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SECOND-ORDER OPTIMALITY CONDITIONS AND DUALITY FOR NONSMOOTH MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. We introduce new notions on second-order generalized convexity types, and establish second-order necessary and sufficient conditions for the local weak efficiency of nonsmooth multiobjective optimization problems involving inequality, equality and set constraints in terms of the Páles-Zeidan second-order directional derivatives. These notions of 2-generalized convexity are advantage for deriving optimality conditions in case that the set constraint is convex. Second-order duality theorems of Mond-Weir type are also given.

Keywords. Second-order necessary efficiency conditions; Páles-Zeidan second-order directional derivatives; First and second-order tangent vectors; Second-order constraint qualifications; 2-generalized convexity.

1. INTRODUCTION

In recent years, second-order necessary optimality conditions have been extensively investigated due to their enlargement upon first order necessary conditions. Recently, many results on second-order optimality conditions for optimization problems with $C^2, C^{1,1}, C^1$, and $C^{0,1}$ data were established; see, e.g., [1]-[16] and the references therein. Second-order optimality conditions were derived by Aghezzaf and Hachimi [1], Ben-Tal [2], and Jiménez and Novo [11] for optimization problems with twice differentiable functions, by Ginchev and Ivanov [6] for continuously differentiable scalar problems, and by Ivanov [9] for scalar problems involving Lipschitz second-order Hadamard differentiable functions, In 2015, Constantin [3] established second-order necessary conditions for the Lipschitz optimization problems with inequality constraints by using second-order directional derivatives. Recently, Luu [13] derived primal and dual second-order necessary conditions for the weak efficiency of nonsmooth vector equilibrium problems involving inequality, equality, and set constraints by using the Páles-Zeidan second order directional derivatives. Second-order Karush-Kuhn-Tucker necessary optimality conditions for weak efficiency were established under suitable second-order constraint qualifications.

The purpose of this paper is to develop new second-order necessary and sufficient optimality conditions for efficiency and some duality results for nonsmooth vector optimization problems involving inequality, equality, and set constraints. The paper is organized as follows. Some basic tools, such as, definitions, lemmas etc are presented in Section 2.

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Section 3 is devoted to developing second-order necessary conditions for local weak efficient solutions and efficient solutions of nonsmooth vector optimization problems involving inequality, equality, and set constraints. In Section 4, we introduce the notions of generalized convexity types involving 2-convexity, 2- qusiconvexity, and 2-pseudoconvexity. These notions are advantage for deriving optimality conditions in case that the set constraint is convex. This section also deals with the second-order sufficient conditions for weak efficiency with the assumptions on the 2-generalized convexity. Section 5, the last section, gives the second-order duality theorems of Mond–Weir type.

2. Preliminaries

Let f be a real-valued function defined on $\mathbb{R}^n$, which is locally Lipschitz at $\bar{x} \in \mathbb{R}^n$. Recall from [5] that the Clarke generalized derivative of f at $\bar{x} \in X$ in a direction $v \in X$ is defined as

$$f^0(\bar{x}; v) := \limsup_{x \to \bar{x}, t \downarrow 0} \frac{f(x+tv) - f(x)}{t}.$$

Following [17], the Páles–Zeidan second-order upper generalized directional derivative of f at $\bar{x}$ in the direction v is defined as

$$f^{00}(\bar{x}; v) := \limsup_{t \downarrow 0} 2\frac{f(\bar{x}+tv) - f(\bar{x}) - tf^0(\bar{x}; v)}{t^2}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be Fréchet differentiable at $\bar{x} \in X$, and let $\nabla f(\bar{x})$ be its gradient at $\bar{x}$. The following limit is called second-order directionally derivative at $\bar{x}$ in the direction $v \in \mathbb{R}^n$

$$f''(\bar{x}; v) := \lim_{t \downarrow 0} 2\frac{f(\bar{x}+tv) - f(\bar{x}) - t\nabla f(\bar{x})v}{t^2}.$$

Remark 2.1. (a) If f is continuously Fréchet differentiable near $\bar{x}$ with the Fréchet derivative $\nabla f(\bar{x})$, then f is Lipchitz near $\bar{x}$, and $f^0(\bar{x}; v) = \nabla f(\bar{x})v$ $(\forall v \in X)$. This is false, when $\nabla f(x)$ is not continuous at $\bar{x}$.

(b) If $f : \mathbb{R}^n \to \mathbb{R}$ is continuously Fréchet differentiable near $\bar{x}$ and second-order directionally differentiable at $\bar{x}$ in a direction $v \in X$, then (see [3])

$$f^{00}(\bar{x}; v) = f^{''}(\bar{x}; v).$$

Let f be a mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$. Recall that f is Gâteaux differentiable at $\bar{x}$ if there exists a continuous linear mapping Λ_1 from X into Y such that

$$f(\bar{x}+tv) = f(\bar{x}) + t\Lambda_1(v) + o(t) \ (\forall\, v \in \mathbb{R}^n),$$

where $\|o(t)\|/t \to 0$ as $t \to 0$. Λ_1 is said to be the Gâteaux derivative of f at $\bar{x}$ and is denoted by $f'_G(\bar{x})$. Note that a mapping, which is Gâteaux differentiable at $\bar{x}$, may be not continuous at $\bar{x}$.

Let us denote $(v)^2 := (v, v) \in \mathbb{R}^n \times \mathbb{R}^n$. The mapping f is twice Gâteaux differentiable at $\bar{x}$ if f is Gâteaux differentiable at $\bar{x}$ and there exist continuous multilinear symmetric mappings Λ_2 from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^m$ (continuous linear symmetric in 2 variables), such that

$$f(\bar{x}+tv) = f(\bar{x}) + t\Lambda_1(v) + \frac{t^2}{2!}\Lambda_2(v)^2 + o(t^2) \ (\forall v \in \mathbb{R}^n),$$

where $\Lambda_1 = f'_G(\bar{x}),\ \|o(t^2)\|/t^2 \to 0$ as $t \to 0$. Note that the symmetric means it does not change under permutation of variables. For the correctness of this definition, the symmetric multilinear mapping Λ_2 should be uniquely determined by the respective form $v \to \Lambda_2(v)^2$ (see [12]). The continuous multilinear symmetric mapping Λ_2 is the second-order Gâteaux derivative of f at $\bar{x}$, and is denoted by $f_G^{(2)}(\bar{x})$. Thus for a function f which is 2-times Gâteaux differentiable at $\bar{x}$, f can be expanded as

$$f(\bar{x}+tv) = f(\bar{x}) + tf'_G(\bar{x})(v) + \frac{t^2}{2!} f_G^{(2)}(\bar{x})(v)^2 + o(t^2) \ (\forall v \in \mathbb{R}^n).$$

In the case that f is twice Fréchet differentiable at $\bar{x}$, we have the following Taylor expansion:

$$f(\bar{x}+v) = f(\bar{x}) + \nabla f(\bar{x})(v) + \frac{1}{2}\nabla^2 f(\bar{x})(v)^2 + r(v),$$

where $\nabla f(\bar{x})$ is the first order Fréchet derivative of f at $\bar{x}$, $\nabla^2 f(\bar{x})(\bar{x})$ is the Hessian or second-order Fréchet derivative of f at $\bar{x}$, and $\|r(v)\|/\|v\|^2 \to 0$, as $v \to 0$. .

Let C be a nonempty subset of $\mathbb{R}^n$. Following [4], an element $u \in X$ is called a tangent vector to C at $\bar{x} \in \mathrm{cl}C$ iff

$$\lim_{t \downarrow 0} \frac{1}{t} d(\bar{x}+tu; C) = 0, \tag{2.1}$$

where $d(x;C)$ stands for the distance from x to C, and cl$C$ is the closure of C. The set of all tangent vectors to C at $\bar{x}$ is denoted by $T_{\bar{x}}C$, and is called the tangent cone to C at $\bar{x}$. Note that $T_{\bar{x}}C$ is a nonempty closed cone containing $0 \in X$. Moreover, (2.1) is equivalent to the existence of a function $\gamma : (0,+\infty) \to X$ with $\gamma(t) \to 0$ as $t \downarrow 0$, and

$$\bar{x} + t(u+\gamma(t)) \in C \quad (\forall t > 0).$$

Adapting the definition in [4], an element $v \in X$ is said to be a second-order tangent vector to C at $\bar{x}$ iff there exist $u \in X$ such that

$$\lim_{t \downarrow 0} \frac{1}{t^2} d(\bar{x}+tu+\frac{t^2}{2}v; C) = 0. \tag{2.2}$$

The vector u satisfying (2.2) is said to be associated with v. Denote by $T_{\bar{x}}^2 C$ the set of all second-order tangent vectors to C at $\bar{x}$. Observe that $v \in T_{\bar{x}}^2 C$ with the associated vector u is equivalent to the existence of a function $\gamma_1 : (0,+\infty) \to X$ with $\gamma_1(t) \to 0$ as $t \downarrow 0$, and

$$\bar{x} + tu + \frac{t^2}{2}(v+\gamma_1(t)) \in C \ (\forall t > 0).$$

Moreover, $v \in T_{\bar{x}}^2 C$ implies that $u \in T_{\bar{x}}C$, and $T_{\bar{x}}^2 C$ is a nonempty cone containig $0 \in X$ (see [4]).

Remark 2.2. It is easy to see that, for $\bar{x} \in C_1 \cap C_2$,
(i) $T_{\bar{x}}(C_1 \cap C_2) \subset T_{\bar{x}}C_1 \cap T_{\bar{x}}C_2$;
(ii) $T_{\bar{x}}^2(C_1 \cap C_2) \subset T_{\bar{x}}^2 C_1 \cap T_{\bar{x}}^2 C_2$.

The following lemma plays an important role in the proof of second-order necessary optimality conditions in Section 3 in the case that C is convex.

Lemma 2.1. *Assume that C is convex. Then, for $\bar{x} \in C$,*
a) $C - \bar{x} \subset T_{\bar{x}}C$;
b) $C - \bar{x} \subset T_{\bar{x}}^2 C$; for $x \in C$, $x - \bar{x} \in T_{\bar{x}}^2 C$ with the associated vector $x - \bar{x} \in T_{\bar{x}}C$.

Proof. a) Since C in convex, for $x \in C, t \in [0,1], \bar{x} + t(x-\bar{x}) \in C$. Hence, for $\gamma(t) \equiv 0$, one obtains $\bar{x} + t(x - \bar{x} + \gamma(t)) \in C$, which means that $x - \bar{x} \in T_{\bar{x}}C$.

b) For t small enough, $t_1 := t + \frac{t^2}{2} \in [0,1]$. For $\gamma_1(t) \equiv 0$, $\bar{x} + t(x-\bar{x}) + \frac{t^2}{2}(x - \bar{x} + \gamma_1(t)) \in C$, which means that $x - \bar{x} \in T^2_{\bar{x}}C$ with the associated vector $x - \bar{x} \in T_{\bar{x}}C$. $\square$

Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^r$. So, $f = (f_1, \ldots, f_r)$. Let $g := (g_1, \ldots, g_m) : \mathbb{R}^n \to \mathbb{R}^m$, $h := (h_1, \ldots, h_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$, and C a subset of $\mathbb{R}^n$. We denote $J := \{1, \ldots, r\}, I := \{1, \ldots, m\}, L := \{1, \ldots, \ell\}$.

We set

$$M := \Big\{ x \in C : \ g_i(x) \leqslant 0, i \in I, h_j(x) = 0, j \in L := \{1, \ldots, \ell\} \Big\}.$$

Let us consider the following vector optimization problem:

$$\min\{f(x) : \ x \in M\}. \tag{MP}$$

To derive second-order necessary conditions for (MP), we introduce the following assumptions.

Assumption 2.1. a) Functions $f_k\,(k \in J)$ and $g_k(k \in I)$ are locally Lipschitz, Gâteaux differentiable, and regular in the Clarke sense at $\bar{x} \in C$ with Gâteaux derivative $f'_{G,k}(\bar{x})$ and $g'_{G,i}(\bar{x})$, respectively;

b) $h_j\,(j \in L)$ are twice continuouosly Fréchet diferentiable at $\bar{x}$ with Hessian $\nabla^2 h_j(\bar{x})$ (h is of the class C^2).

A vector $\bar{x}$ is called a local weak efficient solution (local efficient solution) iff there is a number $\delta > 0$ such that there is no $x \in M \cap B(\bar{x};\delta)$ such that

$$f_k(x) < f_k(\bar{x}) \quad (\forall k \in J),$$

$$\big(\text{resp. } f_k(x) \leq f_k(\bar{x})\,(\forall k \in J),\ f_s(x) < f_s(\bar{x}) \ \text{at least one}\, s \in J\big),$$

where $B(\bar{x};\delta)$ stands for the open ball of radius δ around $\bar{x}$.

We set

$$\begin{aligned} &I(\bar{x}) := \{i \in I : g_i(\bar{x}) = 0\}, \\ &H := \{x \in X : \ h(x) = 0\}, \\ &I_0(\bar{x};u) := \{i \in I(\bar{x}) : g'_{G,i}(\bar{x})u = 0\}. \end{aligned}$$

Adapting the definition in [8], a vector $u \in X$ is called critical at $\bar{x} \in M$ iff

$$\begin{aligned} &f'_{G,k}(\bar{x})u \leqslant 0 \quad (\forall k \in J), \\ &g'_{G,i}(\bar{x})u \leqslant 0 \quad (\forall i \in I(\bar{x})), \\ &\nabla h_j(\bar{x})u = 0 \quad (\forall j \in L), \\ &u \in T_{\bar{x}}C. \end{aligned}$$

3. Second-order Necessary Optimality Conditions

From [16, Theorem 4.1], we can obtain the following second-order Karush-Kuhn-Tucker necessary conditions for efficiency.

Theorem 3.1. (Second-order necessary optimality condition) *Let $\bar{x} \in M$ be a local efficient solution of (MP). Assume that Assumption 2.1 is fulfilled, C is convex, $(C-\bar{x}) \cap Ker\nabla h(\bar{x}) \neq \emptyset$, and the following regularity condition (RC) holds*

$$(RC) \qquad \sum_{j\in L} \nu_j \nabla h_j(\bar{x})v \geqslant 0 \; (\forall v \in T^2_{\bar{x}}C) \Longrightarrow \nu_1 = \cdots = \nu_\ell = 0, \tag{RC}$$

and the following constraint qualification (CQ1) holds: there exist $(u_0, v_0) \in Ker\nabla h(\bar{x}) \times T^2_{\bar{x}}C$ such that

$$\begin{aligned} & f'_{G,s}(\bar{x})v_0 + \frac{1}{2} f_k^{00}(\bar{x}; u_0) < 0 \; (\forall k \in J, k \neq s), \\ & g'_{G,i}(\bar{x})v_0 + \frac{1}{2} g_i^{00}(\bar{x}, u_0) < 0 \; (\forall i \in I_0(\bar{x}; u_0)), \\ & \nabla h_j(\bar{x})v_0 + \frac{1}{2}\nabla^2 h_j(\bar{x})(u_0, u_0) = 0 \; (\forall j \in L). \end{aligned} \tag{CQ1}$$

Then, for every $x \in \bar{x} + (C-\bar{x}) \cap \nabla h(\bar{x})$, there exist $\lambda_k \geq 0\, (\forall k \in J), \lambda \neq 0$, $\mu_i \geq 0\, (i \in I)$ and $\nu_j\; (j \in L)$ such that

$$\begin{aligned} & f'_{G,s}(\bar{x})(x-\bar{x}) + \sum_{k\in J; k\neq s} \lambda_k f'_{G,k}(\bar{x})(x-\bar{x}) + \sum_{i\in I(\bar{x})} \mu_i g'_{G,i}(\bar{x})(x-\bar{x}) \\ & + \sum_{j\in L} \nu_j \nabla h_j(\bar{x})(x-\bar{x}) + \frac{1}{2} f_s^{00}(\bar{x}; x-\bar{x}) + \frac{1}{2} \sum_{k\in J; k\neq s} \lambda_k f_k^{00}(\bar{x}; x-\bar{x}) \\ & + \frac{1}{2} \sum_{i\in I(\bar{x})} \mu_i g_i^{00}(\bar{x}; x-\bar{x}) + \frac{1}{2} \sum_{j\in L} \nu_j \nabla^2 h_j(\bar{x})(x-\bar{x}, x-\bar{x}) \geqslant 0, \end{aligned}$$

$$\begin{aligned} & \mu_i g_i(\bar{x}) = 0 \; (\forall i \in I), \\ & \mu_i g'_{G,i}(\bar{x})(x-\bar{x}) = 0 \; (\forall i \in I(\bar{x})), \\ & \lambda_k f'_{G,k}(\bar{x})(x-x_0) = 0\, (\forall k \in J, k \neq s). \end{aligned}$$

Proof. We first observe that $h_j\, (j \in L)$ is twice continuously Fréchet differentiable at $\bar{x}$. One has $\nabla^2 h_j(\bar{x})(u,u) = h''(\bar{x}, u)$. Since $\bar{x}$ is a local efficient solution of (MP), for $s \in J$, it is a local solution of the following scalar problem:

$$\begin{aligned} & \min f_s(x), \\ \text{s.t.} \quad & f_k(x) \leq f_s(\bar{x})\, (k \in J, k \neq s), \\ & g_i(x) \leq 0\, (i \in I), \\ & h_j(x) = 0\, (j \in L), \\ & x \in C. \end{aligned}$$

We invoke [16, Theorem 4.1] to deduce that, for every critical direction $u \in (C-\bar{x}) \cap \mathrm{Ker}\nabla h(\bar{x})$, there exist $\lambda_k \geq 0\; (\forall k \in J), \lambda \neq 0$, $\mu_i \geq 0\, (i \in I)$, and $\nu_j \in \mathbb{R}\, (j \in L)$ such that

$$f'_{G,k}(\bar{x})v + \sum_{k\in J; k\neq s} \lambda_k f'_{G,k}(\bar{x})v + \sum_{i\in I(\bar{x})} \mu_i g'_{G,i}(\bar{x})v + \sum_{j\in L} \nu_j \nabla h_j(\bar{x})v \geqslant 0 \; (\forall v \in T^2_{\bar{x}}C), \tag{3.1}$$

$$\frac{1}{2} f_s^{00}(\bar{x};; u) + \frac{1}{2} \sum_{k\in J; k\neq s} \lambda_k f_k^{00}(\bar{x}; u) + \frac{1}{2} \sum_{i\in I(\bar{x})} \mu_i g_i^{00}(\bar{x}; u) + \frac{1}{2} \sum_{j\in L} \nu_j \nabla^2 h_j(\bar{x})(u,u) \geqslant 0, \tag{3.2}$$

$$\mu_i g_i(\bar{x}) = 0 \ (\forall i \in I), \tag{3.3}$$

$$\mu_i g'_{G,i}(\bar{x})u = 0 \ (\forall i \in I(\bar{x})), \tag{3.4}$$

and

$$\mu_i g'_{G,i}(\bar{x})(x-\bar{x}) = 0 \ (\forall i \in I(\bar{x})). \tag{3.5}$$

From Lemma 2.1, $C-\bar{x} \subset T_{\bar{x}}C \cap T^2_{\bar{x}}C$. Hence, for every critical direction $u \in (C-\bar{x}) \cap \mathrm{Ker}\nabla h(\bar{x})$, (3.1)-(3.5) hold. From Lemma 2.1, for $x \in C, x-\bar{x} \in T^2_{\bar{x}}C$ with the associated vector $x-\bar{x}$, and (3.1)-(3.5) hold with $u = v = x-\bar{x}$. The conclusion follows. □

We can also prove another second-order Karush-Kuhn-Tucker necessary conditions for weak efficiency.

Theorem 3.2. (Second-order necessary optimality condition) *Let $\bar{x} \in M$ be a local weak efficient solution of (MP). Assume that Assumption 2.1 is fulfilled, C is convex, and the regularity condition (RC) holds, and the constraint qualification (CQ2) holds:*

$$\begin{aligned} &g'_{G,i}(\bar{x})v_0 + \frac{1}{2}g_i^{00}(\bar{x},u_0) < 0 \ (\forall i \in I_0(\bar{x};u_0)), \\ &\nabla h_j(\bar{x})v_0 + \frac{1}{2}\nabla^2 h_j(\bar{x})(u_0,u_0) = 0 \ (\forall j \in L). \end{aligned} \tag{CQ2}$$

Suppose that $(C-\bar{x}) \cap Ker\nabla h(\bar{x}) \neq \emptyset$. Then, for every $x \in \bar{x} + (C-\bar{x}) \cap Ker\nabla h(\bar{x})$, there exist $\lambda_k \geq 0 \ (\forall k \in J), \lambda \neq 0$, $\mu_i \geq 0 \, (i \in I)$, and $\nu_j \in \mathbb{R} \, (j \in L)$ such that

$$\begin{aligned} &\sum_{k\in J} \lambda_k f'_{G,k}(\bar{x})(x-\bar{x}) + \sum_{i\in I(\bar{x})} \mu_i \nabla g_i(\bar{x})(x-\bar{x}) \\ &+ \sum_{j\in L} \nu_j \nabla h_j(\bar{x})(x-\bar{x}) + \frac{1}{2}\sum_{k\in J} \lambda_k f_k^{00}(\bar{x};x-\bar{x}) + \frac{1}{2}\sum_{i\in I(\bar{x})} \mu_i g_i^{00}(\bar{x};x-\bar{x}) \\ &+ \frac{1}{2}\sum_{j\in L} \nu_j \nabla^2 h_j(\bar{x})(x-\bar{x},x-\bar{x}) \geqslant 0, \\ &\qquad \mu_i g_i(\bar{x}) = 0 \ (\forall i \in I), \\ &\qquad \mu_i g'_{G,i}(\bar{x})(x-\bar{x}) = 0 \ (\forall i \in I(\bar{x})). \end{aligned}$$

Proof. It follows from Remark 2.1 that h_j, where $j \in L$, is twice continuously Fréchet differentiable at $\bar{x}$. One has $\nabla^2 h_j(\bar{x})(u,u) = h''(\bar{x},u)$. Since $\bar{x}$ is a local weak efficient solution of (MP), we invoke [16, Theorem 4.1] to deduce that, for every critical direction $u \in \mathrm{Ker}\nabla h(\bar{x})$, there exist $\lambda_k \geq 0 \ (\forall k \in J), \lambda \neq 0$, $\mu_i \geq 0 \, (i \in I)$, and $\nu_j \in \mathbb{R} \, (j \in L)$ such that

$$\sum_{k\in J} \lambda_k f'_{G,k}(\bar{x})v + \sum_{i\in I(\bar{x})} \mu_i g'_{G,i}(\bar{x})v + \sum_{j\in L} \nu_j \nabla h_j(\bar{x})v \geqslant 0 \ (\forall v \in T^2_{\bar{x}}C), \tag{3.6}$$

$$\frac{1}{2}\sum_{k\in J} \lambda_k f_k^{00}(\bar{x};u) + \frac{1}{2}\sum_{i\in I(\bar{x})} \mu_i g_i^{00}(\bar{x};u) + \frac{1}{2}\sum_{j\in L} \nu_j \nabla^2 h_j(\bar{x})(u,u) \geqslant 0, \tag{3.7}$$

$$\mu_i g_i(\bar{x}) = 0 \ (\forall i \in I), \tag{3.8}$$

and

$$\mu_i g'_{G,i}(\bar{x})u = 0 \ (\forall i \in I(\bar{x})). \tag{3.9}$$

By Lemma 2.1, $C-\bar{x}\subset T_{\bar{x}}C\cap T^2_{\bar{x}}C$. Hence, for every critical direction $u\in(C-\bar{x})\cap \mathrm{Ker}\nabla h(\bar{x})$, (3.6)-(3.9) hold. Also by Lemma 2.1, for $x\in C$, $x-\bar{x}\in T^2_{\bar{x}}C$ with the associated vector $x-\bar{x}$, and (3.6)-(3.9) hold with $u=v=x-\bar{x}$. The conclusion follows. □

4. Second-order Sufficient Optimality Conditions

To derive second-order sufficient optimality conditions for (MP), we introduce some notions ofthe 2-generalized convexity (second-order generalized convex types. Let C be a subset of $\mathbb{R}^n$ and $\bar{x}\in C$.

Definition 4.1. (i) A function $f:\mathbb{R}^n\to\mathbb{R}$, which is locally Lipschitz and Gâteaux differentiable at $\bar{x}$, is called 2-convex (second-order convex type) at $\bar{x}$ on $C\subset\mathbb{R}^n$ iff, for all $x\in C$,

$$f(x)-f(\bar{x})\geq f'_G(\bar{x})(x-\bar{x})+\frac{1}{2}f^{00}(\bar{x};x-\bar{x});$$

(ii) The function f is called 2-pseudoconvex (second-order pseudoconvex type) at $\bar{x}$ on $C\subset\mathbb{R}^n$ iff, for all $x\in C$, the following implications hold:

$$f'_G(\bar{x})(x-\bar{x})+\frac{1}{2}f^{00}(\bar{x};x-\bar{x})\geq 0\Longrightarrow f(x)\geq f(\bar{x}).$$

(iii) The function f is called 2-quasiconvex (second-order quasiconvex type) at $\bar{x}$ on C iff, for all $x\in C$, the following implication hold:

$$f(x)\leq f(\bar{x})\Longrightarrow f'_G(\bar{x})(x-\bar{x})+\frac{1}{2}f^{00}(\bar{x};x-\bar{x})\leq 0.$$

Remark 4.1. (a) It should be noted here that a 2-convex function is also 2-quasiconvex and 2-pseudoconvex.

(b) From [16, Remark 2.1 (b)], for $x\in C$, $f^{00}(\bar{x};x-\bar{x})=f''(\bar{x};x-\bar{x})$. If $f''(\bar{x};x-\bar{x})\geq 0$, $\forall x\in C$, then the fact that f is 2-convex at $\bar{x}$ on C implies that it is convex at $\bar{x}$ on C.

In fact, since f is 2-convex at $\bar{x}$ on C,

$$f(x)-f(\bar{x})\geq\nabla f(\bar{x})(x-\bar{x})+f''(\bar{x};x-\bar{x})\geq\nabla f(\bar{x})(x-\bar{x}).$$

Hence, f is convex at $\bar{x}$ on C.

(c) Analogously, in the case that f is continuously twice Fréchet differentiable near $\bar{x}$ and second-order directionally differentiable at $\bar{x}$, if $f''(\bar{x};x-\bar{x})\leq 0$, $\forall x\in C$, then if f is convex at $\bar{x}$ on C implies that it is 2-convex at $\bar{x}$ on C.

Remark 4.2. Recall that Ginchev and Ivanov [6] defined: For a given function $f:\mathbb{R}^n\to\mathbb{R}$, which is Fréchet differentiable at $\bar{x}$ and second-order directionally differentiable at $\bar{x}$ in every direction $x-\bar{x}$ such that $f(x)<f(\bar{x}),\nabla f(\bar{x})(x-\bar{x})=0$, f is said to be second-order pseudoconvex at $\bar{x}$ on $C\subset\mathbb{R}^n$ iff, for all $x\in C$, the following implications hold:

$$(a)\ f(x)<f(\bar{x})\Longrightarrow\nabla f(\bar{x})(x-\bar{x})\leq 0;$$

$$(b)\ f(x)<f(\bar{x})\,\text{and}\,\nabla f(\bar{x})(x-\bar{x})=0\Longrightarrow f''(\bar{x};x-\bar{x})<0.$$

Note that if f is second-order pseudoconvex in the sense of Ginchev and Ivanov [6], then it is 2-pseudoconvex in the sense of our Definition 4.1 (ii).

From Definition 4.1, we have the following remark.

Remark 4.3. (a) If, for all $x \in C$, the following implication holds

$$f'_G(\bar{x}) = 0, \text{and } f^{00}(\bar{x}; x - \bar{x}) \geq 0 \Longrightarrow f(x) \geq f(\bar{x}),$$

then f is 2-pseudoconvex at $\bar{x}$ on C in the sense of Definition 4.1 (i).

(b) If, for all $x \in C$, the following implication holds

$$f(x) \leq f(\bar{x}) \text{ and } f'_G(\bar{x})(x - \bar{x}) = 0 \Longrightarrow f^{00}(\bar{x}; x - \bar{x}) \leq 0.$$

then f is 2-quasiconvex at $\bar{x}$ on C in the sense of Definition 4.1 (ii).

The following examples illustrate Remarks 4.1 (a), and 4.3 (a).

Example 4.1. Let $f(x) = x^2, \bar{x} = 0$, $f'_G(0) = \nabla f(0) = 0$, and $f^{00}(0; x) = \nabla^2 f(0)x^2 = 2x^2$. Then f is 2-convex at $\bar{x} = 0$. We also have $f^{00}(0; x) = 2x^2 \geq 0, \forall x \in \mathbb{R}$. Hence, f is convex at $\bar{x} = 0$ on $\mathbb{R}$.

Example 4.2. Give a function f on $\mathbb{R}$: $f(x) = (x+1)^2, \bar{x} = -1, C = [-2, 2]$. We have $f'_G(-1) = \nabla f(-1) = 0$. $f^{00}(-1; x+1) = f''(-1; x+1) = 2(x+1)^2 \geq 0$, $\forall x \in C$. By Remark 4.1 (a), f is 2-pseudoconvex at $\bar{x} = -1$ on C.

The following example illustrates Remark 4.3 (b).

Example 4.3. Give a function f on $\mathbb{R}$: $f(x) = -(x-1)^2, \bar{x} = 1, C = [-2, 2]$. We have $f'_G(1) = \nabla f(1) = 0$. $f(x) = -(x-1)^2 \leq f(1) = 0$. $f^{00}(1; x-1) = f''(1; x-1) = -2 < 0$. By Remark 4.3 (b), f is 2-quasiconvex at $\bar{x} = 1$ on C.

We now introduce the notion of strictly 2-pseudoconvex (second-order strict pseudoconvex type).

Definition 4.2. A function $f : \mathbb{R}^n \to \mathbb{R}$, which is locally Lipschitz and Gâteaux differentiable at $\bar{x}$, is called strictly 2-pseudoconvex function at $\bar{x}$ on $C \subset \mathbb{R}^n$ iff, for all $x \in C, x \neq \bar{x}$, the following implications hold:

$$f'_G(\bar{x})(x - \bar{x}) + \frac{1}{2} f^{00}(\bar{x}; x - \bar{x}) \geq 0 \Longrightarrow f(x) > f(\bar{x}).$$

Example 4.4. Give a function $f(x) = |x|, \bar{x} = 0, C = [-1, 1]$. Then $f^0(0; x - \bar{x}) = f'(0)x = |x| (\forall x \in \mathbb{R})$. $f^{00}(0; x - \bar{x}) = 0 (\forall x \in \mathbb{R})$. We also have $f(x) = |x| > 0 = f(\bar{x}) (\forall x \neq 0)$. Thus f is strictly 2-pseudoconvex at $\bar{x}$ on C.

Recall that a function f is said to be pseudoconvex at $\bar{x}$ iff

$$x \in \mathbb{R}^n, \nabla f(\bar{x})(x - \bar{x}) \geq 0 \Longrightarrow f(x) \geq f(\bar{x}).$$

By [9], every Fréchet differentiable pseudoconvex function is second-order pseudoconvex in the sense of Ginchev-Ivanov [6]. The converse does not holds in general.

Example 4.5. [9] Consider the following function

$$f(x) = \begin{cases} x^2, & x \geq 0, \\ -x^2, & x < 0, \end{cases}$$

This function is not pseudoconvex at $x = 0$. But it is second-order pseudoconvex at $x = 0$.

We now can formulate a second-order sufficient condition for weakly efficiency of (MP). We can see that the generalized convexity is advantage for establishing second-order optimality conditions for optimizations where the set constraint is convex.

Theorem 4.1. *Let $\bar{x} \in M$. Suppose that $g_,(i \in I(\bar{x})), f_k(k \in J, k \neq s), \pm h_j(j \in L)$ are 2-quasiconvex at $\bar{x}$ on M, C is convex, and $(C - \bar{x}) \cap Ker \nabla h(\bar{x}) \neq \emptyset$. Assume that, for every $x \in C$, there exist $\lambda_k \geq 0\, (\forall k \in J), \lambda \neq 0$, $\mu_i \geq 0\, (i \in I)$, and ν_j $(j \in L)$ such that, for every $x \in M$,*

$$\begin{aligned}
&\sum_{k \in J} \lambda_k f'_{G,k}(\bar{x})(x - \bar{x}) + \sum_{i \in I(\bar{x})} \mu_i g'_{G,i}(\bar{x})(x - \bar{x}) + \sum_{j \in L} \nu_j \nabla h_j(\bar{x})(x - \bar{x}) \\
&+ \frac{1}{2} \sum_{k \in J} \lambda_k f_k^{00}(\bar{x}; x - \bar{x})) + \frac{1}{2} \sum_{i \in I(\bar{x})} \mu_i g_i^{00}(\bar{x}; x - \bar{x})) \\
&+ \frac{1}{2} \sum_{j \in L} \nu_j \nabla^2 h_j(\bar{x})(x - \bar{x}, x - \bar{x}) \geqslant 0,
\end{aligned} \tag{4.1}$$

Moreover, suppose that λf is 2-pseudoconvex at $\bar{x}$ on M. Then $\bar{x}$ is a weakly efficient solution of (MP).

Proof. Denote $\lambda := (\lambda_, \dots, \lambda_r)$ and $f'_G(\bar{x}) := (f'_{G,1}(\bar{x}), \dots, f'_{G,r}(\bar{x}), f^{00} := (f_1^{00}, \dots,\ f_r^{00})$. For each $i \in I(\bar{x})$, we have $g_i(x) \leq 0 = g_i(\bar{x})$, $\forall x \in M$. By virtue of the 2-quasiconvexity of g at $\bar{x}$ on M, it follows that

$$g'_{G,i}(x - \bar{x}) + \frac{1}{2} g_i^{00}(\bar{x}; x - \bar{x}) \leq 0. \tag{4.2}$$

In view of the 2-quasiconvexity of $\pm h_j$ at $\bar{x}$ on M, it holds that

$$\nabla h(\bar{x})(x - \bar{x}) + \frac{1}{2} \nabla^2 h_j(\bar{x})(x - \bar{x}, x - \bar{x}) = 0,\ \forall x \in M. \tag{4.3}$$

From (4.1)-(4.3), we conclude that

$$\lambda f'_G(\bar{x})(x - \bar{x}) + \frac{1}{2} \lambda f^{00}(\bar{x}; x - \bar{x}) \geq 0,\ \forall x \in M.$$

In view of the second-order pseudoconvexity of λf, we obtain that

$$\lambda f(x) \geq \lambda f(\bar{x}),\ \forall x \in M.$$

Thus, $\bar{x}$ is a solution of the following problem

$$\begin{aligned}
(P_1) \qquad & \min \lambda f(x), \\
\text{s.t.} \quad & g_i(x) \leq 0\, (i \in I), \\
& h_j(x) = 0\, (j \in L), \\
& x \in C.
\end{aligned}$$

Note that the feasible set of (P_1) is M. Hence, $\bar{x}$ is a weak solution of problem (MP). The proof is complete. □

From Theorem 4.1, we obtain the following consequence.

Corollary 4.1. *Let $\bar{x} \in M$. Suppose that $g_i(i \in I(\bar{x})), f_k(k \in J), \pm h_j(j \in L)$ are 2-convex at $\bar{x}$ on M, C is convex, and $(C - \bar{x}) \cap Ker \nabla h(\bar{x}) \neq \emptyset$. Assume that, for every $x \in C$, there exist $\lambda_k \geq 0\, (\forall k \in J), \lambda \neq 0$, $\mu_i \geq 0\, (i \in I)$, and $\nu_j\ (j \in L)$ such that*

$$\sum_{k \in J} \lambda_k f'_{G,k}(\bar{x})(x - \bar{x}) + \sum_{i \in I(\bar{x})} \mu_i g'_{G,i}(\bar{x})(x - \bar{x}) + \sum_{j \in L} \nu_j \nabla h_j(\bar{x})(x - \bar{x})$$
$$+ \frac{1}{2} \sum_{k \in J} \lambda_k f_k^{00}(\bar{x}; x - \bar{x})) + \frac{1}{2} \sum_{i \in I(\bar{x})} \mu_i g_i^{00}(\bar{x}; x - \bar{x}))$$
$$+ \frac{1}{2} \sum_{j \in L} \nu_j \nabla^2 h_j(\bar{x})(x - \bar{x}, x - \bar{x}) \geqslant 0,$$

Moreover, suppose that λf is 2- pseudoconvex at $\bar{x}$ on M. Then $\bar{x}$ is a weakly efficient solution of (MP).

Proof. Since f_k, $\forall k \in J$, is second-order convex at $\bar{x}$ on M and $\lambda_k \geq 0$, $\forall k \in J$, λf also is 2-convex at $\bar{x}$ on M. Hence, λf is second-order pseudoconvex at $\bar{x}$ on M. Since g_i, $\forall i \in I(\bar{x})$, and $\pm h_j$, $\forall j \in L$ are 2-convex at $\bar{x}$ on M, they are 2-quasiconvex at $\bar{x}$ on M. Thus all hypotheses of Theorem 4.1 are fulfilled. From Theorem 4.1, we have the desired conclusion immediately. □

By an argument analogous to that used for the proof of Theorem 4.1, we have the following theorem.

Theorem 4.2. *Let $\bar{x} \in M$. Suppose that, for some $s \in J$, f_s is second-order pseudoconvex at $\bar{x}$ on M, g_i, $i \in I(\bar{x})$, $f_k, k \in J, k \neq s, \pm h_j, j \in L$ are 2-quasiconvex at $\bar{x}$ on M, C is convex, and $(C - \bar{x}) \cap Ker \nabla h(\bar{x}) \neq \emptyset$. Assume that, for every $x \in \bar{x} + (C - \bar{x}) \cap Ker \nabla h(\bar{x})$, there exist $\mu_i \geq 0\, (i \in I)$ and $\nu_j\ (j \in L)$ such that*

$$f'_{G,s}(\bar{x})(x - \bar{x}) + \sum_{i \in I(\bar{x})} \mu_i g'_{G,i}(\bar{x})(x - \bar{x}) + \sum_{j \in L} \nu_j \nabla h_j(\bar{x})(\bar{x})$$
$$+ \frac{1}{2} f_s^{00}(\bar{x}; x - \bar{x}) + \frac{1}{2} \sum_{i \in I(\bar{x})} \mu_i g_i^{00}(\bar{x}; x - \bar{x}) + \frac{1}{2} \sum_{j \in L} \nu_j \nabla^2 h_j(\bar{x})(x - \bar{x}, x - \bar{x}) \geqslant 0,$$

Then $\bar{x}$ is a weakly efficient solution of (MP).

Theorem 4.2 can be illustrated by the following example.

Example 4.6. Given the following functions and sets

$$f = (f_1, f_2), f_1(x) = |x|, f_2(x) = -x^3\, (x \in \mathbb{R}), C = [0, \pi].$$
$$g(x) = x(x-1), h(x) = \sin 10x, \bar{x} = 0.$$

Then

$$f'_{G,1}(\bar{x})(x - \bar{x}) = |x|, f_1^{00}(0; x - \bar{x}) = 0;$$
$$f'_{G,2}(\bar{x})(x - \bar{x}) = 0, f_2^{00}(\bar{x}; x - \bar{x}) = 0;$$
$$g'_G(\bar{x})(x - \bar{x}) = -x, g^{00}(\bar{x}; x - \bar{x}) = 2x^2;$$
$$\nabla h(\bar{x})(x - \bar{x}) = x, h^{00}(\bar{x}; x - \bar{x}) = 0.$$

Consider the multiobjective optimization problem:

$$(P_2) \qquad \min f(x)\ s.t.\ g(x) \leq 0, h(x) = 0, x \in C.$$

Then the feasible set of this problem is as: $M = \{0, \frac{\pi}{10}, \frac{2\pi}{10}, \dots, \pi\}$. For $\lambda_1 = 1, \lambda_2 = 0, \mu = \nu = 1$, condition (14) in Theorem 4.1 holds. It is easy to see that for $\lambda = (1,0)$, λf is 2-pseudoconvex at $\bar{x} = 0$ on M, g and $\pm h$ are second-order quasiconvex at $\bar{x} = 0$ on M. Thus all hypotheses of Theorem 4.1 are fulfilled to Problem (P_2). Applying this theorem yields that $\bar{x} = 0$ is a weakly efficient solution of (P_2).

5. Second-Order Duality

We consider the following second-order dual problem of Mond-Weir type (MWD):

$$\begin{aligned}
&\max f(u) := (f_1(u), \dots, f_r(u)),\\
&\text{s.t.} \sum_{k\in J} f'_{G,k}(u)(x-u) + \sum_{i\in I(u)} \mu_i g'_{G,i}(u)(x-u) + \sum_{j\in L} \nu_j \nabla h_j(u)(x-u)\\
&+\frac{1}{2}\sum_{k\in J} \lambda_k f_k^{00}(\bar{x}; x-u) + \frac{1}{2}\sum_{i\in I(u)} \mu_i g_i^{00}(u; x-u)\\
&+\frac{1}{2}\sum_{j\in L} \nu_j \nabla^2 h_j(u)(x-u, x-u) \geqslant 0,\\
&\qquad h(u) = 0, x \in C; \quad \mu_i \geq 0,\ i \in I.
\end{aligned} \tag{MWD}$$

Denote by M_1 the feasible set of (MWD).

To derive duality theorems, we introduce the following assumption.

Assumption 5.1. a) Functions $f_k\,(k \in J := \{1, \dots, r\}; g_i\,(i \in I(\bar{x})$ are locally Lipschitz, Gâteaux differentiable, and regular in the Clarke sense at $u \in C$ with Gâteaux derivative $f'_{G,k}(u), g'_{G,i}(u)$, respectively;

b) $h_j\,(j \in L)$ are twice continuouosly Fréchet diferentiable at u with Hessian $\nabla^2 h_j(u)$ (h is of the class C^2).

We now can state a second-order weak duality theorem for (MP) and (MWD).

Theorem 5.1. (Weak Duality) *Let x and (u, λ, μ, ν) be the feasible points for Problems (MP) and (MWD), respectively. Assume that Assumption 5.1 is fulfilled, and λf is 2-pseudoconvex at u on C, $g_i(i \in I(u)), \pm h_j(j \in L)$ are 2-quasiconvex at u on M, C is convex, and $(C - \bar{x}) \cap \mathrm{Ker}\nabla h(u) \neq \emptyset$. Suppose that h is a function of the class C^2 in a neighborhood of u. Then $f(x) \not< f(u)$.*

Proof. For every $i \in I(u), g_i(x) \leq 0 = g_i(u), \forall x \in M$. By the 2-quasiconvexity of g_i at u on M, one obtains

$$g'_{G,i}(x-u) + \frac{1}{2} g_i^{00}(u; x-u) \leq 0\,(\forall x \in M). \tag{5.1}$$

Thanks to the 2-quasiconvexity of $\pm h_j$ at u on M, we deduce that

$$\nabla h_j(u)(x-u) + \frac{1}{2}\nabla^2 h(u)(x-u, x-u) = 0\,(\forall x \in M),$$

which together with (5.1) yields that

$$\lambda f'_G(u)(x-u) + \frac{1}{2}\lambda f^{00}(u; x-u) \geq 0, \forall x \in M.$$

Thanks to the 2-pseudoconvexity of λf at u on M, we arrive at $\lambda f(x) \geq \lambda f(u)$, $\forall x \in M$. Hence, u is a solution of the following problem

$$\begin{aligned} & \min \lambda f(x), \\ \text{s.t.} \quad & g_i(x) \leq 0\,(i \in I), \\ & h_j(x) = 0\,(j \in L), \\ & x \in C. \end{aligned}$$

Hence, $f(x) \not< f(u)\,(\forall x \in M)$. The proof is complete. □

We give a strong duality theorem for (MP) and (MWD).

Theorem 5.2. (Strong duality) *Let $\bar{x}$ be a local weak efficient solution of (MP). Assume that Assumption 5.1 holds, and all hypotheses of Theorem 3.2 are fulfilled. Then there exist $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \bar{\lambda}_r)$, $\bar{\lambda} \neq 0$, $\bar{\mu} := (\bar{\mu}_1, \ldots, \bar{\mu}_r), \bar{\mu}_i \geqslant 0\ (\forall i \in I), \bar{\nu} := (\bar{\nu}_1, \ldots, \bar{\nu}_\ell), \bar{\nu}_j \in \mathbb{R}\ (\forall j \in L)$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$ is a feasible point of (MWD), and the value of the objective functions of (MP) and (MWD) at $\bar{x}$ and $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$, respectively, are equal. Moreover, if the assumptions of Theorem 5.1 hold, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$ is a weakly efficient solution of (MWD).*

Proof. Since $\bar{x}$ is a local weakly efficient solution of (MP), we can invoke Theorem 3.2 to deduce that there exist $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \bar{\lambda}_r), \bar{\lambda} \neq 0, \bar{\mu} := (\bar{\mu}_1, \ldots, \bar{\mu}_r), \bar{\mu}_i \geqslant 0\,(\forall i \in I), \bar{\nu} := (\bar{\nu}_1, \ldots, \bar{\nu}_\ell), \bar{\nu}_j \in \mathbb{R}$ such that

$$\begin{aligned} & \sum_{k \in J} \bar{\lambda}_k f'_{G,k}(\bar{x})(x - \bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i g'_{G,i}(\bar{x})(x - \bar{x}) + \sum_{j \in L} \bar{\nu}_j \nabla h_j(\bar{x})(x - \bar{x}) \\ & + \frac{1}{2} \sum_{k \in J} \lambda_k f_k^{00}(\bar{x}; x - \bar{x}) + \frac{1}{2} \sum_{i \in I(\bar{x})} \bar{\mu}_i g_i^{00}(\bar{x}; x - \bar{x}) \\ & + \frac{1}{2} \sum_{j \in L} \bar{\nu}_j \nabla^2 h_j(\bar{x})(x - \bar{x}, x - \bar{x}) \geqslant 0, \end{aligned}$$

.

Then, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$ is a feasible point of (MWD), and the value of the objective functions of (MP) and (MWD) at $\bar{x}$ and $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$, respectively, are equal. If the assumptions of Theorem 5.1 hold, we conclude from Theorem 5.1 that, for every feasible points (u, λ, μ, ν) of (MWD), $f(\bar{x}) \not< f(u)$. Hence, there is no $(u, \lambda, \mu, \nu) \in M_1$ such that $f(\bar{x}) < f(u)$. Hence, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$ is a weakly efficient solution of (MWD). □

Theorem 5.3. (Converse duality) *Let $\bar{x}$ and $(\bar{u}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$ be feasible points of (MP) and (MWD), respectively. Assume that Assumption 5.1 holds is fulfilled at $\bar{u}$, $M_2 := (C - \bar{u}) \cap Ker \nabla h(\bar{u}) \neq \emptyset$; $\bar{\lambda} f$ is strict second-order pseudoconvex at $\bar{u}$ on M_2, and $g_i (i \in I(u)), \pm h_j (j \in L)$ are second-order quasiconvex at $\bar{u}$ on M_2. If*

$$\sum_{k=1}^{r} \bar{\lambda}_k f_k(\bar{x}) \leq \sum_{k=1}^{r} \bar{\lambda}_k f_k(\bar{u}), \tag{5.2}$$

then $\bar{u} = \bar{x}$.

Proof. Assume the contrary that $\overline{u} \neq \overline{x}$. Since $(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ is a feasible of (MWD), we have that, for every $x \in \overline{x} + M_2$,

$$\begin{aligned}&\sum_{k\in J}\overline{\lambda}_k f'_{G,k}(\overline{u})(x-\overline{u}) + \sum_{i\in I(\overline{u})}\overline{\mu}_i g'_{G,i}(\overline{u})(x-\overline{u}) + \sum_{j\in L}\overline{\nu}_j \nabla h_j(\overline{u})(x-\overline{u})\\ &\quad + \frac{1}{2}\sum_{k\in J}\overline{\lambda}_k f_k^{00}(\overline{u};x-\overline{u}) + \frac{1}{2}\sum_{i\in I(\overline{u})}\mu_i g_i^{00}(\overline{u};x-\overline{u}) + \frac{1}{2}\sum_{j\in L}\nu_j \nabla^2 h_j(\overline{u})(x-\overline{u},x-\overline{u}) \geqslant 0.\end{aligned} \tag{5.3}$$

We have $g_i(x) \leq 0 = g_i(\overline{u})$ for every $i \in I(\overline{u})$, $x \in \overline{x} + M_2$. In view of the 2-quasiconvexity of g_i at $\overline{u}$ on M_2, we have

$$g'_{G,i}(\overline{u})(x-\overline{u}) + \frac{1}{2} g_i^{00}(\overline{u};x-\overline{u}) \leq 0. \tag{5.4}$$

We also have $h_j(x) = 0 = h_j(\overline{u})$, for all $j \in L$, $x \in \overline{x} + M_2$. By virtue of the 2-quasiconvexity of $\pm h_j$ at $\overline{u}$ on M_2, we have

$$\nabla h_j(\overline{u}) + \frac{1}{2}\nabla^2 h_j(\overline{u})(x-\overline{u},x-\overline{u}) = 0. \tag{5.5}$$

Combining (5.3), (5.4), and (5.5) yields that

$$\sum_{k\in J}\overline{\lambda}_k f'_{G,k}(\overline{u})(x-\overline{u})\frac{1}{2}\sum_{i\in I(\overline{u})} f_i^{00}(\overline{u};x-\overline{u}) \geq 0,$$

which is equivalent to

$$\overline{\lambda} f'(\overline{u})(x-\overline{u}) + \frac{1}{2}\overline{\lambda} f^{00}(\overline{u};x-\overline{u}) \geq 0.$$

This contradicts (5.2). Hence, $\overline{u} = \overline{x}$. The proof is complete. □

6. Conclusion

This paper is the continuity of results presented in [16] to the nonsmooth vector equilibrium problems involving inequality, equality, and set constraints in terms of the Páles-Zeidan second-order generalized directional derivatives. We introduced new notions of 2-generalized convexity, and established second-order necessary conditions for the weak efficiency of the nonsmooth vector optimization problems involving inequality, equality and set constraints. Second-order sufficient optimality conditions are established with assumptions on 2-generalized convexity. It can be seen that these notions of the generalized convexity are advantage for establishing second-order optimality conditions for the optimization problems with the set constraint to be convex. The second-order necessary conditions obtained here are new and significant replenishes of those obtained in [16] for the nonsmooth optimization problems with equality, inequality, and set constraints.

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