10^{-101}	0.000178
200	8.9533×10^{-133}	0.000240

TABLE 3. Computational results for Example 5.1 when $N = 100$.

Dim	Algorithm (3.20) $\|\|x_N\|\|_2$	Algorithm (3.20) Time (secs)	Algorithm (3.21) $\|\|x_N\|\|_2$	Algorithm (3.21) Time (secs)
1	8.0000×10^{-3}	0.000118	3.2480×10^{-8}	0.006067
10	2.5298×10^{-2}	0.001446	1.0271×10^{-7}	0.014259
100	8.0000×10^{-2}	0.003222	3.2480×10^{-7}	0.021392
1,000	2.5298×10^{-1}	0.022713	1.0271×10^{-6}	0.142859
10,000	8.0000×10^{-1}	0.440925	3.2480×10^{-6}	1.631194
100,000	2.5298×10^{0}	4.595126	1.0271×10^{-5}	18.86650

TABLE 4. Computational results for Example 5.1 when $N = 100$.

Dim	Algorithm (3.21) $\|\|x_N\|\|_2$	Algorithm (3.21) Time (secs)
1	3.2480×10^{-8}	0.000120
10	1.0271×10^{-7}	0.002884
100	3.2480×10^{-7}	0.006929
1,000	1.0271×10^{-6}	0.056250
10,000	3.2480×10^{-6}	0.829133
100,000	1.0271×10^{-5}	9.144194

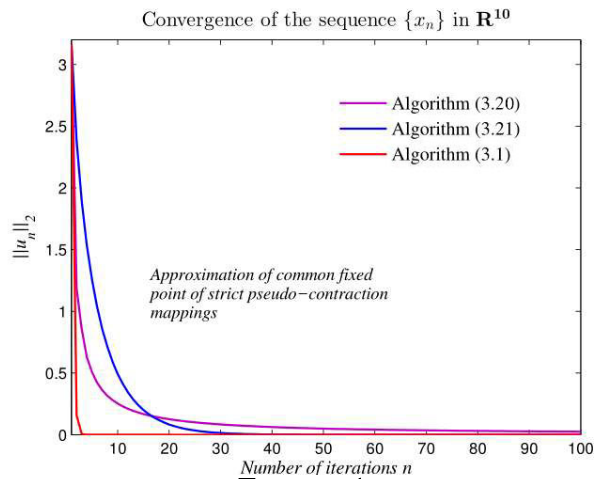

FIGURE 1

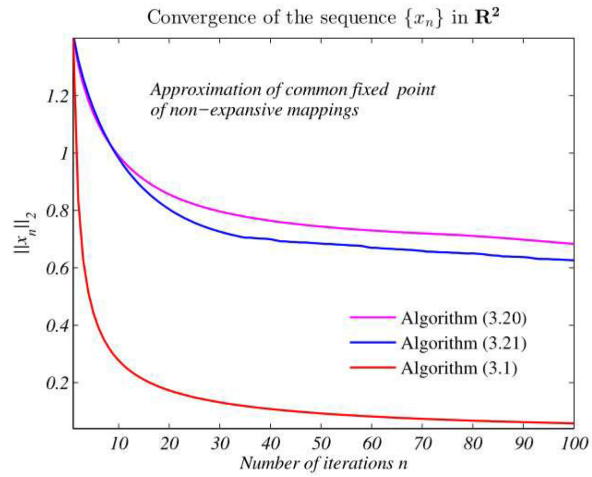

FIGURE 2

Example 5.2. Let N, M > 1 be arbitrary but fixed. Define the mappings $A, T_i, S_j : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$Au = \left(\frac{1}{2}(u_1 - u_2), u_1 - u_2\right), \quad A^*u = \left(\frac{1}{2}(u_1 + u_2), u_2\right), \quad T_i u = \sin u, \quad S_j u = P_{B_r[a]}u,$$

for all $u = (u_1, u_2) \in \mathbb{R}^2$, $i = 1, 2, \ldots, \mathrm{N}$, and $j = 1, 2, \ldots, \mathrm{M}$, where $P_{B_r(a)} : \mathbb{R}^2 \to B_r[a]$ is the metric projection defined by

$$P_{B_r(a)}(v) = \begin{cases} v, & \text{if } v \in B_r(a), \\ a + \dfrac{r}{\|v - a\|}(v - a), & \text{if } v \notin B_r(a), \end{cases} \tag{5.2}$$

where $B_r(a) = \{z \in \mathbb{R}^2 : ||z - a|| \leq r\}$. Clearly, $A^*u = \left(\frac{1}{2}(u_1 + u_2), u_2\right)$. For computational purposes, we choose $a = 0$ and $r = 1$. The map A is linear and bounded with $||A|| = \sqrt{2}$, while T_i and S_j are both nonexpansive. We set $\delta = \frac{1}{2||A||^2}$, $a_{i,n} = \frac{2i}{\mathrm{N}(\mathrm{N}+1)}$, $b_{j,n} = \frac{2j}{\mathrm{M}(\mathrm{M}+1)}$, $\beta_{i,n} = \frac{2n}{3n+1}$, and $\gamma_{j,n} = \frac{5n}{8n+1}$ for each i and j in all the algorithms; $\alpha_n = \frac{1}{n+1}$ in Algorithms (3.20) and (3.1). In Algorithm (3.20), $u = (1, 1) \in \mathbb{R}^2$; in Algorithm (3.1), $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $F(u) = \frac{1}{3}u$ for all $u \in \mathbb{R}^2$ while $\zeta_n = \frac{n}{5n+1}$ in Algorithm (3.21). It is clear that $\Omega := \cap_{i=1}^{\mathrm{N}} \mathrm{Fix}(T_i) \cap A^{-1}\left(\cap_{j=1}^{\mathrm{M}} \mathrm{Fix}(S_j)\right) = \{0\}$.

Tables 5 and 6 illustrate the computational results and the corresponding test time. It can be observed that, from the those tables and Figure 2, the viscosity algorithm (3.1) converges faster and spends a fairly lesser test period.

6. CONCLUSION

In this paper, we proposed the Halpern, viscosity, and hybrid-type algorithms for estimating the common fixed points of strict pseudo-contraction mappings in Hilbert spaces. The sequences generated by these algorithms were proven to converge strongly. Moreover, we present some applications of our main theorems as well as numerical examples to show the efficiency and accuracy of our algorithms. Furthermore, the hybrid-type algorithm consumed the most time due to the explicit projection's structure provided by the Haugazeau method; while the viscosity approach seems to better estimate the solutions when compared to the other two algorithms.

TABLE 5. Computational results for Example 5.2 in $\mathbb{R}^2$.

N	Algorithm (3.20)		Algorithm (3.21)	
	$\|\|x_N\|\|_2$	Time (secs)	$\|\|x_N\|\|_2$	Time (secs)
10	9.9018×10^{-1}	0.020598	9.8165×10^{-1}	0.021217
100	6.8344×10^{-1}	0.026126	6.2657×10^{-1}	0.037273
1,000	3.4863×10^{-1}	0.084964	3.6167×10^{-1}	0.223677
10,000	1.6735×10^{-1}	0.879659	1.7800×10^{-1}	2.229821
100,000	7.8845×10^{-2}	8.635996	8.4405×10^{-2}	22.24061

TABLE 6. Computational results for Example 5.2 in $\mathbb{R}^2$.

N	Algorithm (3.1)	
	$\|\|x_N\|\|_2$	Time (secs)
10	2.7715×10^{-1}	0.011828
100	5.8232×10^{-2}	0.022474
1,000	1.2371×10^{-2}	0.088361
10,000	2.6474×10^{-3}	0.860727
100,000	5.6860×10^{-4}	8.519460

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