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AN INERTIAL FORWARD-BACKWARD METHOD WITH SELF-ADAPTIVE STEP SIZES FOR FINDING MINIMUM-NORM SOLUTIONS OF INCLUSION AND SPLIT EQUILIBRIUM PROBLEMS

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Abstract. In this paper, the problem of finding a common solution of variational inclusion and split equilibrium problems is studied. A modified inertial forward-backward splitting algorithm with self-adaptive step sizes is introduced for solving the problem, and the strong convergence of the algorithm is established in Hilbert spaces. Applications and examples are provided to support our main results.
Keywords. Forward-backward splitting algorithm; Inertial method; Minimum-norm solutions; Split equilibrium problem; Variational inclusion.

1. Introduction

Let $\mathscr{H}$ be a real Hilbert space, and let $B : \mathscr{H} \to \mathscr{H}$ be an operator. The Variational Inclusion Problem (VIP) studied in this paper is defined as find $x^* \in \mathscr{H}$ such that

$$0 \in (B+D)x^*, \tag{1.1}$$

where $D : \mathscr{H} \to 2^{\mathscr{H}}$ is a set-valued operator. The solution set of problem (1.1) is denoted by $(B+D)^{-1}(0)$. The VIP attracted the interest of many researchers due to its wide application to various problems arising in optimal control, mathematical economics, and so on. Specifically, many problems in machine learning, image processing, and linear inverse problems can be modeled mathematically as (1.1); see, e.g., [1, 2, 3, 4, 5, 6] and the references therein. An efficient method for solving the VIP is the following forward-backward splitting method $x_{n+1} = (I+rD)^{-1}(I-rB)x_n, \forall n \geq 1$, where r is a positive real number, $(I-rB)$ is the forward operator, and $(I+rD)^{-1}$ is the resolvent operator, which is also known as the backward operator).

There are numerous algorithms for the VIP. It is a hot topic to improve the convergence rate of various iterative algorithms. In 1964, Polyak [7] studied the convergence of the following inertial extrapolation algorithm $x_{n+1} = x_n + \beta_n(x_n - x_{n-1}) - \alpha_n A x_n$, $\forall n \geq 0$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences.

Recently, there has been an increased interest in studying inertial type algorithms. In 2001, Alvarez and Attouch [8] combined the inertial technique and the proximal point algorithm for

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maximal monotone operators. Their algorithm is known as the inertial proximal point algorithm

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n D)^{-1} y_n, \ n \geq 1. \end{cases}$$

They obtained a weak convergence theorem provided that $\{r_n\}$ is nondecreasing and $\{\theta_n\} \subset [0,1)$ with

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty. \tag{1.2}$$

In 2003, Moudafi and Oliny [9] proposed the following inertial proximal point algorithm for solving the zero problem of the sum of two monotone operators

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n D)^{-1}(y_n - r_n B x_n), \ n \geq 1 \end{cases}$$

They demonstrated the sequence generated by their algorithm converges weakly to the zero of $B + D$ provided $r_n < \frac{2}{L}$, where L is the Lipschitz constant of B and condition (1.2) is satisfied. Note that their algorithm does not take the form of the forward-backward splitting algorithm if $\theta_n > 0$ because B is still evaluated at the point x_n. In 2015, Lorenz and Pock [10] proposed the following inertial forward-backward algorithm for monotone operators

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n D)^{-1}(y_n - r_n B y_n), \ n \geq 1 \end{cases}$$

where $r_n > 0$. They obtained a weak convergence theorem. It is known that strong convergence results are more applicable and desirable than weak convergence results. In 2018, Cholamjiak et al. [11] introduced the following inertial forward-backward splitting algorithm, which combines Halpern and Mann iteration methods for solving the VIP in Hilbert spaces

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = \beta_n u + \varepsilon_n y_n + \mu_n J^D_{\lambda_n}(y_n - \lambda_n B y_n), \ n \geq 1, \end{cases} \tag{1.3}$$

where $B : \mathscr{H} \to \mathscr{H}$ is k-inverse-strongly monotone operator, and $D : H \to 2^{\mathscr{H}}$ is a maximal monotone operator, $J^D_{\lambda_n} = (I + \lambda_n D)^{-1}$, $0 < \lambda_n \leq 2k$, $\{\alpha_n\} \subset [0, \alpha]$ with $\alpha \in [0,1)$, and $\{\beta_n\}, \{\varepsilon_n\}$, and $\{u_n\}$ are sequences in $[0,1]$ with $\beta_n + \varepsilon_n + \mu_n = 1$. Under the following conditions on the control parameters

(1) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$;
(2) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$;
(3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2k$;
(4) $\liminf_{n\to\infty} \mu_n > 0$,

they obtained a strong convergence theorem in the framework of real Hilbert spaces.

Let $\mathscr{C}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathscr{H}$, and let $F : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$ be a bifunction. In 1994, Blum and Oettli [12] investigated the Equilibrium Problem (EP), which is to find $x \in \mathscr{C}$ such that

$$F(x,y) \geq 0, \ \forall \ y \in \mathscr{C}. \tag{1.4}$$

We denote the solution set of (1.4) by $EP(F)$. The EP has received a lot of attention due to its application to problems arising in the field of optimization, economics, physics, traffic networks, and so on; see, e.g., [13, 14] and the references therein. Recently, several authors proposed various iterative algorithms for solving the EP and its extensions; see, e.g., [15, 16, 17, 18, 19] and the references therein.

In 2013, Kazmi and Rizvi [20] introduced and studied the following Split Equilibrium Problem (SEP). Let $\mathscr{C}$ be a nonempty subset of $\mathscr{H}_1$, let $\mathscr{Q}$ be a nonempty subset of $\mathscr{H}_2$. Let $F_1 : \mathscr{C} \times \mathscr{C} \to \mathbb{R}, F_2 : \mathscr{Q} \times \mathscr{Q} \to \mathbb{R}$ be bifunctions. Let $A : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator. They defined the SEP as follows: Find $x \in \mathscr{C}$ such that

$$F_1(x, x^*) \geq 0, \ \forall \, x \in \mathscr{C} \tag{1.5}$$

and such that

$$y = Ax \in \mathscr{Q} \text{ solves } F_2(y, y^*), \ \forall \, y^* \in \mathscr{Q}. \tag{1.6}$$

Observe that problem (1.5) is the classical Equilibrium Problem. Thus, SEP consists of a pair of EPs, which have to be solved so that the image $y = Ax$ under a given bounded linear operator A of the solution of the EP in $\mathscr{H}_1$ is a solution of the other EP in $\mathscr{H}_2$. The solution set of SEP (1.5)-(1.6) is denoted by $\Omega = \{z \in EP(F_1) : \ Az \in EP(F_2)\}$.

In 2019, Cholamjiak et al. [21] proposed the following modified inertial forward-backward splitting method for solving the SEP and the VIP

$$\begin{cases} y_n = x_n + \delta_n(x_n - x_{n-1}), \\ z_n = \alpha_n y_n + (1-\alpha_n) T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A) y_n, \\ x_{n+1} = \beta_n z_n + (1-\beta_n) J_{t_n}^D (I - t_n B) z_n, \ n \geq 1, \end{cases} \tag{1.7}$$

where $J_{t_n}^D = (I + t_n D)^{-1}$, $\{t_n\} \subset (0, 2\alpha)$, $\{\delta_n\} \subset [0, \delta]$, $\delta \in [0,1)$, $\{r_n\} \subset (0, \infty)$ with $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Under the following conditions;

(1) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$;
(2) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(4) $\liminf_{n\to\infty} r_n > 0$;
(5) $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 2\alpha$,

they obtained a weak convergence theorem. However, it was pointed out by the authors that the summability condition (1) of Algorithm 1.3 and Algorithm 1.7 is a drawback in their implementation. Another computational weakness of Algorithm 1.7 is the fact that its step size γ depends on the norm of the operator $\|A\|$, which, in most cases, is unknown or difficult to calculate or even estimate. Moreover, it is required that the operator B in Algorithm 1.7 and the other algorithms above is co-coercive (inverse-strongly monotone), which is a very stringent condition.

Motivated by the above works in the literature, we investigate the problem of finding a common solution of the VIP (1.1) and the SEP (1.5)-(1.6). That is, find a point $x^* \in \mathscr{C}$ such that $x^* \in (B+D)^{-1}(0) \cap \Omega \neq \emptyset$. We construct an inertial forward-backward splitting algorithm with self-adaptive step sizes with the following properties:

- our proposed method requires the underlying operator to be monotone and Lipschitz continuous. This relaxes the strong assumption of co-coerciveness (inverse-strongly monotonicity) on the operator in the literature;
- in contrast to the methods in the literature, the implementation of our method does not depend on the norm of the bounded linear operator nor the Lipschitz constant of the monotone operator involved. This makes our method easier to implement;
- our method employs the inertial technique to improve the speed of convergence of the proposed algorithm. Our algorithm does not require the summability condition $\left(\sum_{n=1}^{\infty}\alpha_n\|x_n-x_{n-1}\|<\infty\right)$, which makes our method easily implementable;
- the sequence generated by our proposed algorithm is strongly convergent to a minimum-norm solution of the common solution problem under mild conditions. Moreover, our convergence analysis is not dependent on the usual "the two-case approach", which was widely used in many papers to guarantee strong convergence.

We carry out some numerical experiments on our proposed method in comparison with other existing methods for solving the VIP (1.1) and SEP (1.5)-(1.6). The results indicate that our method is easier to implement and also outperforms other methods in the literature. The organization of the paper is as follows: Section 2 consists of basic definitions and results, which are needed in subsequent sections. In Section 3, we present and discuss the proposed method. The convergence of this method is investigated in Section 4. The applications of our main results is given in Section 5. In Section 6, we perform some numerical experiments in comparison with other related methods in the literature. Section 7, the last section, concludes this paper.

2. Preliminaries

In this section, we recall some basic definitions, lemmas, and related results, which are needed to prove our strong convergence result. Let $\mathscr{C}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathscr{H}$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ defined by $||x||=\sqrt{\langle x,x\rangle}$, $\forall x\in\mathscr{H}$. We denote the weak and strong convergence of a sequence $\{x_n\}$ to a point x as $x_n\rightharpoonup x$ and $x_n\to x$, respectively. Also, we denote the set of weak limits of $\{x_n\}$ by $w_\omega(x_n)$, that is, $w_\omega(x_n):=\{x\in H: x_{n_j}\rightharpoonup x,$ for some subsequence $\{x_{n_j}\}$ of $\{x_n\}\}$.

Let $A:\mathscr{H}\to\mathscr{H}$ be a mapping. Then, A is said to be

(i) *L-Lipschitz continuous* if there exists $L>0$ such that $||Ax-Ay||\leq L||x-y||$, $\forall x,y\in\mathscr{H}$, if $L\in[0,1)$, then A is called a *contraction mapping;*

(ii) *nonexpansive* if A is $1-$Lipschitz continuous;

(iii) *L-co-coercive (or L-inverse strongly monotone)* if there exists $L>0$ such that $\langle Ax-Ay,x-y\rangle\geq L||Ax-Ay||^2$, $\forall x,y\in\mathscr{H}$;

(iv) *monotone* if $\langle Ax-Ay,x-y\rangle\geq 0$, $\forall x,y\in\mathscr{H}$

It is known that inverse-strongly monotone mappings are monotone and Lipschitz continuous. In general, the converse is not true.

Let $\mathscr{C}$ be a closed and convex subset of $\mathscr{H}$. Recall that the metric projection $P_{\mathscr{C}}$ is a map, which assigns each $x\in\mathscr{H}$ to the unique point $P_{\mathscr{C}}x$, that is, $\|x-P_{\mathscr{C}}x\|=\min\{\|x-y\|:y\in\mathscr{C}\}$.

The metric projection $P_{\mathscr{C}}$ is characterized by the inequality $\langle x-P_{\mathscr{C}}x,y-P_{\mathscr{C}}x\rangle\geq 0$, $\forall y\in\mathscr{C}$. For more details on the properties of the metric projection, we refer to [22, Section 3].

Lemma 2.1. [23, 24] *Let $\mathscr{H}$ be a real Hilbert space. Then the following assertions hold:*

(1) $2\langle x,y\rangle = \|x\|^2+\|y\|^2-\|x-y\|^2 = \|x+y\|^2-\|x\|^2-\|y\|^2,\ \forall x,y\in\mathscr{H}$;
(2) $\|\alpha x+(1-\alpha)y\|^2 = \alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2,\ \forall x,y\in\mathscr{H},\ \alpha\in(0,1)$;
(3) $\|x-y\|^2 \le \|x\|^2+2\langle y,x-y\rangle,\ \forall x,y\in\mathscr{H}$.

Recall that a function $c:\mathscr{H}\to\mathbb{R}$ is said to be convex if, for all $t\in[0,1]$ and $x,y\in\mathscr{H}$, $c(tx+(1-t)y)\le tc(x)+(1-t)c(y)$.

Recall that a convex function $c:\mathscr{H}\to\mathbb{R}$ is said to be subdifferentiable at a point $x\in\mathscr{H}$ if the set

$$\partial c(x) = \{u\in\mathscr{H}\mid c(y)\ge c(x)+\langle u,y-x\rangle,\ \forall y\in\mathscr{H}\} \tag{2.1}$$

is nonempty, where each element in $\partial c(x)$ is called a subgradient of c at x, $\partial c(x)$ is called the subdiferential of c at x, and the inequality in (2.1) is called the subdifferential inequality of c at x. We say that c is subdifferentiable on $\mathscr{H}$ if c is subdifferentiable at each $x\in\mathscr{H}$ [25].

Let $B:\mathscr{H}\to 2^{\mathscr{H}}$ be a multivalued operator on $\mathscr{H}$. Then

(i) the effective domain of B, denoted by $dom(B)$, is given as $dom(B)=\{x\in\mathscr{H}: Bx\neq\emptyset\}$;
(ii) the graph $G(B)$ is defined by $G(B):=\{(x,u)\in\mathscr{H}\times\mathscr{H}: u\in B(x)\}$;
(iii) the operator B is said to be monotone if $\langle x-y,u^*-v^*\rangle\ge 0$ for all $x,y\in dom(B), u^*\in Bx$ and $v^*\in By$.
(iv) a monotone operator B on $\mathscr{H}$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on $\mathscr{H}$.
(v) the resolvent mapping $J_\lambda^B:\mathscr{H}\to\mathscr{H}$ associated with B is defined as $J_\lambda^B(x)=(I+\lambda B)^{-1}(x)$, for some $\lambda>0$, where I is the identity operator on $\mathscr{H}$. For a maximal monotone operator B, the $dom(J_\lambda^B)=\mathscr{H}$ and J_λ^B is a single-valued and firmly nonexpansive mapping.

Recall that a function $g:\mathscr{H}\to\mathscr{R}\cup\{+\infty\}$ is said to be weakly lower semi-continuous (w-lsc) at $x\in\mathscr{H}$, if $\liminf\limits_{k\to\infty} g(x_n)\ge g(x)$ holds for an arbitrary sequence $\{x_n\}_{n=0}^\infty$ in $\mathscr{H}$ satisfying $x_n\rightharpoonup x$.

The following assumptions are required in solving the split equilibrium problem.

Assumption 2.1. [12] Let $F:\mathscr{C}\times\mathscr{C}\to\mathbb{R}$ be a bifunction satisfying the following assumptions:

1) $F(x,x)=0,\ \forall\, x\in\mathscr{C}$;
2) F is monotone, i.e., $F(x,y)+F(y,x)\le 0,\ \forall x\in\mathscr{C}$;
3) for each $x,y,z\in\mathscr{C}$, $\limsup\limits_{t\to 0} F(tz+(1-t)x,y)\le F(x,y)$;
4) for each $x\in\mathscr{C}$, $y\to F(x,y)$ is convex and lower semi continuous.

Lemma 2.2. [26] *Let $F:\mathscr{C}\times\mathscr{C}\to\mathbb{R}$ be a bifunction satisfying Asummption 2.1. For any $r>0$ and $x\in\mathscr{H}$, define a mapping $T_r^F:\mathscr{H}\to\mathscr{C}$ by*

$$T_r^F(x)=\left\{z\in\mathscr{C}: F(z,y)+\frac{1}{r}\Big\langle y-z,z-x\Big\rangle\ge 0,\ \forall y\in\mathscr{C}\right\}.$$

Then,

(1) *T_r^F is nonempty and single valued;*
(2) *T_r^F is firmly nonexpansive, that is, $langleT_r^Fx-T_r^Fy,x-y\rangle\ge\|T_r^Fx-T_r^Fy\|^2$;*
(3) *$F(T_r^F)=EP(F)$ is closed and convex.*

Lemma 2.3. [27] *Let $B:\mathscr{H}\to 2^{\mathscr{H}}$ be a maximal monotone mapping, and $A:\mathscr{H}\to\mathscr{H}$ be a Lipschitz continuous and monotone mapping. Then $A+B$ is a maximal monotone mapping.*

Lemma 2.4. [28] *Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\gamma_n\}$ be a sequence of real numbers in $(0,1)$ with conditions $\sum_{n=1}^{\infty}\gamma_n = \infty$, and $\{d_n\}$ be a sequence of real numbers. Assume that $a_{n+1} \le (1-\gamma_n)a_n + \gamma_n d_n$, $n \ge 1$. If $\limsup_{k\to\infty} d_{n_k} \le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0$, then $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

In this section, we present our proposed method and discuss its features. We begin by defining the following functions

$$g(x) = \frac{1}{2}\|(I - T_r^{F_2})Ax\|^2, \quad h(x) = \frac{1}{2}\|(I - T_r^{F_1})x\|^2,$$

and

$$G(x) = A^*(I - T_r^{F_2})Ax, \quad H(x) = (I - T_r^{F_1})x.$$

It can be easily verified from Aubin [29] that g and h are weakly lower semi-continuous, convex, and differentiable. We now give the following assumptions under which our strong convergence result is obtained.

Assumption 3.1. Suppose that the following conditions hold:

(a) The feasible sets $\mathscr{C}$ and $\mathscr{Q}$ are nonempty, closed, and convex subsets of the real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively.

(b) $F_1 : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$, $F_2 : \mathscr{Q} \times \mathscr{Q} \to \mathbb{R}$ are bifunctions satisfying Assumption (2.1), and F_2 is upper semi-continuous in the first argument.

(c) $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator, $B : \mathscr{H}_1 \to \mathscr{H}_1$ is a Lipschitz continuous and monotone operator, and $D : \mathscr{H}_1 \to 2^{\mathscr{H}_1}$ is a maximal monotone operator such that $\Gamma = (B+D)^{-1}(0) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in \mathscr{C} : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$.

(d) $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying $\beta_n + \gamma_n \le 1, \lim_{n\to\infty}\gamma_n = 0, \sum_{n=1}^{\infty}\gamma_n = \infty, \inf(1-\beta_n-\gamma_n)\beta_n > 0, 0 < a \le \alpha_n, \beta_n \le b < 1, 0 < c \le \tau_n \le d < 4$.

(e) Let $\{\varepsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty}\frac{\varepsilon_n}{\gamma_n} = 0$ and $\{r_n\} \subset (0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$.

Algorithm 3.1.

Step 1: Select initial point $x_0, x_1 \in \mathscr{H}_1$, let $s_1 > 0, \mu \in (0,1), \theta \ge 3$, and set $n = 1$. Given the iterates x_{n-1} and x_n for each $n \ge 1$, choose θ_n such that $0 \le \theta_n \le \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2: Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$.

Step 3: Compute $z_n = T_{r_n}^{F_1}\big(I - \lambda_n A^*(I - T_{r_n}^{F_2})A\big)w_n$ and $y_n = \alpha_n w_n + (1-\alpha_n)z_n$.

Step 4: Compute $u_n = (I + s_n D)^{-1}(I - s_n B)y_n = J_{s_n}^D(I - s_n B)y_n$ and $v_n = u_n - s_n(Bu_n - By_n)$.

Step 5 Compute $x_{n+1} = (1-\beta_n-\gamma_n)w_n + \beta_n v_n$, where

$$\lambda_n := \begin{cases} \tau_n \frac{g(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2}, & \text{if } \|G(w_n)\|^2 + \|H(w_n)\|^2 \neq 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$s_{n+1} = \begin{cases} \min\left\{ \frac{\mu \|y_n - u_n\|}{\|By_n - Bu_n\|}, s_n \right\} & \text{if} \quad By_n - Bu_n \neq 0, \\ s_n, & \text{otherwise}. \end{cases} \tag{3.2}$$

Set $n := n+1$ and go back to **Step 1**.

Remark 3.1. By conditions (d) and (e), one can easily verify from (3.1) that

$$\lim_{n\to\infty} \theta_n ||x_n - x_{n-1}|| = 0 \text{ and } \lim_{n\to\infty} \frac{\theta_n}{\gamma_n} ||x_n - x_{n-1}|| = 0.$$

Remark 3.2. From (3.2), we have that $s_{n+1} \leq s_n$ for all $n \geq 1$. From our conditions, we also have that B is L-Lipschitz continuous. Thus if $By_n \neq Bu_n$ in Algorithm 3.1, then

$$s_{n+1} = \min\left\{ \frac{\mu ||y_n - u_n||}{||By_n - Bu_n||},\ s_n \right\} \geq \min\left\{ \frac{\mu}{L},\ s_n \right\}.$$

By induction, we obtain that $\{s_n\}$ is bounded below by $\min\left\{ \frac{\mu}{L},\ s_1 \right\}$. Also, since $\{s_n\}$ is monotone nonincreasing, we have that the limit exists, and $\lim_{n\to\infty} s_n \geq \min\{\frac{\mu}{L}, s_1\} > 0$.

Remark 3.3. From the definition of $\{s_n\}$ in (3.2), we have

$$\|By_n - Bu_n\| \leq \frac{\mu}{s_{n+1}} \|y_n - u_n\|, \quad \forall\, n. \tag{3.3}$$

Inequality (3.3) holds if $By_n = Bu_n$. In the case that $By_n \neq Bu_n$, we have

$$s_{n+1} = \min\left\{ \frac{\mu \|y_n - u_n\|}{\|By_n - Bu_n\|}, s_n \right\} \leq \frac{\mu \|y_n - u_n\|}{\|By_n - Bu_n\|},$$

which implies that

$$\|By_n - Bu_n\| \leq \frac{\mu}{s_{n+1}} \|y_n - u_n\|.$$

Hence, (3.3) holds for both cases $By_n \neq Bu_n$ and $By_n = Bu_n$.

4. Convergence Analysis

Lemma 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 under Assumption 3.1. Then,*

$$\|v_n - p\|^2 \leq \|w_n - p\|^2 - \tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2} - \alpha_n(1-\alpha_n)\|w_n - z_n\|^2$$
$$- \left(1 - s_n^2 \cdot \frac{\mu^2}{s_{n+1}^2}\right)\|y_n - u_n\|^2.$$

Proof. Let $p \in \Gamma$. Since $G(w_n) = A^*(I - T_{r_n}^{F_2})Aw_n,\ p = T_{r_n}^{F_2} p$, and $I - T_{r_n}^{F_2}$ is firmly nonexpansive, we have

$$\begin{aligned} \langle G(w_n), w_n - p \rangle &= \langle (I - T_{r_n}^{F_2})Aw_n,\ Aw_n - Ap \rangle \\ &= \langle (I - T_{r_n}^{F_2})Aw_n - (I - T_{r_n}^{F_2})Ap,\ Aw_n - Ap \rangle \\ &\geq \|(I - T_{r_n}^{F_2})Aw_n\|^2 \\ &= 2g(w_n). \end{aligned} \tag{4.1}$$

From the definition of z_n in Step 3, the nonexpansivity of $T_{r_n}^{F_1}$, and (4.1), we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|(I - \lambda_n A^*(I - T_{r_n}^{F_2})A)w_n - p\|^2 \\
&= \|w_n - p - \lambda_n G(w_n)\|^2 \\
&= \|w_n - p\|^2 + \lambda_n^2 \|G(w_n)\|^2 - 2\lambda_n \langle G(w_n), w_n - p\rangle \\
&\leq \|w_n - p\|^2 + \lambda_n^2 \|G(w_n)\|^2 - 4\lambda_n g(w_n) \\
&\leq \|w_n - p\|^2 - \tau_n(4 - \tau_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2}.
\end{aligned} \tag{4.2}$$

From the condition on τ_n, we obtain $\|z_n - p\| \leq \|w_n - p\|$. In view of the definition of y_n and (4.2), we obtain

$$\begin{aligned}
\|y_n - p\|^2 &= \alpha_n \|w_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|w_n - z_n\|^2 \\
&\leq \alpha_n \|w_n - p\|^2 + (1-\alpha_n)\left[\|w_n - p\|^2 - \tau_n(4-\tau_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2}\right] \\
&- \alpha_n(1-\alpha_n)\|w_n - z_n\|^2 \\
&= \|w_n - p\|^2 - \tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2} - \alpha_n(1-\alpha_n)\|w_n - z_n\|^2.
\end{aligned} \tag{4.3}$$

From the condition on τ_n and α_n, we have $\|y_n - p\| \leq \|w_n - p\|$. Applying (3.3) and (4.3), we have

$$\begin{aligned}
\|v_n - p\|^2 &= \|u_n - p\|^2 + s_n^2\|Bu_n - By_n\|^2 - 2s_n\langle u_n - p, Bu_n - By_n\rangle \\
&= \|y_n - p\|^2 + \|u_n - y_n\|^2 + 2\langle u_n - y_n,\ y_n - p\rangle + s_n^2\|Bu_n - By_n\|^2 \\
&\quad - 2s_n\langle u_n - p, Bu_n - By_n\rangle \\
&= \|y_n - p\|^2 + \|u_n - y_n\|^2 + 2\langle u_n - y_n,\ u_n - p\rangle - 2\|u_n - y_n\|^2 + s_n^2\|Bu_n - By_n\|^2 \\
&\quad - 2s_n\langle u_n - p, Bu_n - By_n\rangle \\
&= \|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\langle u_n - y_n,\ u_n - p\rangle + s_n^2\|Bu_n - By_n\|^2 \\
&\quad - 2s_n\langle u_n - p, Bu_n - By_n\rangle \\
&= \|y_n - p\|^2 - \|u_n - y_n\|^2 + s_n^2\|Bu_n - By_n\|^2 - 2\langle u_n - p,\ y_n - u_n - s_n(By_n - Bu_n)\rangle \\
&\leq \|w_n - p\|^2 - \tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2} - \alpha_n(1-\alpha_n)\|w_n - z_n\|^2 \\
&\quad - \left(1 - s_n^2 \cdot \frac{\mu^2}{s_{n+1}^2}\right)\|y_n - u_n\|^2 - 2\langle u_n - p,\ y_n - u_n - s_n(By_n - Bu_n)\rangle.
\end{aligned} \tag{4.4}$$

From Step 4, we have that $u_n = (I + s_n D)^{-1}(I - s_n B)y_n$. Hence $(I - s_n B)y_n \in (I + s_n D)u_n$. Since D is maximal monotone, there exists $k_n \in Du_n$ such that $(I - s_n B)y_n = u_n + s_n k_n$, which implies that

$$k_n = \frac{1}{s_n}(y_n - s_n B y_n - u_n). \tag{4.5}$$

Observe that $0 \in (B+D)p$ and $Bu_n + k_n \in (B+D)u_n$. Since $B+D$ is maximal monotone we obtain

$$\langle Bu_n + k_n,\ u_n - p\rangle \geq 0. \tag{4.6}$$

Substituting (4.5) into (4.6) we have $\frac{1}{s_n}\langle s_n B u_n + y_n - s_n B y_n - u_n, u_n - p\rangle \geq 0$, which implies that

$$\langle y_n - u_n - s_n(By_n - Bu_n),\ u_n - p\rangle \ \geq 0. \tag{4.7}$$

Applying (4.7) in (4.4), we have

$$\begin{aligned}\|v_n - p\|^2 \leq \|w_n - p\|^2 - \tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2} - \alpha_n(1-\alpha_n)\|w_n - z_n\|^2 \\ - \left(1 - s_n^2 \cdot \frac{\mu^2}{s_{n+1}^2}\right)\|y_n - u_n\|^2,\end{aligned} \tag{4.8}$$

which is the desired result. □

Lemma 4.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 under Assumption 3.1. Then, $\{x_n\}$ is bounded.*

Proof. Let $p \in \Gamma$. By applying the triangular inequality, we conclude from Step 2 that $\|w_n - p\| \leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|$. From Remark 3.1, we have $\lim_{n\to\infty}\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| = 0$. Hence, there exists a constant $M_1 > 0$ such that $\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| \leq M_1$ for all $n \geq 1$. This implies that

$$\|w_n - p\| \leq \|x_n - p\| + \gamma_n M_1. \tag{4.9}$$

Also, since $\lim\limits_{n\to\infty}\left(1 - \mu^2\frac{s_n^2}{s_{n+1}^2}\right) = 1 - \mu^2 > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$1 - \mu^2\frac{s_n^2}{s_{n+1}^2} > 0, \quad \forall\, n \geq n_0.$$

which together with the conditions on τ_n and α_n in (4.8) yields $\|v_n - p\| \leq \|w_n - p\|$, $\forall n \geq n_0$. From the definition of x_{n+1} in Step 4, we have, for all $n \geq n_0$,

$$\|x_{n+1} - p\| \leq \|(1-\beta_n-\gamma_n)(w_n - p) + \beta_n(v_n - p)\| + \gamma_n\|p\| \tag{4.10}$$

and

$$\begin{aligned}\|(1-\beta_n-\gamma_n)(w_n-p) + \beta_n(v_n-p)\|^2 &= (1-\beta_n-\gamma_n)^2\|w_n-p\|^2 + \beta_n^2\|v_n-p\|^2 \\ &\quad + 2(1-\beta_n-\gamma_n)\beta_n\langle w_n - p, v_n - p\rangle \\ &\leq (1-\gamma_n)^2\|w_n - p\|^2.\end{aligned}$$

Substituting the last inequality into (4.10) and applying (4.9), we have, for all $n \geq n_0$,

$$\begin{aligned}
\|x_{n+1}-p\| &\leq (1-\gamma_n)\|w_n-p\|+\gamma_n\|p\| \\
&\leq (1-\gamma_n)\left[\|x_n-p\|+\gamma_n M_1\right]+\gamma_n\|p\| \\
&\leq (1-\gamma_n)\|x_n-p\|+\gamma_n(M_1+\|p\|) \\
&\leq \max\Big\{\|x_n-p\|,\, M_1+\|p\|\Big\} \\
&\;\vdots \\
&\leq \max\Big\{\|x_{n_0}-p\|,\, M_1+\|p\|\Big\},
\end{aligned}$$

which implies that $\{x_n\}$ is bounded. Consequently, $\{w_n\},\{z_n\},\{y_n\},\{u_n\}$, and $\{v_n\}$ are also bounded. $\square$

Lemma 4.3. *Let $\{u_n\}$ and $\{y_n\}$ be the sequences generated by Algorithm 3.1 under Assumption 3.1 such that $\lim_{j\to\infty}\|u_{n_j}-y_{n_j}\|=0$ for some subsequences $\{u_{n_j}\}$ and $\{y_{n_j}\}$ of $\{u_n\}$ and $\{y_n\}$, respectively. If $\{y_{n_j}\}$ converges weakly to some $x^*\in\mathscr{H}_1$ as $j\to\infty$, then $x^*\in(B+D)^{-1}(0)$.*

Proof. Let $(u,v)\in G(B+D)$, that is, $v-Bu\in Du$. Since $u_{n_j}=(I+s_{n_j}D)^{-1}(I-s_{n_j}B)y_{n_j}$, then $(I-s_{n_j}B)y_{n_j}\in(I+s_{n_j}D)u_{n_j}$. This implies that $\frac{1}{s_{n_j}}\Big(y_{n_j}-u_{n_j}-s_{n_j}By_{n_j}\Big)\in Du_{n_j}$. Since D is maximal monotone, we have

$$\Big\langle u-u_{n_j},\, v-Bu-\frac{1}{s_{n_j}}\Big(y_{n_j}-u_{n_j}-s_{n_j}By_{n_j}\Big)\Big\rangle \geq 0,$$

which implies that

$$\langle u-u_{n_j},\, v\rangle-\Big\langle u-u_{n_j},\, Bu+\frac{1}{s_{n_j}}\Big(y_{n_j}-u_{n_j}-s_{n_j}By_{n_j}\Big)\Big\rangle \geq 0.$$

Thus

$$\begin{aligned}
\langle u-u_{n_j},\, v\rangle &\geq \Big\langle u-u_{n_j},\, Bu+\frac{1}{s_{n_j}}\Big(y_{n_j}-u_{n_j}-s_{n_j}By_{n_j}\Big)\Big\rangle \\
&= \langle u-u_{n_j},\, Bu-By_{n_j}\rangle+\Big\langle u-u_{n_j},\, \frac{1}{s_{n_j}}(y_{n_j}-u_{n_j})\Big\rangle \\
&= \langle u-u_{n_j},\, Bu-Bu_{n_j}\rangle+\langle u-u_{n_j},\, Bu_{n_j}-By_{n_j}\rangle+\Big\langle u-u_{n_j},\, \frac{1}{s_{n_j}}(y_{n_j}-u_{n_j})\Big\rangle \\
&\geq \langle u-u_{n_j},\, Bu_{n_j}-By_{n_j}\rangle+\Big\langle u-u_{n_j},\, \frac{1}{s_{n_j}}(y_{n_j}-u_{n_j})\Big\rangle.
\end{aligned}$$

Now, since $\lim_{j\to\infty}\|u_{n_j}-y_{n_j}\|=0$, and B is Lipschitz continuous, we have $\lim\limits_{j\to\infty}\|Bu_{n_j}-By_{n_j}\|=0$. Moreover, $\lim\limits_{n\to\infty}s_n=s\geq\min\{s_0,\frac{\mu}{L}\}$ implies

$$\langle u-x^*,\, v\rangle=\lim_{j\to\infty}\langle u-u_{n_j},\, v\rangle\geq 0.$$

By the maximal monotonicity of $B+D$, we conclude that $x^*\in(B+D)^{-1}(0)$. $\square$

Lemma 4.4. *The following inequality holds for all $p \in \Gamma$ and $n \in \mathbb{N}$*

$$\begin{aligned}\|x_{n+1}-p\|^2 \le (1-\gamma_n)\|x_n-p\|^2 + \gamma_n\Big(3\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2 + \|p\|^2\Big) \\ - \beta_n\tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2+\|H(w_n)\|^2} - \beta_n\alpha_n(1-\alpha_n)\|w_n-z_n\|^2 \\ - \beta_n\Big(1-s_n^2\cdot\frac{\mu^2}{s_{n+1}^2}\Big)\|y_n-u_n\|^2 - (1-\beta_n-\gamma_n)\beta_n\|w_n-v_n\|^2\end{aligned}$$

Proof. Let $p \in \Gamma$. From Lemma 2.1 and the Cauchy-Schwartz inequality, we obtain from Step 2 that

$$\begin{aligned}\|w_n-p\|^2 &= \|x_n-p\|^2 + \theta_n^2\|x_n-x_{n-1}\|^2 + 2\theta_n\langle x_n-p, x_n-x_{n-1}\rangle \\ &\le \|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|\left[\theta_n\|x_n-x_{n-1}\| + 2\|x_n-p\|\right] \\ &\le \|x_n-p\|^2 + 3\gamma_n\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2, \qquad (4.11)\end{aligned}$$

for some $M_2 > 0$. In view of Lemma 2.1, (4.8), and (4.11), we have from Step 5 that

$$\begin{aligned}&\|x_{n+1}-p\|^2 \\ &= \|(1-\beta_n-\gamma_n)(w_n-p) + \beta_n(v_n-p) + \gamma_n(-p)\|^2 \\ &\le (1-\beta_n-\gamma_n)\|w_n-p\|^2 + \beta_n\|v_n-p\|^2 + \gamma_n\|p\|^2 - (1-\beta_n-\gamma_n)\beta_n\|w_n-v_n\|^2 \\ &\le (1-\gamma_n)\|w_n-p\|^2 - \beta_n\tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2+\|H(w_n)\|^2} \\ &\quad - \beta_n\alpha_n(1-\alpha_n)\|w_n-z_n\|^2 - \beta_n\Big(1-s_n^2\cdot\frac{\mu^2}{s_{n+1}^2}\Big)\|y_n-u_n\|^2 + \gamma_n\|p\|^2 \\ &\quad - (1-\beta_n-\gamma_n)\beta_n\|w_n-v_n\|^2 \\ &\le (1-\gamma_n)\left[\|x_n-p\|^2 + 3\gamma_n\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2\right] - \beta_n\tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2+\|H(w_n)\|^2} \\ &\quad - \beta_n\alpha_n(1-\alpha_n)\|w_n-z_n\|^2 - \beta_n\Big(1-s_n^2\cdot\frac{\mu^2}{s_{n+1}^2}\Big)\|y_n-u_n\|^2 + \gamma_n\|p\|^2 \\ &\quad - (1-\beta_n-\gamma_n)\beta_n\|w_n-v_n\|^2 \\ &\le (1-\gamma_n)\|x_n-p\|^2 + \gamma_n\Big(3\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2 + \|p\|^2\Big) \\ &\quad - \beta_n\tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2+\|H(w_n)\|^2} - \beta_n\alpha_n(1-\alpha_n)\|w_n-z_n\|^2 \\ &\quad - \beta_n\Big(1-s_n^2\cdot\frac{\mu^2}{s_{n+1}^2}\Big)\|y_n-u_n\|^2 - (1-\beta_n-\gamma_n)\beta_n\|w_n-v_n\|^2,\end{aligned}$$

which is the desired result. □

Lemma 4.5. *The following inequality holds for all $p \in \Gamma$, $M_2 > 0$ and $n \geq n_0$*

$$\|x_{n+1}-p\|^2 \leq (1-\gamma_n)\|x_n-p\|^2 + \gamma_n\Big[3\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2 + 2\beta_n\|v_n-w_n\|\|p-x_{n+1}\| + 2\langle p, p-x_{n+1}\rangle\Big].$$

Proof. Let $p \in \Gamma$ and $d_n = (1-\beta_n)w_n + \beta_n v_n$. Then $\|d_n - w_n\| = \beta_n\|v_n - w_n\|$. Applying Lemma 2.1, we have $\|d_n - p\| \leq (1-\beta_n)\|w_n - p\| + \beta_n\|v_n - p\| \leq \|w_n - p\|$, $\forall\, n \geq n_0$. Also, from the definition of x_{n+1} in Step 5, we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&= \|(1-\beta_n)w_n + \beta_n v_n - \gamma_n w_n - p\|^2 \\
&= \|d_n - \gamma_n w_n - p\|^2 \\
&= \|(1-\gamma_n)(d_n-p) - \gamma_n(w_n-d_n) - \gamma_n p\|^2 \\
&\leq (1-\gamma_n)^2\|d_n-p\|^2 - 2\langle \gamma_n(w_n-d_n) + \gamma_n p,\ x_{n+1}-p\rangle \\
&\leq (1-\gamma_n)\|d_n-p\|^2 + 2\gamma_n\|w_n-d_n\|\|p-x_{n+1}\| + 2\langle \gamma_n p,\ p-x_{n+1}\rangle \\
&\leq (1-\gamma_n)\Big[\|x_n-p\|^2 + 3\gamma_n\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2\Big] + 2\gamma_n\|w_n-d_n\|\|p-x_{n+1}\| + 2\gamma_n\langle p,\ p-x_{n+1}\rangle \\
&\leq (1-\gamma_n)\|x_n-p\|^2 + \gamma_n\Big[3\frac{\theta_n}{\gamma_n}\|x_n-x_{n+1}\|M_2 + 2\beta_n\|v_n-w_n\|\|p-x_{n+1}\| + 2\langle p, p-x_{n+1}\rangle\Big],
\end{aligned}$$

that is the desired result. □

We now prove our strong convergence theorem for Algorithm 3.1.

Theorem 4.1. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces, and let $A : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator. Let $\{x_n\}$ be a sequence generated by Algorithm 3.1, and suppose that Assumption 3.1 is satisfied. Then, $\{x_n\}$ converges strongly to a point $q \in \Gamma$, where $\|q\| = \min\{\|z\| : z \in \Gamma\}$.*

Proof. Since $\|q\| = \min\{\|z\| : z \in \Gamma\}$, then $q = P_\Gamma(0)$. It follows that $q \in \Gamma$. Now, from Lemma 4.5, we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq (1-\gamma_n)\|x_n-q\|^2 + \gamma_n\Big[3\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2 \\
&\qquad + 2\beta_n\|v_n-w_n\|\|q-x_{n+1}\| + 2\langle q, q-x_{n+1}\rangle\Big] \\
&= (1-\gamma_n)\|x_n-q\|^2 + \gamma_n a_n, \qquad (4.12)
\end{aligned}$$

where $a_n = 3\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2 + 2\beta_n\|v_n-w_n\|\|q-x_{n+1}\| + 2\langle q, q-x_{n+1}\rangle$.

Now, we claim that $\{\|x_n - q\|\}$ converges to zero. To establish this, it suffices from Lemma 2.4 to show that $\limsup_{k\to\infty} a_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - q\|\}$ of $\{\|x_n - q\|\}$ satisfying the condition

$$\liminf_{k\to\infty}\Big(\|x_{n_{k+1}}-q\| - \|x_{n_k}-q\|\Big) \geq 0. \qquad (4.13)$$

Suppose that $\{\|x_{n_k}-q\|\}$ is a subsequence of $\{\|x_n-q\|\}$ such that (4.13) holds. We obtain from Lemma 4.4 that

$$\begin{aligned}&\beta_{n_k}\tau_{n_k}(4-\tau_{n_k})(1-\alpha_{n_k})\frac{g^2(w_{n_k})}{\|G(w_{n_k})\|^2+\|H(w_{n_k})\|^2}+\beta_{n_k}\alpha_{n_k}(1-\alpha_{n_k})\|w_{n_k}-z_{n_k}\|^2\\&\quad+\beta_{n_k}\Big(1-s_{n_k}^2\cdot\frac{\mu^2}{s_{n_k+1}^2}\Big)\|y_{n_k}-u_{n_k}\|^2+(1-\beta_{n_k}-\gamma_{n_k})\beta_{n_k}\|w_{n_k}-v_{n_k}\|^2\\&\quad\le(1-\gamma_{n_k})\|x_{n_k}-q\|^2-\|x_{n_k+1}-q\|^2+\gamma_{n_k}\Big(3\frac{\theta_{n_k}}{\gamma_{n_k}}\|x_{n_k}-x_{n_k-1}\|M_2+\|q\|^2\Big).\end{aligned}$$

By applying (4.13) and the fact that $\lim_{k\to\infty}\gamma_{n_k}=0$, we have that

$$\lim_{k\to\infty}\beta_{n_k}\tau_{n_k}(4-\tau_{n_k})(1-\alpha_{n_k})\frac{g^2(w_{n_k})}{\|G(w_{n_k})\|^2+\|H(w_{n_k})\|^2}=0,$$

$$\lim_{k\to\infty}\beta_{n_k}\alpha_{n_k}(1-\alpha_{n_k})\|w_{n_k}-z_{n_k}\|^2=0,$$

$$\lim_{k\to\infty}\beta_{n_k}\Big(1-s_{n_k}^2\cdot\frac{\mu^2}{s_{n_k+1}^2}\Big)\|y_{n_k}-u_{n_k}\|^2=0,$$

and

$$\lim_{k\to\infty}(1-\beta_{n_k}-\gamma_{n_k})\beta_{n_k}\|w_{n_k}-v_{n_k}\|^2=0.$$

From the conditions on the control parameters, it follows that

$$\lim_{k\to\infty}\frac{g^2(w_{n_k})}{\|G(w_{n_k})\|^2+\|H(w_{n_k})\|^2}=\lim_{k\to\infty}\|w_{n_k}-z_{n_k}\|=\lim_{k\to\infty}\|y_{n_k}-u_{n_k}\|=\lim_{k\to\infty}\|w_{n_k}-v_{n_k}\|=0. \tag{4.14}$$

Since G and H are Lipschitz continuous, we have that $\lim_{k\to\infty}g(w_{n_k})=0$. From the definition of $g(w_{n_k})$, we have $\lim_{k\to\infty}g(w_{n_k})=\lim_{k\to\infty}\frac{1}{2}\|(I-T_{r_{n_k}}^{F_2})Aw_{n_k}\|^2=0$, which implies that

$$\lim_{k\to\infty}\|(I-T_{r_{n_k}}^{F_2})Aw_{n_k}\|=0. \tag{4.15}$$

Consequently,

$$\lim_{k\to\infty}\|A^*(I-T_{r_{n_k}}^{F_2})Aw_{n_k}\|\le\|A^*\|\|(I-T_{r_{n_k}}^{F_2})Aw_{n_k}\|=\|A\|\|(I-T_{r_{n_k}}^{F_2})Aw_{n_k}\|=0. \tag{4.16}$$

From the definition of y_{n_k} and (4.14), we have $\|y_{n_k}-z_{n_k}\|\le\alpha_{n_k}\|w_{n_k}-z_{n_k}\|+(1-\alpha_{n_k})\|z_{n_k}-z_{n_k}\|\to0$ as $k\to\infty$. This together with (4.14) yields that

$$\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=\lim_{k\to\infty}\|w_{n_k}-u_{n_k}\|=\lim_{k\to\infty}\|z_{n_k}-u_{n_k}\|=0. \tag{4.17}$$

From Step 5, and (4.14) together with the fact that $\lim_{k\to\infty}\gamma_{n_k}=0$, we have $\|x_{n_{k+1}}-w_{n_k}\|=\|\beta_{n_k}(v_{n_k}-w_{n_k})-\gamma_{n_k}w_{n_k}\|\le\beta_{n_k}\|v_{n_k}-w_{n_k}\|+\gamma_{n_k}\|w_{n_k}\|\to0$ as $k\to\infty$. From Remark 3.1, we have $\|w_{n_k}-x_{n_k}\|=\theta_{n_k}\|x_{n_k}-x_{n_{k-1}}\|\to0$ as $k\to\infty$. Combing these, we have

$$\lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0 \tag{4.18}$$

Thus

$$\lim_{k\to\infty}\|x_{n_k}-z_{n_k}\|=0,\ \lim_{k\to\infty}\|x_{n_k}-v_{n_k}\|=0,\ \lim_{k\to\infty}\|x_{n_k}-u_{n_k}\|=0,\ \lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|=0. \quad (4.19)$$

Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $x^*\in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$ as $k\to\infty$. From (4.19), we obtain $y_{n_k}\rightharpoonup x^*$ as $k\to\infty$. By invoking Lemma 4.3, it follows from (4.14) that $x^*\in(B+D)^{-1}(0)$. Since x^* is an arbitrary element in $w_\omega(x_n)$, it follows that $w_\omega(x_n)\subset(B+D)^{-1}(0)$.

Next, we prove that $w_\omega(x_n)\subset\Omega$. First, we establish that $w_\omega(x_n)\subset EP(F_1)$. From $z_{n_k}=T^{F_1}_{r_{n_k}}(I-\lambda_{n_k}A^*(I-T^{F_2}_{r_{n_k}})A)w_{n_k}$, we have $F_1(z_{n_k},y)+\frac{1}{r_{n_k}}\langle y-z_{n_k},z_{n_k}-w_{n_k}+\lambda_{n_k}G(w_{n_k})\rangle\geq 0$ for all $y\in\mathscr{C}$, which implies that

$$F_1(z_{n_k},y)+\frac{1}{r_{n_k}}\Big\langle y-z_{n_k},\ z_{n_k}-w_{n_k}\Big\rangle+\frac{1}{r_{n_k}}\Big\langle y-z_{n_k},\ \lambda_{n_k}G(w_{n_k})\Big\rangle\geq 0,\ \forall\, y\in\mathscr{C}.$$

We have from Assumption 2.1 (2) that

$$\frac{1}{r_{n_k}}\Big\langle y-z_{n_k},\ z_{n_k}-w_{n_k}\Big\rangle+\frac{1}{r_{n_k}}\Big\langle y-z_{n_k},\ \lambda_{n_k}G(w_{n_k})\Big\rangle\geq F_1(y,\ z_{n_k}),\ \forall\, y\in\mathscr{C}.$$

Since $z_{n_k}\rightharpoonup x^*$, then by applying (4.14), (4.16), Assumption 2.1 (4) and the fact that $\liminf\limits_{k\to\infty} r_{n_k}>0$, we obtain $F_1(y,x^*)\leq 0$, $\forall y\in\mathscr{C}$. Let $y_t=ty+(1-t)x^*$, $\forall t\in(0,1]$ and $y\in\mathscr{C}$. This implies that $y_t\in\mathscr{C}$. Thus $F_1(y_t,x^*)\leq 0$. Applying Assumption 2.1 (1)-(4), we have

$$0=F_1(y_t,y_t)\leq tF_1(y_t,y)+(1-t)F_1(y_t,x^*)\leq tF_1(y_t,y).$$

Hence, $F_1(y_t,y)\geq 0$, $\forall y\in\mathscr{C}$. Letting $t\to 0$ and applying Assumption 2.1 (3), we have $F_1(x^*,y)\geq 0$, $\forall y\in\mathscr{C}$, which implies that $x^*\in EP(F_1)$.

Next, we show that $Ax^*\in EP(F_2)$. Since A is a bounded linear operator and $w_{n_k}\rightharpoonup x^*$, we have $Aw_{n_k}\rightharpoonup Ax^*$. Consequently, it follows from (4.15) that $T^{F_2}_{r_{n_k}}Aw_{n_k}\rightharpoonup Ax^*$ as $k\to\infty$. From the definition of $T^{F_2}_{r_{n_k}}Aw_{n_k}$, we have

$$F_2(T^{F_2}_{r_{n_k}}Aw_{n_k},y)+\frac{1}{r_{n_k}}\Big\langle y-T^{F_2}_{r_{n_k}}Aw_{n_k},\ T^{F_2}_{r_{n_k}}Aw_{n_k}-Aw_{n_k}\Big\rangle\geq 0,\ \forall\, y\in\mathscr{Q}.$$

Since F_2 is upper semicontinuous in the first argument, it follows from (4.15) and the fact that $\liminf\limits_{k\to\infty} r_{n_k}>0$ that $F_2(Ax^*,y)\geq 0$, $\forall y\in\mathscr{Q}$, which implies that $Ax^*\in EP(F_2)$. Thus $w_\omega(x_n)\subset\Gamma$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ converging weakly to $\hat{x}$ such that $\limsup_{k\to\infty}\langle q,q-x_{n_k}\rangle=\lim_{j\to\infty}\langle q,q-x_{n_{k_j}}\rangle$. Since $q=P_\Gamma(0)$, we have

$$\limsup_{k\to\infty}\Big\langle q,\ q-x_{n_k}\Big\rangle=\lim_{j\to\infty}\langle q,\ q-x_{n_{k_j}}\rangle=\Big\langle q,\ q-\hat{x}\Big\rangle\leq 0,$$

which together with (4.18) implies that $\lim_{k\to\infty}\langle q,q-x_{n_{k+1}}\rangle\leq 0$. Using (4.14) and the fact that $\lim_{n\to\infty}\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|=0$, we have $\limsup_{k\to\infty}a_{n_k}\leq 0$. Aapplying Lemma 2.4 to (4.12), we have that $\lim_{n\to\infty}\|x_n-q\|=0$. Thus $\{x_n\}$ converges strongly to q. □

5. Applications

In this section, we apply our result to study certain optimization problems.

5.1. **Variational inclusion and split variational inequality problems.** In this subsection, we apply our result to approximate the common solution of variational inclusion and split variational inequality problems.

Let $\mathscr{C}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathscr{H}$, and let $M : \mathscr{H} \to \mathscr{H}$ be a single-valued mapping. The *Variational Inequality Problem* ($\mathrm{VI}_q\mathrm{P}$) is formulated as follows:

$$\text{Find } x^* \in \mathscr{C} \text{ such that } \langle y - x^*, Mx^* \rangle \geq 0, \quad \forall\, y \in \mathscr{C}.$$

We denote the solution set of the $\mathrm{VI}_q\mathrm{P}$ by $VI(\mathscr{C}, P)$. The variational inequality theory was first introduced independently by Fichera [30] and Stampacchia [31]. The $\mathrm{VI}_q\mathrm{P}$ is a useful mathematical model, which unifies several important concepts in applied mathematics, such as complementarity problems, network equilibrium problems, necessary optimality conditions, etc. Various solution methods were investigated for finding solutions of $\mathrm{VI}_q\mathrm{P}$; see, e.g., [20, 32, 33, 34] and hte references therein. Here, we apply our result to the following split variational inequality problem ($\mathrm{SVI}_q\mathrm{P}$) with constraint of variational inclusion problem:

$$\text{Find } x^* \in (B+D)^{-1}(0) \text{ such that } \langle x - x^*, M_1 x^* \rangle \geq 0, \quad \forall\, x \in \mathscr{C}. \tag{5.1}$$

and

$$y^* = Ax^* \in \mathscr{Q} \text{ solves } \langle y - y^*, M_2 y^* \rangle \geq 0, \quad \forall\, y \in \mathscr{Q}, \tag{5.2}$$

where $\mathscr{C}$ and $\mathscr{Q}$ are nonempty, closed, and convex subsets of real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator, $B : \mathscr{H}_1 \to \mathscr{H}_1$ is a Lipschitz continuous and monotone operator, and $D : \mathscr{H}_1 \to 2^{\mathscr{H}_1}$ is a maximal monotone operator. We denote the solution set of problem (5.1)-(5.2) by Ω_1 and assume that $(B+D)^{-1}(0) \cap \Omega_1 \neq \emptyset$. Taking $F_i(x,y) := \langle y - x, M_i x \rangle, i = 1,2$, one sees that the $\mathrm{SVI}_q\mathrm{P}$ (5.1)-(5.2) becomes the problem of finding a solution of the SEP (1.5)-(1.6), which is also a solution of variational inclusion problem (1.1). Furthermore, all the conditions of Theorem 4.1 hold. Thus Theorem 4.1 provides a strong convergence theorem for approximating a common solution of VIP (1.1) and $\mathrm{SVI}_q\mathrm{P}$ (5.1)-(5.2).

5.2. **Convex minimization and split equilibrium problems.** Let $\mathscr{H}$ be a real Hilbert space, $f : \mathscr{H} \to \mathbb{R}$ be a convex function, and $F : \mathscr{H} \to \mathbb{R}$ be a proper convex and lower semicontinuous function. We consider the following *convex minimization problem*:

$$\min_{x \in H} f(x) + F(x), \tag{5.3}$$

which is equivalent to finding $x \in \mathscr{H}$ such that $0 \in \nabla f(x) + \partial F(x)$, where ∇f is the gradient of f and ∂F is the subdifferential of F. It is known that if ∇f is L-Lipschitz continuous, then it is $\frac{1}{L}$-inverse strongly monotone (co-coercive), and hence it is L-Lipschitz continuous and monotone. Also, it is known that ∂F is maximal monotone (see [35]). The solution set of (5.3) we denote by Ω_2. So, by setting $B = \nabla f$ and $D = \partial F$ in Theorem 4.1, we obtain the following result for approximating a common solution of convex minimization problem (5.3) and split equilibrium problem (1.5)-(1.6) in Hilbert spaces.

Theorem 5.1. *Let $\mathscr{C}$ and $\mathscr{Q}$ be nonempty, closed, and convex subsets of real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Let $F_1 : \mathscr{C} \times \mathscr{C} \to \mathbb{R}, F_2 : \mathscr{Q} \times \mathscr{Q} \to \mathbb{R}$ be bifunctions such that Assumption (2.1) holds and F_2 is upper semi-continuous in the first argument. Let $f : \mathscr{H} \to \mathbb{R}$ be a convex and differentiable function such that ∇f is L-Lipschitz continuous and $G : \mathscr{H} \to \mathbb{R}$ be a proper convex and lower semicontinuous function. Let $\{x_n\}$ be a sequence generated as follows*

Algorithm 5.1.

Step 1: Select initial point $x_0, x_1 \in \mathscr{H}_1$, let $s_1 > 0, \mu \in (0,1), \theta \geq 3$, and set $n = 1$. Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases}$$

Step 2: Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$.
Step 3: Compute $z_n = T_{r_n}^{F_1}\big(I - \lambda_n A^*(I - T_{r_n}^{F_2})A\big)w_n$ and $y_n = \alpha_n w_n + (1-\alpha_n)z_n$.
Step 4: Compute $u_n = J_{s_n}^{\partial F}(I - s_n \nabla f)y_n$ and $v_n = u_n - s_n(\nabla f u_n - \nabla f y_n)$.
Step 5 Compute $x_{n+1} = (1 - \beta_n - \gamma_n)w_n + \beta_n v_n$, where

$$\lambda_n := \begin{cases} \tau_n \frac{g(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2}, & \text{if } \|G(w_n)\|^2 + \|H(w_n)\|^2 \neq 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$s_{n+1} = \begin{cases} \min\left\{\frac{\mu\|y_n - u_n\|}{\|\nabla f y_n - \nabla f u_n\|}, s_n\right\} & \text{if } \quad \nabla f y_n - \nabla f u_n \neq 0, \\ s_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go back to **Step 1**.

Then, $\{x_n\}$ converges strongly to a point $q \in \Omega_2 \cap \Omega$, where $\|q\| = \min\{\|z\| : z \in \Omega_2 \cap \Omega\}$.

6. Numerical Experiments

In this section, using some test examples, we discuss the numerical behavior of Algorithm 3.1 and compare it with the standard forward-backward method ($\theta_n = 0$), the Algorithm 1.7 proposed by Cholamjiak et al. [21], the shrinking projection method of Cholamjiak et al. [21] (see Appendix 7.1), Appendix 7.2, and Appendix 7.3. We perform all implementations using Matlab 2016 (b), installed on a personal computer with Intel(R) Core(TM) i5-2600 CPU@2.30GHz and 8.00 Gb-RAM running on Windows 10 operating system. In Tables 1-2, "No. of Iter." means the number of iterations.

In our computations, we choose $\alpha_n = \frac{n+1}{2n+1}, \gamma_n = \frac{1}{n+2}, \beta_n = \frac{1}{2} - \gamma_n,\ \varepsilon_n = \frac{1}{(n+2)^2},\ \theta = 7,\ s_1 = 0.65,\ \mu = 0.8,\ \tau_n = \frac{6n+2}{2n+1}, r_n = \frac{3n+1}{2n+1}$, and we take $\delta_n = \frac{n+1}{2n+3}, t_n = 0.1$ in Algorithm 1.7 and Appendix 7.1. Also, for Appendix 7.3, we choose $f(x) = \frac{x}{2}$. Furthermore, in the implementation, we define $\text{TOL}_n := \|x_{n+1} - x_n\|$ and use the stopping criterion $\text{TOL}_n < 10^{-2}$ for the iterative processes.

Example 6.1. Let $\mathscr{H}_1 = \mathscr{H}_2 = \mathbb{R}$, the set of all real numbers with the inner product defined by $\langle x, y\rangle = xy,\ \forall\, x, y \in \mathbb{R}$ and induced norm $|\cdot|$. For $r > 0$, consider $\mathscr{C} = [-10, 10]$ and $\mathscr{Q} = [0, 20]$, and define the bifunctions $F_1 : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$ and $F_2 : \mathscr{Q} \times \mathscr{Q} \to \mathbb{R}$ by $F_1 = -2x^2 + xy + y^2$ and $F_2 = -x^2 + xy$. From Lemma 2.2, we see that $T_r^{F_1}(u) = \frac{u}{3r+1}, \forall u \in \mathscr{C}$ and $T_r^{F_2}(v) = \frac{v}{r+1}, \forall v \in \mathscr{Q}$. Let $A : \mathscr{H}_1 \to \mathscr{H}_2$ be defined by $Ax = 2x$ and $B : \mathscr{H}_1 \to \mathscr{H}_1$ be defined by $Bx = x + \sin x$. Let $D : \mathscr{H}_1 \to \mathscr{H}_1$ be defined by $Dx = 5x$, where $x \in \mathscr{H}_1$. Clearly, we see that B is $\frac{1}{3}$-inverse strongly monotone and D is maximal monotone. We consider the following cases and choose $\gamma = \frac{3}{20}$

Case I: Take $x_0 = -\frac{17}{33}$ and $x_1 = \frac{2}{15}$.
Case II: Take $x_0 = -\frac{19}{35}$ and $x_1 = \frac{1}{6}$.
Case III: Take $x_0 = \frac{11}{20}$ and $x_1 = \frac{1}{7}$.

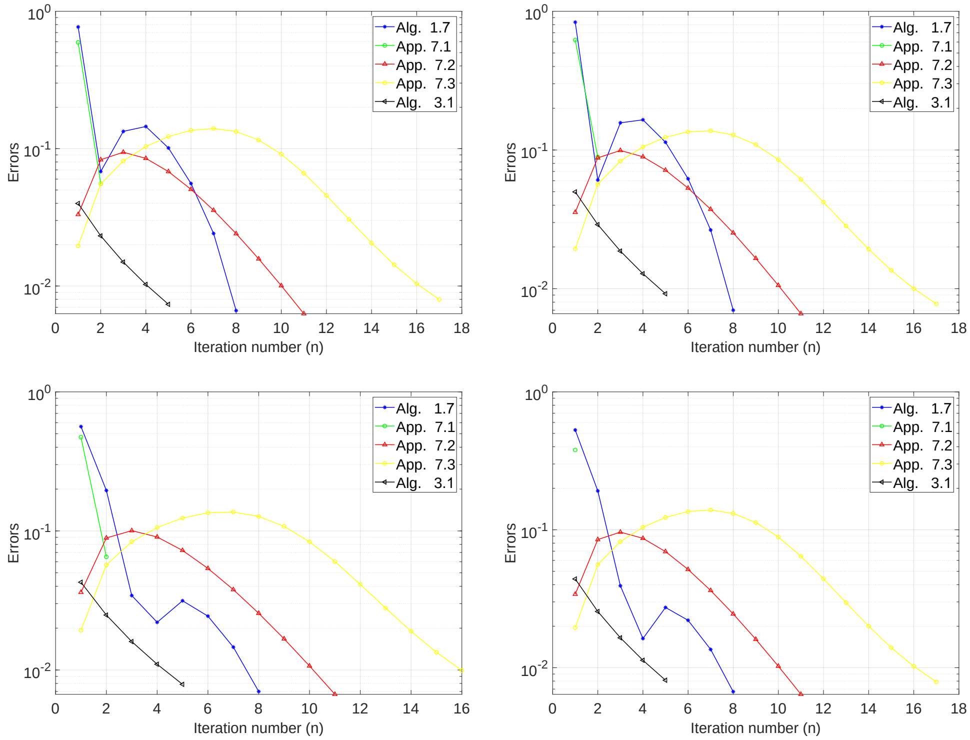

FIGURE 1. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

Case IV: Take $x_0 = -\frac{10}{19}$ and $x_1 = -\frac{5}{34}$.

We compare the performance of our Algorithm 3.1 with Algorithm 1.7, Appendix 7.1, Appendix 7.2, and Appendix 7.3. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Table 1 and Figure 1.

TABLE 1. **Numerical Results for Example 6.1**

		Alg. 1.7	App 7.1	App. 7.2	App. 7.3	Alg. 3.1
Case I	No. of Iter. CPU time (sec)	8 0.0099	3 0.0071	11 0.0090	17 0.0063	5 0.0114
Case II	No. of Iter. CPU time (sec)	8 0.0095	3 0.0075	11 0.0080	17 0.0072	5 0.0155
Case III	No. of Iter. CPU time (sec)	8 0.0103	3 0.0081	11 0.0093	16 0.0084	5 0.0125
Case IV	No. of Iter. CPU time (sec)	8 0.0098	3 0.0071	11 0.0082	17 0.0070	5 0.0111

Example 6.2. Let $\mathscr{H}_1 = \mathscr{H}_2 = (l_2(\mathbb{R}), \|\cdot\|_2)$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^2 < \infty\}, ||x||_2 = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$, and $\langle x, y\rangle = \sum_{j=1}^{\infty} x_j y_j$ for all $x \in \ell_2(\mathbb{R})$. We define $F_1 : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$ and $F_2 : \mathscr{Q} \times \mathscr{Q} \to \mathbb{R}$ by $F_1(x,y) = \langle L_1 x, y - x\rangle$ and $F_2(x,y) = \langle L_2 x, y - x\rangle$, where $L_1 x = \frac{x}{3}$ and $L_2 x = \frac{x}{2}$. One can easily verify that F_1 and F_2 satisfy Assumption 2.1. Let $A : \mathscr{H}_1 \to \mathscr{H}_2$ be defined by $Ax = \frac{x}{3}$ and $A^* y = \frac{y}{3}$. Then, A is a bounded linear operator. After simple calculation and applying Lemma 2.2, we obtain $T_r^{F_1}(u) = \frac{3u}{r+3}$, $\forall u \in \mathscr{C}$, and $T_s^{F_2}(v) = \frac{2v}{s+2}$, $\forall v \in \mathscr{Q}$. Let $B : \mathscr{H}_1 \to \mathscr{H}_1$ be defined by $Bx = \frac{1}{3}x$, and let $D : \mathscr{H}_1 \to \mathscr{H}_1$ be defined by $Dx = 3x$, where $x \in \mathscr{H}_1$. Clearly, we see that B is $\frac{1}{2}$-inverse-strongly monotone and D is maximal monotone. Consider different initial values as follows:
Case I: $x_0 = (-23, 1, -\frac{1}{23}, \cdots)$, $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$;
Case II: $x_0 = (37, 1, \frac{1}{37}, \cdots)$, $x_1 = (\frac{5}{18}, \frac{5}{36}, \frac{5}{72}, \cdots)$;
Case III: $x_0 = (-29, 1, -\frac{1}{29}, \cdots)$, $x_1 = (-\frac{3}{8}, \frac{3}{16}, -\frac{1}{32}, \cdots)$;
Case IV: $x_0 = (25, 1, \frac{1}{25}, \cdots)$, $x_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{18})$.

We compare the performance of our Algorithm 3.1 with Appendix 7.1, Appendix 7.2, and Appendix 7.3. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Table 2 and Figure 2.

TABLE 2. **Numerical results for Example 6.2**

		Alg. 1.7	App 7.1	App. 7.2	App. 7.3	Alg. 3.1
Case I	No. of Iter.	13	3	14	11	6
	CPU time (sec)	0.0137	0.0148	0.0145	0.0092	0.0175
Case II	No. of Iter.	13	3	14	11	6
	CPU time (sec)	0.0114	0.0123	0.0120	0.0097	0.0170
Case III	No. of Iter.	13	3	14	11	6
	CPU time (sec)	0.0113	0.0135	0.0154	0.0082	0.0150
Case IV	No. of Iter.	13	3	14	11	6
	CPU time (sec)	0.0126	0.0162	0.0139	0.0122	0.0245

7. CONCLUSION

In this paper, we studied the problem of finding the common solutions of VIP (1.1) and the SEP (1.5)-(1.6). We proposed a modified inertial forward-backward splitting algorithm with self-adaptive step sizes for approximating the solution of the problem in Hilbert spaces. Our method does not require the Lipschitz constant to be known, which makes our method easier to implement than some other methods in the literature that require knowledge of the Lipschitz constant. We proved that the sequence generated by our proposed method converges strongly to the minimum-norm solution of the problem without following the usual the two-case approach widely used in many papers. Furthermore, we applied our result to certain optimization

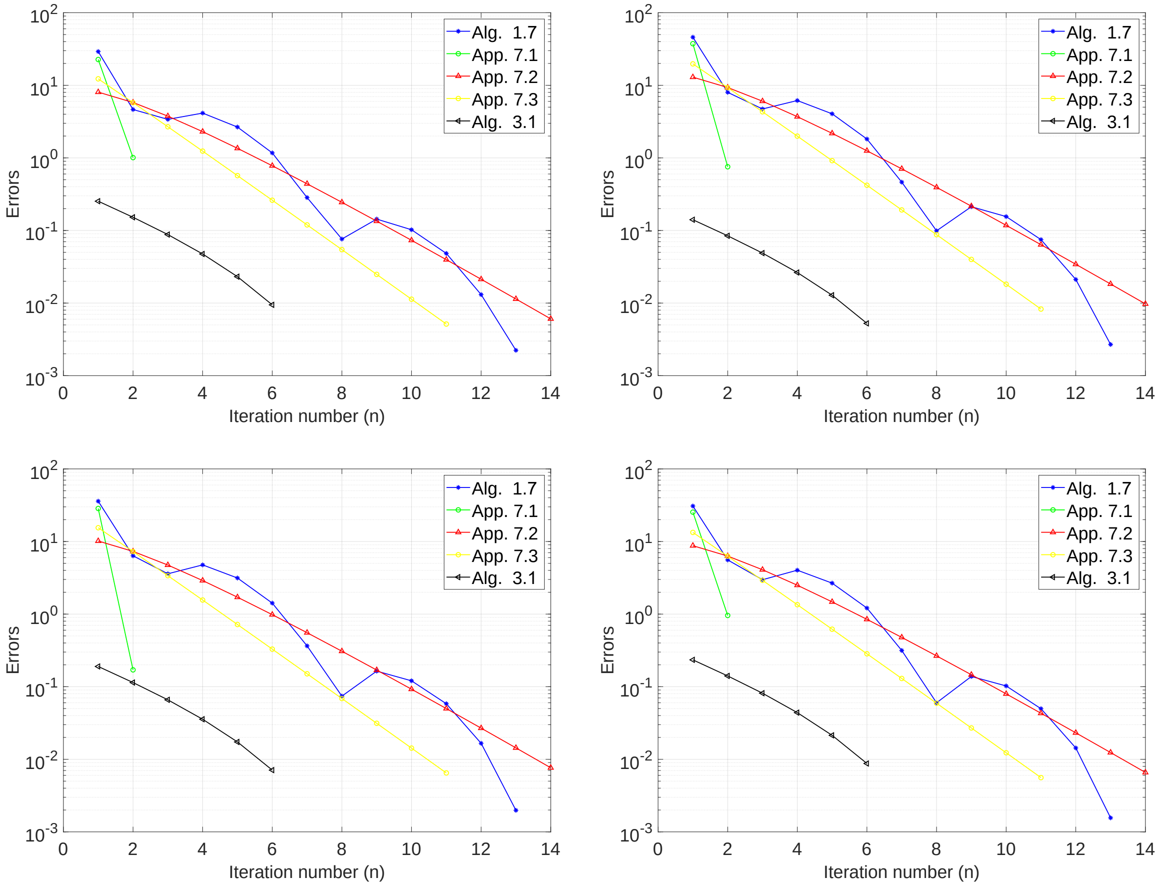

FIGURE 2. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

problems. Finally, we carried out some numerical experiments on our proposed method in comparison with other existing methods in the literature. The results show that our method is easier to implement and also outperforms other methods in the literature.

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Appendix 7.1. The Algorithm in [21].

Initialization: Give $\gamma \in (0, \frac{1}{L})$ and let $x_0, x_1 \in \mathscr{H}_1$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:

$$\begin{cases} y_n = x_n + \delta_n(x_n - x_{n-1}) \\ z_n = \alpha_n y_n + (1-\alpha_n) T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)y_n, \\ w_n = \beta_n z_n + (1-\beta_n) J_{t_n}^{D}(I - t_n B)z_n, \\ C_{n+1} = \{z \in C_n : \|w_n - z\|^2 \le \|x_n - z\|^2 + 2{\delta_n}^2\|x_n - x_{n-1}\|^2 - 2\delta\langle x_n - z,\, x_{n-1} - x_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \;\; n \ge 1, \end{cases}$$

where $J_{t_n}^{D} = (I + t_n D)^{-1}$, $\{t_n\} \subset (0, 2\alpha)$, $\{\delta_n\} \subset [0, \delta]$, $\delta \in [0,1)$, $\{r_n\} \subset (0, \infty)$ with $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$.
Set $n := n+1$ and return to **Step 1**.

Appendix 7.2. The Algorithm in [1].

Initialization: Give $s_1 > 0, \mu \in (0,1)$ and let $x_1 \in \mathscr{H}$ be arbitrary.
Iterative Steps: Given the current iterates x_n, calculate the next iterate as follows:
Step 1. $y_n = (I + s_n D)^{-1}(I - s_n B)x_n$. If $x_n = y_n$, then stop and y_n is a solution of (1.1). Otherwise,
Step 2. Compute $z_n = y_n - s_n(By_n - Bx_n)$ and $x_{n+1} = (1 - \beta_n - \gamma_n)x_n + \gamma_n z_n$. Update

$$s_{n+1} = \begin{cases} \min\left\{\frac{\mu\|x_n - y_n\|}{\|Bx_n - By_n\|}, s_n\right\} & \text{if} \quad Bx_n - By_n \ne 0, \\ s_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and return to **Step 1**.

Appendix 7.3. The Algorithm in [1].

Initialization: Give $s_1 > 0, \mu \in (0,1)$ and let $x_1 \in \mathscr{H}$ be arbitrary.
Iterative Steps: Given the current iterates x_n, calculate the next iterate as follows:
Step 1. $y_n = (I + s_n D)^{-1}(I - s_n B)x_n$. If $x_n = y_n$, then stop and y_n is a solution of (1.1). Otherwise,
Step 2. Compute $z_n = y_n - s_n(By_n - Bx_n)$ and $x_{n+1} = \gamma_n f(x_n) + (1 - \gamma_n)z_n$. Update

$$s_{n+1} = \begin{cases} \min\left\{\frac{\mu\|x_n - y_n\|}{\|Bx_n - By_n\|}, s_n\right\} & \text{if} \quad Bx_n - By_n \ne 0, \\ s_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and return to **Step 1**.
