Appl. Set-Valued Anal. Optim. 4 (2022), No. 2, pp. 223-237
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.4.2022.2.06

STRONG SUBDIFFERENTIAL CALCULUS FOR CONVEX SET-VALUED MAPPINGS AND APPLICATIONS TO SET OPTIMIZATION

EL MAHJOUB ECHCHAABAOUI*, MOHAMED LAGHDIR

Department of Mathematics, Faculty of Sciences, Chouaïb Doukkali University, El-Jadida, Morocco

Abstract. The aim of this paper is to establish the strong vector subdifferential of the convex set-valued mapping $F+G\circ H$, where F, G, and H are convex set-valued mappings, and G is nondecreasing. An application is given to deal with optimality conditions for a constrained convex set-valued optimization problem.

Keywords. Normal cone; Optimality conditions; Set-valued vector optimization; Strong subdifferential; Topological vector space.

1. INTRODUCTION

It is known that strong vector subdifferential calculus plays an important role in vector optimization problems; see, e.g., [1, 2, 3, 4] for its theoretical development and real applications. In [1], Théra established a formula for the strong vector subdifferential of the sum of two vector valued mappings in the framework of ordered complete topological vector spaces by using the so-called sandwich theorem. Recently, Laghdir et al. in [5] established the strong vector subdifferential calculus of the composed convex operator $f+g\circ h$ when f, g, and h are vector valued convex mappings, and g is nondecreasing. Our main objective in this paper is to establish the sum and composition rules for the strong vector subdifferential in the setting of set-valued convex mappings. To our knowledge, it seems that this problem has not been explored previously.

The paper is structured as follows. In Section 2, we present some preliminaries which are needed in the sequel. Section 3 is devoted to stating the strong subdifferential calculus rules of the sum and the composition of convex set-valued mappings. In Section 4, the last section, we derive from the obtained formulas the optimality conditions for a set-valued convex constrained optimization problem.

2. PRELIMINARIES

In this section, we give some basic definitions and results. In what follows, let X and Y be two Hausdorff locally convex topological vector spaces and $Y_+\subset Y$ be a pointed ($Y_+\cap -Y_+=\{0_Y\}$) closed and convex cone with nonempty topological interior inducing a partial order in Y, which is defined by $y_1\leq_{Y_+} y_2 \Longleftrightarrow y_2-y_1\in Y_+$, for $y_1,y_2\in Y$ (see [2, 6]). We adjoin to Y an abstract

*Corresponding author.
E-mail addresses: mahjoubmath@gmail.com (M. Echchaabaoui), laghdirm@gmail.com (M. Laghdir).
Received December 26, 2021; Accepted March 9, 2022.

maximal element $+\infty_Y$ such that, for any $y \in Y$, $y \leq_{Y_+} +\infty_Y$. If u and v are two elements of Y and $u \leq_{Y_+} v$, then the set $[u,v] := \{w \in Y : u \leq_{Y_+} w \leq_{Y_+} v\}$ is the order interval between u and v. A subset C of Y is order bounded if there exist u and v in Y such that $C \subseteq [u,v]$. A subset C of Y is majorized (resp. minorized) if there exists an element $w \in C$ such that $a \leq_{Y_+} w$ (resp. $w \leq_{Y_+} a$) for all $a \in C$. For the subset $C \subset Y$, if there exists $w \in Y$ such that

(i) $a \leq_{Y_+} w$ for all $a \in C$;
(ii) $w \leq_{Y_+} v$ whenever $a \leq_{Y_+} v$ for all $a \in C$,

then w is called the supremum of C, and we write $w = \sup C$. We write the infimum of C, as $\inf C$. We say that (Y,Y_+) is order complete if every minorized subset of Y has an infimum. This is, in fact, equivalent to saying that every majorized subset of Y has a supremum. In addition, (Y,Y_+) is order complete lattice if (Y,Y_+) is order complete and for any pair of elements u, v in Y , $\sup(u,v)$ and $\inf(u,v)$ exists in Y. We often assume that Y_+ is normal, i.e., there exists a basis of open neighbourhoods $\mathfrak{B}$ of the origin such that $W = (W - Y_+) \cap (W + Y_+)$, $\forall W \in \mathfrak{B}$. Let $F : X \rightrightarrows Y$ be a set-valued mapping. The effective domain, graph, and image are defined respectively by

$$\begin{aligned} \mathrm{dom}F &:= \{x \in X : F(x) \neq \emptyset\}, \\ \mathrm{gr}F &:= \{(x,y) \in X \times Y : y \in F(x)\}, \\ \mathrm{Im}F &:= \bigcup_{x \in X} F(x). \end{aligned}$$

If we define the set-valued mapping $F + Y_+$ from X into Y by $(F+Y_+)(x) := F(x) + Y_+$ for any $x \in X$, then the set $\mathrm{epi}F := \mathrm{gr}(F+Y_+) = \{(x,y) \in X \times Y : y \in F(x) + Y_+\}$ is called the epigraph of F.

Definition 2.1. [7] The set-valued mapping F is said to be

1) Y_+-convex if its epigraph is a convex subset of $X \times Y$;
2) proper if its effective domain $\mathrm{dom}F \neq \emptyset$.

Let us recall the concept of connectedness and the concept of the continuity of a set-valued mapping.

Definition 2.2. [8, 9] Let $F : X \rightrightarrows Y$ be a set-valued mapping.

1) F is said to be connected at $x_0 \in X$ if there exists a mapping $h : X \to Y$ such that $h(v) \in F(v)$ for all v in some neighborhood of x_0, and h is continuous at x_0.
2) F is said to be upper-semicontinuous at x_0 if, for any open subset $V \supseteq F(x_0)$, there exists a neighborhood U of x_0 such that $F(U) \subseteq V$.
3) F is said to be lower-semicontinuous at x_0 if, for any open subset V such that$V \cap F(x_0) \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(U) \cap V \neq \emptyset$.

We say that F is continuous at x_0 if it is upper-semicontinuous and lower-semicontinuous at x_0, and we say that F is continuous on X if it is continuous at each point $x \in X$.

We now recall the notion of strongly minimal points.

Definition 2.3. [10] Let $A \subset Y$ be a nonempty subset. $\bar{y} \in A$ is said to be a strongly minimal point of the subset A if $A \subseteq \bar{y} + Y_+$.

Now consider the following set-valued optimization problem:

$$(\mathrm{P}) \qquad \begin{cases} \min \mathrm{F(x)}, \\ x \in S, \end{cases}$$

where $F : X \supseteq S \rightrightarrows Y$ is a set-valued mapping, and $S \subset X$. Using Definition (2.3), a pair $(\bar{x}, \bar{y}) \in X \times Y$ is said to be a strong minimum solution to (P) if $\bar{x} \in S$, and $\bar{y}$ is a strongly minimal point of $F(S)$, i.e., $\bar{y} \leq_{Y_+} y$, $\forall x \in S$, $\forall y \in F(x)$.

Definition 2.4. [8, 11] Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \mathrm{gr}F$. The strong subdifferential of F at $(\bar{x}, \bar{y})$ is defined by

$$\partial^s F(\bar{x}, \bar{y}) := \{T \in L(X,Y) : T(x - \bar{x}) \leq_{Y_+} y - \bar{y}, \forall (x,y) \in \mathrm{gr}F\},$$

where $L(X,Y)$ is the set of all continuous linear operators from X into Y.

By convention, we take $\partial^s F(\bar{x}, \bar{y}) = \emptyset$ if $(\bar{x}, \bar{y}) \notin \mathrm{gr}F$, and we say that F is strongly subdifferentiable at $(\bar{x}, \bar{y})$ if $\partial^s F(\bar{x}, \bar{y}) \neq \emptyset$. Give a convex subset $C \subset X$ with $\bar{x} \in C$. From [2], the vector normal cone to C at $\bar{x}$ is defined as

$$N^v(\bar{x}; C) := \{T \in L(X,Y) : T(x - \bar{x})\rangle \leq_{Y_+} 0_Y, \forall x \in C\}.$$

It is easy to check that the subdifferential $\partial^s F(\bar{x}, \bar{y})$ of a convex set-valued mapping at $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ can be represented geometrically as

$$\partial^s F(\bar{x}, \bar{y}) = \{T \in L(X,Y) : (T, -id_Y) \in N^v((\bar{x}, \bar{y}); \mathrm{epi}\, F)\},$$

where id_Y is the identity mapping.

Now, we prove an existence theorem for the strong subdifferential. Let $F : X \rightrightarrows Y$ be a set-valued mapping, and let us consider the following condition

$$\forall x \in \mathrm{dom}F,\ \inf F(x) \text{ exists and belongs to } F(x). \tag{H_1}$$

If $x \notin \mathrm{dom}F$, we set $\inf F(x) := +\infty_Y$. Under condition (H_1), we associate to the set-valued mapping F a single vector mapping $\varphi_F : X \to Y \cup \{+\infty_Y\}$ defined by

$$\varphi_F(x) := \begin{cases} \inf F(x), & \text{if } x \in \mathrm{dom}F, \\ +\infty_Y, & \text{otherwise.} \end{cases}$$

Theorem 2.1. *Let $F : X \rightrightarrows Y$ be a Y_+-convex set-valued mapping satisfying condition (H_1), and $\bar{x} \in \mathrm{dom}F$. Suppose that Y_+ is normal, and F is continuous at $\bar{x}$. Then φ_F is Y_+-convex and continuous at $\bar{x}$.*

Proof. By using condition (H_1), we easily check that $\mathrm{epi}\varphi_F = \mathrm{epi}F$. Hence, the Y_+-convexity of φ_F follows from the Y_+-convexity of F. Suppose now that F is continuous at $\bar{x}$. Since Y_+ is a normal cone, i.e., there exists a basis of open neighbourhoods $\mathfrak{B}$ of the origin such that $W = (W - Y_+) \cap (W + Y_+)$, $\forall W \in \mathfrak{B}$. Let $W \in \mathfrak{B}$, as $F(\bar{x}) \subseteq F(\bar{x}) + W$ and F is upper-semicontinuous at $\bar{x}$, then there exists a neighborhood U of $\bar{x}$ such that $F(x) \subseteq F(\bar{x}) + W$, $\forall x \in U$. By virtue of condition (H_1), we have $\varphi_F(x) \in F(x)$ for all $x \in \mathrm{dom}F$. Hence $\varphi_F(x) \in F(\bar{x}) + W$, $\forall x \in U$. Moreover, as $F(\bar{x}) \subseteq \varphi_F(\bar{x}) + Y_+$, we have

$$\varphi_F(x) \in \varphi_F(\bar{x}) + W + Y_+, \quad \forall x \in U. \tag{2.1}$$

On other hand, as F is lower-semicontinuous at $\bar{x}$ and $(\varphi_F(\bar{x})+W)\cap F(\bar{x})\neq\emptyset$, we have that there exists a neighborhood V of $\bar{x}$ such that $F(x)\cap(\varphi_F(\bar{x})+W)\neq\emptyset$, $\forall x\in V$. Since $F(x)\subseteq\varphi_F(x)+Y_+$ for all $x\in\mathrm{dom}F$, we obtain $(\varphi_F(x)+Y_+)\cap(\varphi_F(\bar{x})+W)\neq\emptyset$, $\forall x\in V$, which yields that

$$\varphi_F(x)\in\varphi_F(\bar{x})+W-Y_+,\quad \forall x\in V. \tag{2.2}$$

From (2.1), (2.2), and the normality of Y_+, we have $\varphi_F(x)\in\varphi_F(\bar{x})+W$, $\forall x\in U\cap V$. Therefore, φ_F is continuous at $\bar{x}$. □

Theorem 2.2. [4] *Let $f:X\to Y\cup\{+\infty_Y\}$ be a Y_+-convex single vector mapping. Suppose that the following condition holds*

$$\left\{\begin{array}{l}(Y,Y_+)\textit{ is a normal order complete lattice Hausdorff locally convex topological}\\ \textit{vector space, and } f \textit{ is continuous at } \bar{x}\in\mathrm{dom}f.\end{array}\right. \tag{H_2}$$

Then, $\partial^s f(\bar{x})\neq\emptyset$.

Theorem 2.3. *Under the assumptions of Theorem 2.1, we assume, in addition, that the condition (H_2) holds. Then $\partial^s F(\bar{x},\varphi_F(\bar{x}))\neq\emptyset$.*

Proof. From Theorem 2.1, φ_F is Y_+-convex and continuous at $\bar{x}$. Hence by Theorem 2.2, it follows that φ_F is strongly subdifferentiable at $\bar{x}$. Let us now prove that $\partial^s F(\bar{x},\varphi_F(\bar{x}))=\partial^s\varphi_F(\bar{x})$. If $T\in\partial^s F(\bar{x},\varphi_F(\bar{x}))$, then $T(x-\bar{x})\leq_{Y_+} y-\varphi_F(\bar{x})$, $\quad\forall x\in\mathrm{dom}F$, $\forall y\in F(x)$. As $\varphi_F(x)\in F(x)$ for all $x\in\mathrm{dom}F=\mathrm{dom}\varphi_F$, it follows that $T(x-\bar{x})\leq_{Y_+}\varphi_F(x)-\varphi_F(\bar{x})$, $\forall x\in\mathrm{dom}\varphi_F$. Therefore $T(x-\bar{x})\leq_{Y_+}\varphi_F(x)-\varphi_F(\bar{x})$, $\forall x\in X$, i.e., $T\in\partial^s\varphi_F(\bar{x})$. For the reverse inclusion, let $T\in\partial^s\varphi_F(\bar{x})$, i.e., $T(x-\bar{x})\leq_{Y_+}\varphi_F(x)-\varphi_F(\bar{x})$, $\forall x\in X$. Hence, for all $x\in\mathrm{dom}\varphi_F=\mathrm{dom}F$ and $y\in\varphi_F(x)+Y_+$, we have

$$T(x-\bar{x})\leq_{Y_+} y-\varphi_F(\bar{x}), \tag{2.3}$$

For all $y\in F(x)$, we have $y\in\varphi_F(x)+Y_+$. It follows from (2.3) that $T(x-\bar{x})\leq_{Y_+} y-\varphi_F(\bar{x})$, $\forall(x,y)\in\mathrm{gr}F$, which yields that $T\in\partial^s F(\bar{x},\varphi_F(\bar{x}))$. The proof is complete. □

3. Strong Subdifferential Calculus Rules

In this section, we are concerned with the subdifferential calculus of the sum and composition of convex set-valued mappings.

3.1. **Addition.** As mentioned in [1], Théra established in the framework of ordered complete topological vector spaces, the following theorem, which plays an important role in proving our main results.

Theorem 3.1. [1] *Let $f,g:X\to Y\cup\{+\infty_Y\}$ be two Y_+-convex single vector valued mappings. Suppose that the following conditions is satisfied.*

$$\left\{\begin{array}{l}(Y,Y_+)\textit{ is a normal order complete Hausdorff locally convex topological vector space},\\ f\textit{ is continuous at } \bar{x}\in\mathrm{dom}f\cap\mathrm{dom}g.\end{array}\right.$$

Then, $\partial^s(f+g)(\bar{x})=\partial^s f(\bar{x})+\partial^s g(\bar{x})$.

Let us consider the vector indicator mapping $\delta^v_C:X\to Y\cup\{+\infty_Y\}$ of a nonempty subset $C\subseteq X$, defined by

$$\delta^v_C(x):=\left\{\begin{array}{ll}0_Y, & \text{if } x\in C,\\ +\infty_Y, & \text{else.}\end{array}\right.$$

It is easy to see that, for all $\bar{x} \in C$, $\partial^s \delta_C^v(\bar{x}) = N^v(\bar{x}, C)$. Let us note that epi $\delta_C^v = C \times Y_+$ and thus the Y_+-convexity of δ_C^v follows from the convexity of C and Y_+.

The following corollary represents the vector version of the known normal cone intersection formula for two convex subsets under an interior-point-like condition.

Corollary 3.1. *Let C and D be two convex subsets of X with $\bar{x} \in C \cap D$. Suppose that (Y, Y_+) is a normal order complete Hausdorff locally convex topological vector space and $(\mathrm{int}C) \cap D \neq \emptyset$. Then, we have the vector normal cone intersection rule $N^v(\bar{x}; C \cap D) = N^v(\bar{x}; C) + N^v(\bar{x}; D)$.*

Proof. First, we claim that vector indicator mapping δ_C^v is continuous on int$C$. Indeed, we have for any $\bar{x} \in \mathrm{int}C$ and for any neighborhood V of the origin 0_Y that $\delta_C^v(\mathrm{int}C) = \{0_Y\} \subseteq V$. Let us observe that, for any $x \in C \cap D$, $\delta_{C\cap D}^v(x) = \delta_C^v(x) + \delta_D^v(x)$. The vector indictor mappings δ_C^v and δ_D^v satisfy together all the assumptions of Theorem 3.1. Hence,

$$\begin{aligned} N^v(\bar{x}; C \cap D) &= \partial^s \delta_{C\cap D}^v(\bar{x}) \\ &= \partial^s(\delta_C^v + \delta_D^v)(\bar{x}) \\ &= \partial^s \delta_C^v(\bar{x}) + \partial^s \delta_D^v(\bar{x}) \\ &= N^v(\bar{x}; C) + N^v(\bar{x}; D). \end{aligned}$$

The proof is complete. □

Theorem 3.2. *Let $F, G : X \rightrightarrows Y$ be two Y_+-convex set-valued mappings, $\bar{x} \in \mathrm{dom}F \cap \mathrm{dom}G$, $\bar{u} \in F(\bar{x})$, and $\bar{v} \in G(\bar{x})$. Suppose that the following condition is satisfied.*

- *(i) (Y, Y_+) is order complete Hausdorff locally convex topological vector space and $\mathrm{int}(\mathrm{epi}F) \cap \mathrm{epi}G \neq \emptyset$.*

Then, $\partial^s(F+G)(\bar{x}, \bar{u}+\bar{v}) = \partial^s F(\bar{x}, \bar{u}) + \partial^s G(\bar{x}, \bar{v})$.

Proof. Let $A \in \partial^s F(\bar{x}, \bar{u})$ and $B \in \partial^s G(\bar{x}, \bar{v})$, that is,

$$A(x-\bar{x}) \leq_{Y_+} u - \bar{u}, \quad \forall (x,u) \in \mathrm{gr}\, F, \tag{3.1}$$

and

$$B(x-\bar{x}) \leq_{Y_+} v - \bar{v}, \quad \forall (x,v) \in \mathrm{gr}\, G. \tag{3.2}$$

By adding (3.1) and (3.2), we obtain for any $u \in F(x)$ and $v \in G(x)$ that

$$(A+B)(x-\bar{x}) \leq_{Y_+} u + v - (\bar{u}+\bar{v}),$$

which means that $A+B \in \partial^s(F+G)(\bar{x}, \bar{u}+\bar{v})$. For the reverse inclusion, let $T \in \partial^s(F+G)(\bar{x}, \bar{u}+\bar{v})$, that is, $y - (\bar{u}+\bar{v}) - T(x-\bar{x}) \geq_{Y_+} 0_Y$, $\forall (x,y) \in \mathrm{gr}(F+G)$, which yields that, for any $x \in \mathrm{dom}F \cap \mathrm{dom}G$, $u \in F(x)$, $v \in G(x)$ and $\alpha, \beta \in Y_+$, $u + \alpha + v + \beta - (\bar{u}+\bar{v}) - T(x-\bar{x}) \geq_{Y_+} 0_Y$. Thus it follows that, for any $(x,u) \in \mathrm{epi}\, F$ and $(x,v) \in \mathrm{epi}\, G$,

$$T(x-\bar{x}) - (u-\bar{u}) - (v-\bar{v}) \leq_{Y_+} 0_Y. \tag{3.3}$$

Define the following convex subsets of $X \times Y \times Y$ by

$$\begin{aligned} C &:= \{(x,u,v) \in X \times Y \times Y : \ (x,u) \in \mathrm{epi}\, F\}, \\ D &:= \{(x,u,v) \in X \times Y \times Y : \ (x,v) \in \mathrm{epi}\, G\}. \end{aligned}$$

Letting $(x,u,v) \in C \cap D$, we have $(x,u) \in \mathrm{epi}\, F$ and $(x,v) \in \mathrm{epi}\, G$. Hence it follows from relation (3.3) that $(T, -id_Y, -id_Y) \in N^v((\bar{x}, \bar{u}, \bar{v}); C \cap D)$. Condition (i) implies that $(\mathrm{int}C) \cap D \neq \emptyset$.

Indeed, since $\mathrm{int}(\mathrm{epi}\, F) \cap \mathrm{epi}\, G \neq \emptyset$ and $\mathrm{int} C = \mathrm{int}(\mathrm{epi}\, F) \times Y$, it follows that, for any $(x,y) \in \mathrm{int}(\mathrm{epi}\, F) \cap \mathrm{epi}\, G$, $(x,y,y) \in (\mathrm{int} C) \cap D$, which ensures that $(\mathrm{int} C) \cap D \neq \emptyset$. Therefore, according to Corollary 3.1, we deduce that $(T, -id_Y, -id_Y) \in N^v((\bar{x},\bar{u},\bar{v});C) + N^v((\bar{x},\bar{u},\bar{v});D)$. Hence we assert that there exist $(T_1,A_1,B_1) \in N^v((\bar{x},\bar{u},\bar{v});C)$ and $(T_2,A_2,B_2) \in N^v((\bar{x},\bar{u},\bar{v});D)$ such that

$$\begin{cases} (T, -id_Y, -id_Y) = (T_1,A_1,B_1) + (T_2,A_2,B_2), \\ T_1(x-\bar{x}) + A_1(u-\bar{u}) + B_1(v-\bar{v}) \leq_{Y_+} 0_Y, \quad \forall (x,u,v) \in C, & (3.4) \\ T_2(x-\bar{x}) + A_2(u-\bar{u}) + B_2(v-\bar{v}) \leq_{Y_+} 0_Y, \quad \forall (x,u,v) \in D. & (3.5) \end{cases}$$

By taking $x = \bar{x}$ and $u = \bar{u}$ in (3.4), we obtain $B_1(v-\bar{v}) \leq_{Y_+} 0_Y$ for any $v \in Y$. Since Y_+ is pointed ($Y_+ \cap -Y_+ = \{0_Y\}$), it follows that $B_1 = 0$. Similarly, by taking $x = \bar{x}$ and $v = \bar{v}$ in (3.5), we obtain $A_2(u-\bar{u}) \leq_{Y_+} 0_Y$ for all $u \in Y$. Thus $A_2 = 0$. Consequently, $A_1 = -id_Y$, $B_2 = -id_Y$, and

$$\begin{cases} T_1(x-\bar{x}) - (u-\bar{u}) \leq_{Y_+} 0_Y, \quad \forall (x,u) \in \mathrm{epi}\, F, \\ T_2(x-\bar{x}) - (v-\bar{v}) \leq_{Y_+} 0_Y, \quad \forall (x,v) \in \mathrm{epi}\, G, \end{cases}$$

which yields that

$$(T_1, -id_Y) \in N^v((\bar{x},\bar{u});\mathrm{epi}\, F) \quad \text{and} \quad (T_2, -id_Y) \in N^v((\bar{x},\bar{v});\mathrm{epi}\, G),$$

that is, $T_1 \in \partial^s F(\bar{x},\bar{u})$, $T_2 \in \partial^s G(\bar{x},\bar{v})$, and $T = T_1 + T_2$. Thus we obtain

$$\partial^s (F+G)(\bar{x},\bar{u}+\bar{v}) \subseteq \partial^s F(\bar{x},\bar{u}) + \partial^s G(\bar{x},\bar{v}).$$

So, we obtain the desired result. The proof is complete. □

The following theorem gives us the sum rule for two set-valued mappings under the connectedness assumption.

Theorem 3.3. *Let $F, G : X \rightrightarrows Y$ be two set-valued mappings. Assume that the following condition holds.*

$$\begin{cases} (Y,Y_+) \textit{ is a normal order complete lattice Hausdorff locally convex topological} \\ \textit{vector space, } F \textit{ and } G \textit{ are } Y_+\textit{-convex,} \\ F \textit{ is connected at some point } x_0 \in \mathrm{dom} F \cap \mathrm{dom} G. \end{cases}$$

Then, for any $(\bar{x},\bar{u}) \in \mathrm{gr} F$ and $(\bar{x},\bar{v}) \in \mathrm{gr} G$, $\partial^s (F+G)(\bar{x},\bar{u}+\bar{v}) = \partial^s F(\bar{x},\bar{u}) + \partial^s G(\bar{x},\bar{v})$.

Proof. First, we prove that $\mathrm{int}(\mathrm{epi} F) \neq \emptyset$. Since F is connected at x_0, there exist some neighborhood U_0 of x_0 and a mapping $h : X \to Y$ continuous at x_0 such that $h(x) \in F(x)$ for all $x \in U_0$. Let $y_0 \in h(x_0) + \mathrm{int}\, Y_+$, i.e., $h(x_0) \in y_0 - \mathrm{int}\, Y_+ \subset y_0 - Y_+$, which yields that $y_0 - Y_+$ is a neighborhood of $h(x_0)$. Thus it follows from the continuity of h at x_0 that $h^{-1}(y_0 - Y_+)$ is a neighborhood of x_0. By putting $U = U_0 \cap h^{-1}(y_0 - Y_+)$, which is a neighborhood of x_0, we obtain that $y_0 \in F(x) + Y_+$ for any $x \in U$. On other hand, as $y_0 - h(x_0) \in \mathrm{int}\, Y_+$, there exists a neighbourhood V of 0_Y such that $y_0 - h(x_0) + V \subset Y_+$. By using the fact that $Y_+ + Y_+ = Y_+$, we obtain, for any $x \in U$ and $y \in y_0 + y_0 - h(x_0) + V$, that $y \in F(x) + y_0 - h(x_0) + V + Y_+ \subset F(x) + Y_+$, which yields that $(x_0, y_0 + y_0 - h(x_0)) \in \mathrm{int}\,(\mathrm{epi}\, F)$.

Second, we prove that $\mathrm{int}(\mathrm{epi}\, F) \cap \mathrm{epi} G \neq \emptyset$. We proceed by contradiction. Suppose that $\mathrm{int}(\mathrm{epi} F) \cap \mathrm{epi} G = \emptyset$. By the separation theorem [12, Theorem 1.1.3], there exist a nonzero $(x^*, y^*, \beta) \in X^* \times Y^* \times \mathbb{R}$ such that

$$\langle x^*, x \rangle + \langle y^*, y \rangle \leq \beta \leq \langle x^*, x' \rangle + \langle y^*, y' \rangle, \qquad \forall (x,y) \in \mathrm{epi} F, \ \forall (x',y') \in \mathrm{epi} G \tag{3.6}$$

As $x_0 \in \operatorname{dom} F \cap \operatorname{dom} G$, we claim that there exists $z_0 \in Y$ such that $(x_0, z_0) \in \operatorname{epi} F \cap \operatorname{epi} G$. Let $y_1 \in F(x_0)$ and $y_2 \in G(x_0)$. B taking $z_0 := \sup(y_1, y_2)$, we obtain $z_0 \in F(x_0) + Y_+$ and $z_0 \in G(x_0) + Y_+$ i.e. $(x_0, z_0) \in \operatorname{epi} F \cap \operatorname{epi} G$. Observe that $(x_0, z_0 + u) \in \operatorname{epi} F \cap \operatorname{epi} G$ for any $u \in Y_+$. By taking in relation (3.6) $x = x' = x_0$, $y = z_0$, and $y' = z_0 + u$ (resp. $x = x' = x_0$, $y = z_0 + u$, and $y' = z_0$), we obtain $\langle y^*, u\rangle \geq 0$ for all $u \in Y_+$ (resp. $\langle y^*, u\rangle \leq 0$, for all $u \in Y_+$), which yields that $y^* = 0$ since $\operatorname{int} Y_+ \neq \emptyset$. It follows from (3.6) that $\langle x^*, u\rangle \leq 0$ for all $u \in (\operatorname{dom} F - \operatorname{dom} G)$. As F is connected at $x_0 \in \operatorname{dom} F \cap \operatorname{dom} G$, one can easily see that $0_X \in \operatorname{int}(\operatorname{dom} F - \operatorname{dom} G)$, which yields that $x^* = 0_{X^*}$ and this leads to a contradiction. $\square$

3.2. **Composition.** In this subsection, we develop the strong subdifferential calculus of the composition of two set-valued mappings. The approach that we use for computing the strong subdifferential of the composed set-valued mappings is to transform it as the strong subdifferential of the sum of two set-valued mappings. In what follows, Z is a real locally convex topological vector space equipped with a nonempty pointed convex cone Z_+. We work also with the following definitions: for $(x,z) \in X \times Z$, $(A,B) \in L(X,Y) \times L(Z,Y)$, we set $(A,B)(x,z) := A(x) + B(z)$. Let $F : X \rightrightarrows Y$, $H : X \rightrightarrows Z$ and $G : Z \rightrightarrows Y$ be three set-valued mappings.

The composed set-valued mapping $G \circ H : X \rightrightarrows Y$ is defined by

$$(G \circ H)(x) = G(H(x)) := \begin{cases} \bigsqcup\limits_{z \in H(x)} G(z), & \text{if } x \in \operatorname{dom} H, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We have $\operatorname{dom}(G \circ H) = H^{-1}(\operatorname{dom} G) \cap \operatorname{dom} H$, where $H^{-1}(\operatorname{dom} G) := \{x \in X : H(x) \cap \operatorname{dom} G \neq \emptyset\}$. For a nonempty subset $S \subseteq X$, the set-valued indicator mapping $R_S^v : X \rightrightarrows Y$ is defined by

$$R_S^v(x) := \begin{cases} \{0_Y\}, & \text{if } x \in S, \\ \emptyset, & \text{else.} \end{cases}$$

Definition 3.1. [7] Let $G : Z \rightrightarrows Y$ be a set-valued mapping, and $S \subseteq Z$. G is said to be (Z_+, Y_+)-nondecreasing over S if, for any $(z_1, z_2) \in S \times S$ satisfying $z_1 \leq_{Z_+} z_2$, $G(z_2) \subseteq G(z_1) + Y_+$.

In what follows, we need the following definition $L_+(Z,Y) := \{B \in L(Z,Y) : B(Z_+) \subseteq Y_+\}$.

Lemma 3.1. *Let $G : Z \rightrightarrows Y$ be a (Z_+, Y_+)-nondecreasing set-valued mapping, and $(\bar{z}, \bar{y}) \in \operatorname{gr} G$. Then $\partial^s G(\bar{z}, \bar{y}) \subseteq L_+(Z,Y)$.*

Proof. If $A \in \partial^s G(\bar{z}, \bar{y})$, then

$$G(z) \subseteq \bar{y} + A(z - \bar{z}) + Y_+, \quad \forall z \in Z. \tag{3.7}$$

Letting $w \in Z_+$ and taking $z = \bar{z} - w$ in (3.7), we obtain

$$G(\bar{z} - w) \subseteq \bar{y} - A(w) + Y_+. \tag{3.8}$$

As G is (Z_+, Y_+)-nondecreasing we have $G(\bar{z}) \subseteq G(\bar{z} - w) + Y_+$. Since $\bar{y} \in G(\bar{z})$ and the fact $Y_+ + Y_+ = Y_+$, it follows from (3.8) that $A(w) \in Y_+$, $\forall w \in Z_+$, which yields that $A(Z_+) \subseteq Y_+$. $\square$

Corollary 3.2. *Let $H : X \rightrightarrows Z$ and $G : Z \rightrightarrows Y$ be two set-valued mappings. If H is Z_+-convex and G is (Z_+, Y_+)-nondecreasing and Y_+-convex, the,n for any $A \in \partial^s G(\bar{z}, \bar{y})$, the mapping $A \circ H$ is Y_+-convex.*

Proof. It suffices to observe that, for any $A \in L_+(Z,Y)$, A is (Z_+,Y_+)-nondecreasing. □

Let us consider the following auxiliary set-valued mappings

$$\begin{array}{rcl} \tilde{F}: X\times Z & \rightrightarrows & Y \\ (x,z) & \mapsto & F(x)+R^{v}_{\mathrm{epi}H}(x,z), \\ \tilde{G}: X\times Z & \rightrightarrows & Y \\ (x,z) & \mapsto & G(z). \end{array}$$

Note that $\mathrm{dom}\tilde{F} = (\mathrm{dom}F \times Z)\cap \mathrm{epi}H$, $\mathrm{dom}\tilde{G} = X\times \mathrm{dom}G$, and $\mathrm{gr}\tilde{G} = X\times \mathrm{gr}G$. In addition

$$\begin{array}{rcl} \mathrm{epi}\tilde{G} & = & \{(x,z,y)\in X\times Z\times Y: \quad y\in \tilde{G}(x,z)+Y_+\} \\ & = & \{(x,z,y)\in X\times Z\times Y: \quad y\in G(z)+Y_+\} \\ & = & X\times \mathrm{epi}G, \end{array}$$

and

$$\begin{array}{rcl} \mathrm{epi}\tilde{F} & = & \{((x,z,y)\in X\times Z\times Y: \quad y\in \tilde{F}(x,z)+Y_+\} \\ & = & \{((x,z,y)\in X\times Z\times Y: \quad y\in F(x)+R^{v}_{\mathrm{epi}H}(x,z)+Y_+\} \\ & = & \{((x,z,y)\in X\times Z\times Y: \quad (x,z)\in \mathrm{epi}H \ \ and \ \ (x,y)\in \mathrm{epi}F\} \\ & = & (\mathrm{epi}H\times Y)\cap \varphi^{-1}(\mathrm{epi}F), \end{array}$$

where φ is a continuous function defined from $X\times Z\times Y$ into $X\times Y$ by $\varphi(x,z,y) := (x,y)$ for all $(x,z,y)\in X\times Z\times Y$. It is easy to see that if F and G are Y_+-convex and H is Z_+-convex, then $\tilde{F}$ and $\tilde{G}$ are Y_+-convex. Now, we are in a position show that the study of the formula $\partial^s(F+G\circ H)$ can be reduced to that for $\partial^s(\tilde{F}+\tilde{G})$. For this, we need the following lemma, which demonstrates the relationship among the subdifferentials of $\tilde{F}$, $\tilde{G}$, and the subdifferentials of F, H, and G, respectively.

Lemma 3.2. *Let $\bar{x}\in \mathrm{dom}F\cap \mathrm{dom}(G\circ H)$, $\bar{u}\in F(\bar{x})$, $\bar{z}\in H(\bar{x})$, and $\bar{v}\in G(\bar{z})$, we have*

(i) if G is (Z_+,Y_+)-nondecreasing, then

$$A\in \partial^s(F+G\circ H)(\bar{x},\bar{u}+\bar{v}) \quad \Longleftrightarrow \quad (A,0)\in \partial^s(\tilde{F}+\tilde{G})((\bar{x},\bar{z}),\bar{u}+\bar{v}); \tag{3.9}$$

(ii) $\partial^s\tilde{G}((\bar{x},\bar{z}),\bar{v}) = \{0\}\times \partial^s G(\bar{z},\bar{v})$;

(iii) if G is connected at $\bar{z}\in H(\bar{x})$, then $\tilde{G}$ is connected at $(\bar{x},\bar{z})$.

Proof. (i) Let $A\in \partial^s(F+G\circ H)(\bar{x},\bar{u}+\bar{v})$. Then

$$F(x)+(G\circ H)(x)-\bar{u}-\bar{v}-A(x-\bar{x})\subseteq Y_+, \ \ \forall x\in X,$$

and then $F(x)+R^{v}_{\mathrm{epi}H}(x,z)+(G\circ H)(x)-\bar{u}-\bar{v}-A(x-\bar{x})\subseteq Y_+$, $\forall (x,z)\in X\times Z$, which imply that

$$\tilde{F}(x,z)+(G\circ H)(x)-\bar{u}-\bar{v}-A(x-\bar{x})\subseteq Y_+, \ \ \forall (x,z)\in \mathrm{epi}H. \tag{3.10}$$

As G is (Z_+,Y_+)-nondecreasing, then, for any $(x,z)\in \mathrm{epi}H$, $G(z)\subset (G\circ H)(x)+Y_+$. It follows from relation (3.10) that $\tilde{F}(x,z)+G(z)-\bar{u}-\bar{v}-A(x-\bar{x})\subseteq Y_+ + Y_+\subseteq Y_+$. Hence,

$$\tilde{F}(x,z)+\tilde{G}(x,z)-\bar{u}-\bar{v}-A(x-\bar{x})\subseteq Y_+, \ \ \forall (x,z)\in X\times Z,$$

which yields $(A,0) \in \partial^s(\tilde{F}+\tilde{G})((\bar{x},\bar{z}),\bar{u}+\bar{v})$. Conversely, let us take any $(A,0) \in \partial^s(\tilde{F}+\tilde{G})((\bar{x},\bar{z}),\bar{u}+\bar{v})$. It follows that $\tilde{F}(x,z)+\tilde{G}(x,z)-\bar{u}-\bar{v}-A(x-\bar{x}) \subseteq Y_+$, $\forall (x,z) \in X \times Z$, i.e.,

$$F(x)+R_{\mathrm{epi}H}(x,z)+G(z)-\bar{u}-\bar{v}-A(x-\bar{x}) \subseteq Y_+, \ \ \forall (x,z) \in X \times Z.$$

Therefore, for all $(x,z) \in \mathrm{epi}H$, $F(x)+G(z)-\bar{u}-\bar{v}-A(x-\bar{x}) \subseteq Y_+$, which implies that, for all $x \in X$, $F(x)+\bigcup_{z\in H(x)} G(z)-\bar{u}-\bar{v}-A(x-\bar{x}) \subseteq Y_+$, i.e.,

$$F(x)+(G\circ H)(x)-\bar{u}-\bar{v}-A(x-\bar{x}) \subseteq Y_+, \ \ \forall x \in X.$$

Finally, $A \in \partial^s(F+G\circ H)(\bar{x},\bar{u}+\bar{v})$.

(ii) Let $(A,B) \in \partial^s\tilde{G}((\bar{x},\bar{z}),\bar{y})$. For all $((x,z),y) \in \mathrm{gr}\tilde{G} = X \times \mathrm{gr}G$,

$$A(x-\bar{x})+B(z-\bar{z}) \leq_{Y_+} y-\bar{y}. \tag{3.11}$$

By taking $z=\bar{z}$ and $y=\bar{y}$ in (3.11), it follows that, for all $x\in X$, $A(x-\bar{x}) \leq_{Y_+} 0_Y$. Hence $A=0$. Consequently, $\partial^s\tilde{G}((\bar{x},\bar{z}),\bar{y}) \subseteq \{0\}\times\partial^s G(\bar{z},\bar{y})$. For the reverse inclusion, let $B \in \partial^s G(\bar{z},\bar{y})$, i.e., $B(z-\bar{z}) \leq_{Y_+} y-\bar{y}$, $\forall (z,y) \in \mathrm{gr}G$. As $\mathrm{gr}\tilde{G} = X\times \mathrm{gr}G$, we deduce that $\{0\}\times\partial^s G(\bar{z},\bar{y}) \subseteq \partial^s\tilde{G}((\bar{x},\bar{z}),\bar{y})$.

(iii) As G is connected at $\bar{z} \in H(\bar{x})$, there exists a neighborhood V of $\bar{z}$ and a mapping $g: Z \to Y$ such that $g(z) \in G(z)$ for all $z \in V$, and g is continuous at $\bar{z}$. Define the following function

$$\begin{array}{rcl} \tilde{g}: X\times Z & \to & Y \\ (x,z) & \mapsto & g(z) \end{array}$$

It is clear that $\tilde{g}$ is continuous at $(\bar{x},\bar{z})$, and $\tilde{g}(x,z) \in \tilde{G}(x,v)$ for all $(x,z) \in X \times V$. Hence $\tilde{G}$ is connected at $(\bar{x},\bar{z})$. $\square$

Now, we are ready to state our main results in this subsection.

Theorem 3.4. *Let $F: X \rightrightarrows Y$, $H: X \rightrightarrows Z$, and $G: Z \rightrightarrows Y$ be three set-valued mappings, $(\bar{x},\bar{u}) \in \mathrm{gr}F$, $(\bar{x},\bar{z}) \in \mathrm{gr}H$, and $(\bar{z},\bar{v}) \in \mathrm{gr}G$. Suppose that the following condition holds*

$$\left\{\begin{array}{l} (Y,Y_+) \textit{ is a normal order complete lattice Hausdorff locally convex topological} \\ \textit{vector space, } (Z,Z_+) \textit{ is a Hausdorff locally convex space,} \\ F,\ G \textit{ are } Y_+\textit{-convex and } H \textit{ is } Z_+\textit{-convex ,} \\ G \textit{ is } (Z_+,Y_+)\textit{-nondecreasing,} \\ \exists a \in \mathrm{dom}F\cap\mathrm{dom}H \textit{ such that } G \textit{ is connected at some point } b \in H(a). \end{array}\right. \tag{MR1}$$

Then,

$$\partial^s(F+G\circ H)(\bar{x},\bar{u}+\bar{v}) = \bigcup_{A\in\partial^s G(\bar{z},\bar{v})} \partial^s(F+A\circ H)(\bar{x},\bar{u}+A(\bar{z})).$$

Proof. First, let us prove

$$\partial^s(F+G\circ H)(\bar{x},\bar{u}+\bar{v}) \supseteq \bigcup_{A\in\partial^s G(\bar{z},\bar{v})} \partial^s(F+A\circ H)(\bar{x},\bar{u}+A(\bar{z})).$$

Let $A \in \partial^s G(\bar{z},\bar{v})$ and $B \in \partial^s(F+A\circ H)(\bar{x},\bar{u}+A(\bar{z}))$. It follows that

$$(F+A\circ H)(x)-\bar{u}-A(\bar{z})-B(x-\bar{x}) \subseteq Y_+, \quad \forall x \in X,$$

which means that, for any $(x,u) \in \mathrm{gr}F$ and $(x,z) \in \mathrm{gr}H$,

$$B(x-\bar{x}) \leq_{Y_+} u-\bar{u}+A(z-\bar{z}). \tag{3.12}$$

As $A \in \partial^s G(\bar{z},\bar{v})$, we have $G(z) - \bar{v} - A(z-\bar{z}) \subseteq Y_+$, $\forall z \in Z$, which yields that $\cup_{z\in H(x)}(G-A)(z) - \bar{v} + A(\bar{z}) \subseteq Y_+$, $\forall x \in X$, i.e., $(G\circ H)(x) - (A\circ H)(x) - \bar{v} + A(\bar{z}) \subseteq Y_+$, $\forall x \in X$. Therefore,

$$A(z-\bar{z}) \leq_{Y_+} v - \bar{v}, \quad \forall (x,z) \in \mathrm{gr}H, (z,v) \in \mathrm{gr}G. \tag{3.13}$$

From (3.12) and (3.13), we have $B(x-\bar{x}) \leq_{Y_+} u+v-\bar{u}-\bar{v}$, $\forall (x,u+v) \in \mathrm{gr}(F+G\circ H)$, i.e., $B \in \partial^s(F+G\circ H)(\bar{x},\bar{u}+\bar{v})$. For the reverse inclusion, let us take any $B \in \partial^s(F+G\circ H)(\bar{x},\bar{u}+\bar{v})$. According to Lemma 3.2 (i), we have $(B,0) \in \partial^s(\tilde{F}+\tilde{G})((\bar{x},\bar{z}),\bar{u}+\bar{v})$. Under condition (MR1), by virtue of Lemma 3.2 (iii), the mappings $\tilde{F}$ and $\tilde{G}$ satisfy all the assumptions of Theorem 3.3. Hence we $\partial^s(\tilde{F}+\tilde{G})((\bar{x},\bar{z}),\bar{u}+\bar{v}) = \partial^s\tilde{F}((\bar{x},\bar{z}),\bar{u}) + \partial^s\tilde{G}((\bar{x},\bar{z}),\bar{v})$. Then there exists $(T,A) \in \partial^s\tilde{G}((\bar{x},\bar{z}),\bar{v})$ such that $(B-T,-A) \in \partial^s\tilde{F}((\bar{x},\bar{z}),\bar{u})$. By virtue of Lemma 3.2 (ii), we obtain that $T=0$ and $A \in \partial^s G(\bar{z},\bar{v})$. Now, let us show that $B \in \partial^s(F+A\circ H)(\bar{x},\bar{u}+A(\bar{z}))$. As $(B,-A) \in \partial^s\tilde{F}((\bar{x},\bar{z}),\bar{u})$, we have, for all $(x,z) \in X\times Z$, $F(x) + R^v_{\mathrm{epi}H}(x,z) - \bar{u} - B(x-\bar{x}) + A(z-\bar{z}) \subseteq Y_+$, which implies that $F(x) - \bar{u} - B(x-\bar{x}) + A(z-\bar{z}) \subseteq Y_+$, $\forall (x,z) \in \mathrm{epi}H$. Hence, for all $x \in X$, we have $F(x) + \bigcup_{z\in H(x)} A(z) - (\bar{u}+A(\bar{z})) - B(x-\bar{x}) \subseteq Y_+$, i.e., $(F+A\circ H)(x) - (\bar{u}+A(\bar{z})) - B(x-\bar{x}) \subseteq Y_+$. Therefore, $B \in \partial^s(F+A\circ H)(\bar{x},\bar{u}+A(\bar{z}))$. □

Corollary 3.3. *Let $H : X \rightrightarrows Z$ and $G : Z \rightrightarrows Y$ be two set-valued mappings, $\bar{z} \in H(\bar{x})$, and $\bar{y} \in G(\bar{z})$. Suppose that the following condition holds*

$$\begin{cases} (Y,Y_+) \textit{ is a normal order complete lattice Hausdorff locally convex topological} \\ \textit{vector space, } (Z,Z_+) \textit{ is a Hausdorff locally convex space,} \\ G \textit{ is } Y_+\textit{-convex and } H \textit{ is } Z_+\textit{-convex ,} \\ G \textit{ is } (Z_+,Y_+)\textit{-nondecreasing,} \\ G \textit{ is connected at some point of } \mathrm{Im}H. \end{cases}$$

Then

$$\partial^s(G\circ H)(\bar{x},\bar{y}) = \bigcup_{A\in\partial^s G(\bar{z},\bar{y})} \partial^s(A\circ H)(\bar{x},A(\bar{z})).$$

Now consider the case of composition with a linear operator. Let $A : X \to Z$ be a linear operator and $G : Z \rightrightarrows Y$ be a Y_+-convex set-valued mapping. By putting $Z_+ = \{0_Z\}$, the function G is obviously (Z_+,Y_+)-nondecreasing, and A is Z_+-convex. So applying Corollary 3.3, one has the following result.

Corollary 3.4. *Let $\bar{x} \in X$ and $\bar{y} \in G(A(\bar{x}))$. Assume that the following condition holds*

$$\begin{cases} (Y,Y_+) \textit{ is a normal order complete lattice Hausdorff locally convex topological} \\ \textit{vector space, } (Z,Z_+) \textit{ is a Hausdorff locally convex topological vector space,} \\ G \textit{ is } Y_+\textit{-convex,} \\ G \textit{ is connected at some point of } \mathrm{Im}A. \end{cases}$$

Then $\partial^s(G\circ A)(\bar{x},\bar{y}) = \partial^s G(A(\bar{x}),\bar{y})\circ A$.

Proof. It follows from Corollary 3.3 that

$$\partial^s(G\circ A)(\bar{x},\bar{y}) = \bigcup_{B\in\partial^s G(A(\bar{x}),\bar{y})} \partial^s(B\circ A)(\bar{x},A(\bar{x})). \tag{3.14}$$

Moveover, from the definition of the strong subdifferential and the fact that Y_+ is pointed, we have $\partial^s(B\circ A)(\bar{x},A(\bar{x}))=\{B\circ A\}$. It follows from (3.14) that

$$\begin{aligned}\partial^s(G\circ A)(\bar{x},\bar{y}) &= \{B\circ A: \quad B\in\partial^s G(A(\bar{x}),\bar{y})\},\\ &= \partial^s G(A(\bar{x}),\bar{y})\circ A.\end{aligned}$$

□

Corollary 3.5. *Under the assumptions of Theorem 3.4, if F or H is assumed to connected at some point of* $\mathrm{dom}F\cap\mathrm{dom}H$, *then*

$$\partial^s(F+G\circ H)(\bar{x},\bar{u}+\bar{v})=\partial^s F(\bar{x},\bar{u})+\bigcup_{A\in\partial^s G(\bar{z},\bar{v})}\partial^s(A\circ H)(\bar{x},A(\bar{z})).$$

Proof. According to Theorem 3.4, we have

$$\partial^s(F+G\circ H)(\bar{x},\bar{u}+\bar{v})=\bigcup_{A\in\partial^s G(\bar{z},\bar{v})}\partial^s(F+A\circ H)(\bar{x},\bar{u}+A(\bar{z})).$$

Observe that $A\circ H$ is Y_+-convex since $A\in L_+(Z,Y)$. As F or H is connected at some point of $\mathrm{dom}F\cap\mathrm{dom}H$, we can easily see that F or $A\circ H$ is connected at some point of $\mathrm{dom}F\cap\mathrm{dom}(A\circ H)=\mathrm{dom}F\cap\mathrm{dom}H$. Hence, from Theorem 3.3, we have

$$\partial^s(F+G\circ H)(\bar{x},\bar{u}+\bar{v})=\partial^s F(\bar{x},\bar{u})+\bigcup_{A\in\partial^s G(\bar{z},\bar{v})}\partial^s(A\circ H)(\bar{x},A(\bar{z})).$$

□

The following corollary is a result obtained in [5].

Corollary 3.6. *Let $f:X\to Y\cup\{+\infty_Y\}$, $h:X\to Z\cup\{+\infty_Z\}$ and $g:Z\to Y\cup\{+\infty_Y\}$ be three set-valued mappings, $\bar{x}\in\mathrm{dom}F\cap\mathrm{dom}h$ such that $h(\bar{x})\in\mathrm{dom}g$. Suppose also that the following condition holds.*

$$\left\{\begin{array}{l}(Y,Y_+)\textit{ is a normal order complete lattice Hausdorff locally convex topological}\\ \textit{vector space, }(Z,Z_+)\textit{ is a Hausdorff locally convex topological vector space,}\\ f,\ g\textit{ are }Y_+\textit{-convex and }h\textit{ is }Z_+\textit{-convex ,}\\ g\textit{ is }(Z_+,Y_+)\textit{-nondecreasing,}\\ g\textit{ is continuous at some point of }h(\mathrm{dom}f\cap\mathrm{dom}h).\end{array}\right.$$

Then

$$\partial^s(f+g\circ h)(\bar{x})=\bigcup_{A\in\partial^s g(\bar{z})}\partial^s(f+A\circ h)(\bar{x}).$$

Proof. Let us consider the following set-valued mappings

$$F(x)=\left\{\begin{array}{ll}\{f(x)\}, & \text{if } x\in\mathrm{dom}\,f,\\ \emptyset, & \text{otherwise,}\end{array}\right.\qquad H(x)=\left\{\begin{array}{ll}\{h(x)\}, & \text{if } x\in\mathrm{dom}\,h,\\ \emptyset, & \text{otherwise,}\end{array}\right.$$

$$G(x)=\left\{\begin{array}{ll}\{g(x)\}, & \text{if } x\in\mathrm{dom}\,g,\\ \emptyset, & \text{otherwise.}\end{array}\right.$$

It is easy to check that if g is continuous at some point of $h(\mathrm{dom} f \cap \mathrm{dom} h)$, then the set-valued mapping g is connected at some point of $H(\mathrm{dom}F \cap \mathrm{dom}H)$ and also we easily check that the set-valued mappings F, H, and G satisfy all the assumptions of Theorem 3.4. Therefore,

$$\partial^s(F+G\circ H)(\bar{x}, f(\bar{x})+g(h(\bar{x}))) = \bigcup_{A\in\partial^s G(h(\bar{x}),g(h(\bar{x}))} \partial^s(F+A\circ H)(\bar{x},\bar{u}+A(h(\bar{x}))).$$

□

4. Application to Vector Set Optimization Problems

In this section, we consider the following constrained set-valued optimization problem

$$\begin{cases} \text{minimize F(x)}, \\ x\in S, \end{cases} \tag{P1}$$

where $F : X \rightrightarrows Y$ is a set-valued mapping, and S is a nonempty convex closed subset of X. By using the set-valued indicator mapping $R^v_S : X \rightrightarrows Y$ of the nonempty subset $S \subseteq X$, problem (P1) becomes equivalent to the unconstrained set-valued minimization problem

$$\begin{cases} \text{minimize } (F+R^v_S)(x), \\ x\in X. \end{cases} \tag{P2}$$

Lemma 4.1. *(i) If S is convex and closed, then R^v_S is Y_+-convex and, for all $\bar{x}\in S$, $\partial^s R^v_S(\bar{x},0_Y) = N^v_S(\bar{x})$, where $N^v_S(\bar{x}) := \{A \in L(X,Y) : A(x-\bar{x}) \leq_{Y_+} 0_Y, \forall x \in S\}$ is the vector normal cone at $\bar{x} \in S$.*
(ii) If $\mathrm{int}(S) \neq \emptyset$, then R^v_S is connected on $\mathrm{int}(S)$.

Proof. (i) The epigraph of R^v_S is given by $\mathrm{epi}R^v_S = \{(x,y)\in X\times Y : y\in R^v_S(x)+Y_+\} = S\times Y_+$, and its Y_+-convexity easily follows from the convexity of S and Y_+.

(ii) Let us consider the single following mapping $h : X \longrightarrow Y$ defined by $h(x) := 0_Y$ for all $x\in X$. Since $0_Y \in R^v_S(x)$ for any $x\in S$, it follows that $h(x)\in R^v_S(x)$ for any $x\in \mathrm{int}(S)$, which ensures that R^v_S is connected on $\mathrm{int}(S)$. □

We are now ready to establish optimality conditions for (P1).

Theorem 4.1. *Let $F : X \rightrightarrows Y$ be a set-valued mapping, S be a nonempty convex closed subset of X, and $(\bar{x},\bar{y}) \in \mathrm{gr}F$ with $\bar{x}\in S$. If the following qualification condition holds*

$$\begin{cases} (Y,Y_+) \text{ is a normal order complete lattice Hausdorff locally convex topological} \\ \text{vector space, } F \text{ is } Y_+\text{-convex,} \\ \mathrm{dom}F\cap\mathrm{int}(S)\neq\emptyset \text{ or } F \text{ is connected at some point of } \mathrm{dom}F\cap S. \end{cases} \tag{MR2}$$

Then $(\bar{x},\bar{y})$ is a strong minimizer for (P1) with respect to Y_+ if and only if there exists $A \in \partial^s F(\bar{x},\bar{y})$ such that $-A\in N^v_S(\bar{x})$.

Proof. Since (P1) is equivalent to unconstrained set-valued minimization problem (P2), we have that $(\bar{x},\bar{y})$ is a strong minimizer to (P1) if and only if

$$0\in\partial^s(F+R^v_S)(\bar{x},\bar{y}+0_Y). \tag{4.1}$$

Conditions (MR2) and Lemma 4.1 together show that mappings F and R^v_S satisfy all the hypotheses of Theorem 3.3. Hence, $0\in\partial^s F(\bar{x},\bar{y})+\partial^s R^v_S(\bar{x},0_Y)$, i.e., there exists $A\in\partial^s F(\bar{x},\bar{y})$ such that $-A\in\partial^s R^v_S(\bar{x},0_Y) = N^v_S(\bar{x})$. □

Let us consider the following general convex set-valued mathematical programming problem

$$\begin{cases} \text{minimize F(x)}, \\ H(x)\cap -Z_+ \neq \emptyset, \\ x\in C. \end{cases} \tag{P3}$$

where $F : X \rightrightarrows Y$ and $H : X \rightrightarrows Z$ are two set-valued mappings, Z is a real locally convex topological vector space, Z_+ is a closed convex pointed cone with nonempty topological interior, and C is a nonempty, closed, and convex set of X. For establishing the optimality conditions of this problem, we need the following lemma.

Lemma 4.2. *(i) If Z is a real locally convex topological vector space, and $Z_+ \subseteq Z$ is a closed convex cone, then the strong subdifferential of the indicator set-valued mapping $R^v_{-Z_+} : Z \rightrightarrows Y$ is given by $\partial^s R^v_{-Z_+}(\bar{z}, 0_Y) = \{A \in L_+(Z,Y) : A(\bar{z}) = 0_Y\}$.*
(ii) The indicator set-valued mapping $R^v_{-Z_+}$ is (Z_+,Y_+)-nondecreasing on Z.

Proof. (i) Let $\bar{z} \in -Z_+$. Obviously, $(\bar{z}, 0_Y) \in \mathrm{gr}R^v_{-Z_+} = -Z_+ \times \{0_Y\}$. From Lemma 4.1 (i), we have $\partial^s R^v_{-Z_+}(\bar{z}, 0_Y) = N^v_{-Z_+}(\bar{z})$. Now, let us prove the first inclusion $\partial^s R^v_{-Z_+}(\bar{z}, 0_Y) \subseteq \{A \in L_+(Z,Y) : \ A(\bar{z}) = 0_Y\}$. Observe that

$$A \in \partial^s R^v_{-Z_+}(\bar{z}, 0_Y) \Longleftrightarrow A(z-\bar{z}) \leq_{Y_+} 0_Y, \ \forall z \in -Z_+. \tag{4.2}$$

By successively taking $z = 0$ and $z = 2\bar{z}$ in (4.2), we have $A(\bar{z}) \in Y_+ \cap -Y_+ = \{0_Y\}$, i.e., $A(\bar{z}) = 0_Y$. Consequently, we deduce from (4.2) that $A(z) \in Y_+$ for all $z \in Z_+$, which means that $A \in L_+(Z,Y)$. For the reverse inclusion, since $A(\bar{z}) = 0_Y$ and $A \in L_+(Z,Y)$, then it follows that $A(z-\bar{z}) = A(z) \leq_{Y_+} 0_Y$ for all $z \in -Z_+$. Hence the equality holds.
(ii) was proved in [13]. □

Theorem 4.2. *Let $F : X \rightrightarrows Y$ and $H : X \rightrightarrows Z$ be two set-valued mappings, $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ with $\bar{x} \in C$ and $\bar{z} \in H(\bar{x}) \cap (-Z_+) \neq \emptyset$. If the following condition holds*

$$\begin{cases} (Y,Y_+) \text{ is a normal order complete lattice Hausdorff locally convex topological} \\ \text{vector space, } (Z,Z_+) \text{ is a Hausdorff locally convex topological vector space,} \\ F \text{ is } Y_+\text{-convex and } H \text{ is } Z_+\text{-convex,} \\ \mathrm{int}(-Z_+) \cap H(C \cap \mathrm{dom}F \cap \mathrm{dom}H) \neq \emptyset. \end{cases}$$

Then $(\bar{x}, \bar{y})$ is a strong minimiser solution to problem (P3) if and only if there exists $A \in L_+(Z,Y)$ such that

(a) $A(\bar{z}) = 0_Y$;
(b) $0 \in \partial^s(F + A\circ H + R^v_C)(\bar{x}, \bar{y})$.

Proof. The feasible set associated to problem (P3) is given by $S = \{x \in X : H(x) \cap -Z_+ \neq \emptyset\} \cap C$, and it is easy to check that $R^v_S = R^v_C + R^v_{-Z_+} \circ H$. Hence problem (P3) becomes equivalent to the unconstrained set-valued minimization problem

$$\begin{cases} \text{minimize}(F + R^v_C + R^v_{-Z_+} \circ H)(x), \\ x \in X. \end{cases}$$

Thus $(\bar{x}, \bar{y})$ is a strong minimiser solution to problem (P3) if and only if $0 \in \partial^s(F + R^v_C + R^v_{-Z_+} \circ H)(\bar{x}, \bar{y})$. Observe that $\mathrm{epi}(F + R^v_C) = \mathrm{epi}F \cap (C \times Y)$, which asserts that the convexity of the set-valued mapping $F + R^v_C$ follows from the convexity of the epigraph of F and the convexity of the subset C. Also, let us note that the conditions $(\bar{x},\bar{y}) \in \mathrm{gr}F$ with $\bar{x} \in C$ and $\bar{z} \in H(\bar{x}) \cap (-Z_+) \neq \emptyset$ can be written equivalently as $(\bar{x},\bar{y}) \in \mathrm{gr}(F+R^v_C)$, $(\bar{x},\bar{z}) \in \mathrm{gr}H$, and $(\bar{z},0_Y) \in \mathrm{gr}R^v_{-Z_+}$. According to Lemma 4.1 and Lemma 4.2, the set-valued mappings $F+R^v_C$, H, and $R^v_{-Z_+}$ satisfy all the assumptions of Theorem 3.4. Thus we obtain that $(\bar{x},\bar{y})$ is a strong minimiser if and only if there exists $A \in \partial^s R^v_{-Z_+}(\bar{z},0_Y) = \{A \in L_+(Z,Y): \ A(\bar{z}) = 0_Y\}$ such that $0 \in \partial^s(F+R^v_C+A\circ H)(\bar{x},\bar{y}+A(\bar{z}))$. The proof of theorem is complete. □

Corollary 4.1. *Under the assumptions of Theorem 4.2, assume, in addition, that F is connected at some point of C, and H is connected at some point of C. Then $(\bar{x},\bar{y})$ is a strong minimiser to problem (P3) if and only if there exist $A \in L_+(Z,Y)$, $B \in \partial^s F(\bar{x},\bar{y})$ and $T \in \partial^s(A\circ H)(\bar{x},0_Y)$ such that*

(a) $A(\bar{z}) = 0_Y$.
(b) $-T-B \in N^v_C(\bar{x})$.

Proof. According to Theorem 4.2, we have that $(\bar{x},\bar{y})$ is a strong minimiser to problem (P3) if and only if there exists $A \in L_+(Z,Y)$ such that $A(\bar{z}) = 0_Y$ and $0 \in \partial^s(F+A\circ H+R^v_C)(\bar{x},\bar{y})$. The fact that $A \in L_+(Z,Y)$ yields that $A\circ H$ is Y_+-convex. As H is connected at some point of C, it is easy to check that $A\circ H$ is connected at some point of C. Since $\mathrm{epi}(A\circ H+R^v_C) = \mathrm{epi}(A\circ H)\cap(C\times Y)$. Hence the convexity of the set-valued mapping $A\circ H+R^v_C$ follows from the convexity of the epigraph of $A\circ H$ and the convexity of the subset C. The set-valued mappings F and $A\circ H+R^v_C$ together satisfy all the assumptions of Theorem 3.3. Thus

$$0 \in \partial^s(F+A\circ H+R^v_C)(\bar{x},\bar{y}) = \partial^s(A\circ H+R^v_C)(\bar{x},0_Y)+\partial^s F(\bar{x},\bar{y}).$$

On other hand, we claim that the set-valued mappings $A\circ H$ and R^v_C together satisfy all the hypothesis of Theorem 3.3. Hence,

$$0 \in \partial^s(F+A\circ H+R^v_C)(\bar{x},\bar{y}) = \partial^s(A\circ H)(\bar{x},0_Y)+\partial^s R^v_C(\bar{x},0_Y)+\partial^s F(\bar{x},\bar{y}),$$

i.e., there exist $B \in \partial^s F(\bar{x},\bar{y})$ and $T \in \partial^s(A\circ H)(\bar{x},0_Y)$ such that $-T-B \in \partial^s R^v_C(\bar{x},0_Y) = N^v_C(\bar{x})$. □

REFERENCES

[1] M. Théra, Subdifferential calculus for convex operators, J. Math. Anal. Appl. 80 (1981), 78-91.
[2] M. El Maghri, M. Laghdir, Pareto subdifferential caculus for convex vector mappings and applications to vector optimization, SIAM. J. Optim. 19 (2008), 1970-1994.
[3] M. Valadier, Sous-différentiabilité de fonctions convexes à valeurs dans un espace vectoriel ordonné, Math. Scand. 30 (1972), 65-74.
[4] J. Zowe, Subdifferentiability of convex functions with values in an ordered vector space, Math. Scand. 34 (1974), 69-83.
[5] M. Laghdir, A Rekouane, A note on subdifferential of composed convex operator, Int. J. Math. Anal. 8 (2014), 1999-2008.
[6] Q.H. Ansari, P.K. Sharma, Set Order Relations, Set Optimization, and Ekeland's Variational Principle, In: Laha V., Maréchal P., Mishra S.K. (eds) Optimization, Variational Analysis and Applications, Springer Proceedings in Mathematics and Statistics, vol 355, Springer, Singapore, 2021.
[7] A. Taa, On subdifferential calculus for set-valued mappings and optimality conditions. Nonlinear Anal. 74 (2011), 7312-7324.
[8] A. Khan, C. Tammer, Zalinescu, Set-valued Optimization. An Introduction with Applications, Springer, Berlin, 2015.

[9] P.H. Sach. Moreau-Rockafellar theorems for nonconvex set-valued maps, J. Optim. Theory Appl. 133 (2007), 213–227.
[10] D.T. Luc, Theory of Vector Optimization, Lecture Notes Econom. Math. Systems 319, Springer-Verlag, Berlin, 1989.
[11] T. Tanino, Conjugate duality in vector optimization, J. Math. Anal. Appl. 167 (1992), 84–97.
[12] C. Zalinescu, Convex Analysis in General Vector Spaces, World Scientfic, Singapore, 2002.
[13] A. Taa, Subdifferential calculus for set-valued mappings and optimality conditions for multiobjective optimization problems, J. Optim. Theory Appl. 180 (2019), 428-441.