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A POROSITY RESULT REGARDING FIXED POINTS FOR A CLASS OF NONEXPANSIVE MAPPINGS

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Abstract. In one of our recent papers, we considered a complete metric space of nonexpansive mappings taking a bounded and closed subset of a complete hyperbolic space into the space so that the boundary of this subset is mapped back into the subset itself. Using the Baire category approach, we proved that most of these mappings possess a unique fixed point which attracts all their iterates. In the present paper, we improve upon this result by showing that the complement of the set of mappings which have a fixed point is not only of the first Baire category, but also is σ-porous.
Keywords. Complete metric space; Fixed point; Iterate; Nonexpansive mapping; Porous set.

1. Introduction

For nearly sixty years, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings; see, for example, [1]-[16] and the references cited therein. This activity stems from Banach's classical fixed point theorem [17] concerning the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, the studies of feasibility and common fixed point problems, which find important applications in the engineering and medical sciences; see, for example, [15, 16] and [18]-[26].

In our recent paper, we considered a complete metric space of nonexpansive mappings taking a bounded and closed subset of a complete hyperbolic space into the space so that the boundary of the subset is mapped back into the subset itself. Using the Baire category approach, we proved that most of these mappings possess a unique fixed point, which attracts all their iterates. In the present paper, we improve upon this result by showing that the complement of the set of mappings which have a fixed point is not only of the first Baire category, but also is σ-porous.

Let (X,ρ) be a metric space, and let R^1 denote the real line. We say that a mapping $c : R^1 \to X$ is a metric embedding of R^1 into X if $\rho(c(s),c(t)) = |s-t|$ for all real s and t. The image of R^1 under a metric embedding is called a metric line. The image of a real interval $[a,b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping is called a metric segment.

Assume that (X,ρ) contains a family M of metric lines such that, for each pair of distinct points x and y in X, there exists a unique metric line in M, which passes through x and y. This

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metric line determines a unique metric segment joining x and y. We denote this segment by $[x,y]$. For each $0 \le t \le 1$, there exists a unique point z in $[x,y]$ such that

$$\rho(x,z) = t\rho(x,y) \text{ and } \rho(z,y) = (1-t)\rho(x,y).$$

This point is denoted by $(1-t)x \oplus ty$. We say that X, or more precisely (X,ρ,M), is a hyperbolic space if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \le \frac{1}{2}\rho(y,z)$$

for all x,y and z in X. An equivalent requirement is that

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z) \le \frac{1}{2}(\rho(x,w)+\rho(y,z))$$

for all x,y,z and w in X. A set $K \subset X$ is called ρ-convex if $[x,y] \subset K$ for all x and y in K.

It is clear that all normed linear spaces are hyperbolic. A discussion of more examples of hyperbolic spaces and, in particular, of the Hilbert ball can be found, for example, in [6, 27].

2. Preliminaries

Let (X,ρ,M) be a complete hyperbolic space, and let K be a nonempty, closed, and bounded subset of X.

For each point $x \in X$ and each number $r > 0$, set

$$B(x,r) := \{y \in X : \ \rho(x,y) \le r\}.$$

Let

$$\partial K := \{x \in K : \ B(x,\varepsilon) \cap (X \setminus K) \ne \emptyset \text{ for each } \varepsilon > 0\}$$

and

$$\mathrm{Int}(K) := K \setminus \partial K.$$

Assume that $\theta \in K$ and that the following assumption holds.

(A) For each $\gamma \in (0,1)$ and each $x \in K$, $\gamma\theta \oplus (1-\gamma)x \in K$.

Note that this assumption does hold if K is a convex subset of a Banach space.

Denote by $\mathscr{A}$ the set of all mappings $T : K \to X$ such that

$$\rho(T(x),T(y)) \le \rho(x,y) \text{ for each } x,y \in K \tag{2.1}$$

and $T(\partial K) \subset K$. Set

$$\mathrm{diam}(K) := \sup\{\rho(x,y) : \ x,y \in K\}.$$

For each $A,B \in \mathscr{A}$, let

$$d(A,B) := \sup\{\rho(A(x),B(x)) : \ x \in K\}. \tag{2.2}$$

In view of (2.1) and the boundedness of K, for each $A,B \in \mathscr{A}$, $d(A,B)$ is finite. It is not difficult to see that $(\mathscr{A},d)$ is a complete metric space.

The following result was established in [1].

Theorem 2.1. *Assume that $T \in \mathscr{A}$, $c \in (0,1)$ and that $\rho(T(x),T(y)) \le c\rho(x,y)$ for all $x,y \in K$. Then T has a fixed point.*

Note that in [1] this result was, in fact, obtained for set-valued mappings.

In [28], using Theorem 2.1 and the Baire category approach, we obtained that a generic (typical) mapping in $\mathscr{A}$ possesses a unique fixed point, which attracts all its iterates.

The following generalization of Theorem 2.1 was obtained in [29].

Theorem 2.2. *Assume that $T \in \mathscr{A}$ and that $\phi : [0,\infty) \to [0,1]$ is a decreasing function such that*

$$\phi(t) < 1 \text{ for all } t > 0$$

and

$$\rho(T(x),T(y)) \le \phi(\rho(x,y))\rho(x,y) \text{ for all } x,y \in K.$$

Then T has a fixed point.

The mapping T in Theorem 2.2 has a unique fixed point which attracts all its iterates.

In this connection, see also [30, Corollary 3].

In the present paper, using Theorem 2.2, we show that the complement of the set of mappings which have a fixed point is not only of the first Baire category, but also is σ-porous.

At this point we recall that a mapping T which satisfies the contractive condition of Theorem 2.2 is called a Rakotch contraction [13].

We now recall the notion of porosity. Let (Y,D) be a complete metric space. We denote by $B(y,r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called porous in (Y,D) [13] if there exist $\alpha \in (0,1)$ and $r_0 > 0$ such that, for each $r \in (0,r_0]$ and each $y \in Y$, there exists $z \in Y$ for which

$$B(z,\alpha r) \subset B(y,r) \setminus E.$$

A subset of the space Y is called σ-porous in (Y,D) if it is a countable union of porous subsets of (Y,D).

Since porous sets are nowhere dense, all σ-porous sets are of the first Baire category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have Lebesgue measure 0 and are of the first Baire category.

3. The Porosity Result

In this section, we first state and then prove our main result.

Theorem 3.1. *There exists a set $\mathscr{F} \subset \mathscr{A}$ such that the set $\mathscr{A} \setminus \mathscr{F}$ is σ-porous and, for each $T \in \mathscr{F}$, there exists a decreasing function $\phi : [0,\infty) \to [0,1]$ such that*

$$\phi(t) < 1 \text{ for all } t > 0$$

and

$$\rho(T(x),T(y)) \le \phi(\rho(x,y))\rho(x,y) \text{ for all } x,y \in K.$$

In this connection, see also [31].

We precede the proof of our main result with the following lemma.

Lemma 3.1. *Assume that $T \in \mathscr{A}$ and that $z,x \in K$. Then $\rho(z,T(x)) \le 3\mathrm{diam}(K)$.*

Proof. Let $y \in \partial K$. Then $T(y) \in K$ and $\rho(y,T(y)) \le \mathrm{diam}(K)$. This implies that

$$\rho(z,T(y)) \le 2\mathrm{diam}(K).$$

It is clear that

$$\rho(T(x),T(y)) \le \rho(x,y) \le \mathrm{diam}(K)$$

and so $\rho(z,T(x)) \le 3\mathrm{diam}(K)$. This completes the proof of this lemma. □

Proof of Theorem 3.1. Let $n \geq 1$ be an integer. Define

$$\begin{aligned}\mathscr{F}_n := \{T \in \mathscr{A} :\ &\sup\{\rho(T(x),T(y))\rho(x,y)^{-1} : x,y \in K \\ &\text{and } \rho(x,y) \geq n^{-1}(\operatorname{diam}(K)+1)^{-1}\} < 1\}.\end{aligned} \tag{3.1}$$

Set $\mathscr{F} := \cap_{n=1}^{\infty} \mathscr{F}_n$ and assume $S \in \mathscr{F}$, $t > 0$. Define

$$\begin{aligned}\phi(t) := \sup\{\rho(S(x),S(y))\rho(x,y)^{-1} &: x,y \in K \\ &\text{and } \rho(x,y) \geq \min\{t, 2^{-1}(\operatorname{diam}(K)+1)^{-1}\}\}.\end{aligned}$$

It is clear that $\phi(t)$ is well defined and that ϕ is a decreasing function. Set $\phi(0) := 1$. In view of (3.1), one has $\phi(t) < 1$, $\forall t \in (0,\infty)$. It is not difficult to see that

$$\rho(S(x),S(y)) \leq \rho(x,y)\phi(\rho(x,y)),\ x,y \in K.$$

Let $n \geq 1$ be an integer. We claim that the set $\mathscr{A} \setminus \mathscr{F}_n$ is porous. To see this, fix a positive number α such that

$$36\alpha(\operatorname{diam}(K)+1)^2 n \leq 1. \tag{3.2}$$

Assume that $T \in \mathscr{A}$, $r \in (0,1]$. Set

$$\gamma = 4\alpha r(\operatorname{diam}(K)+1)n. \tag{3.3}$$

It is clear that $\gamma < 1$. Define

$$T_\gamma(x) := (1-\gamma)T(x) \oplus \gamma\theta, \quad \forall x \in K. \tag{3.4}$$

Evidently, $T_\gamma(K) \subset X$. Assumption (A) and (3.3) imply that $T_\gamma(x) \in K$ for each $x \in \partial K$. In view of (3.3) for each $x,y \in K$, we have

$$\rho(T_\gamma(x),T_\gamma(y)) \leq (1-\gamma)\rho(T(x),T(y)) \leq (1-\gamma)\rho(x,y)$$

and $T_\gamma \in \mathscr{A}$. Lemma 3.1 and relations (2.2) and (3.4) imply that

$$\begin{aligned}d(T_\gamma,T) &\leq \sup\{\rho(T(x),(1-\gamma)T(x) \oplus \gamma\theta) :\ x \in K\} \\ &\leq \gamma \sup\{\rho(\theta,T(x)) :\ x \in K\} \\ &\leq 3\gamma \operatorname{diam}(K).\end{aligned} \tag{3.5}$$

Assume now that $S \in \mathscr{A}$ satisfies

$$d(S,T_\gamma) \leq \alpha r. \tag{3.6}$$

It follows from (3.2), (3.3), (3.5), and (3.6) that

$$\begin{aligned}d(S,T) &\leq d(S,T_\gamma) + d(T_\gamma,T) \\ &\leq \alpha r + d(T_\gamma,T) \\ &\leq \alpha r + 3\gamma \operatorname{diam}(K) \\ &\leq r.\end{aligned}$$

Assume that $x,y \in K$ satisfy

$$\rho(x,y) \geq n^{-1}(\operatorname{diam}(K)+1)^{-1}. \tag{3.7}$$

Relations (2.1), (3.4), and (3.6) imply that

$$\begin{aligned}\rho(S(x),S(y)) &\leq \rho(T_\gamma(x),T_\gamma(y))+2\alpha r\\ &\leq \rho((1-\gamma)T(x)\oplus\gamma\theta,(1-\gamma)T(y)\oplus\gamma\theta)+2\alpha r\\ &\leq (1-\gamma)\rho(T(x),T(y))+2\alpha r\\ &\leq (1-\gamma)\rho(x,y)+2\alpha r\\ &= \rho(x,y)-\gamma\rho(x,y)+2\alpha r.\end{aligned}$$

This together with (3.3) and (3.7) implies that

$$\begin{aligned}\rho(S(x),S(y))\rho(x,y)^{-1} &\leq 1-\gamma+2\alpha r\rho(x,y)^{-1}\\ &\leq 1-\gamma+2\alpha r(n(\mathrm{diam}(K)+1)\\ &= 1-\gamma/2.\end{aligned}$$

Thus $S\in\mathscr{F}_n$. This implies that

$$\{S\in\mathscr{A}:\ d(S,T_\gamma)\leq\alpha r\}\subset\{S\in\mathscr{A}:\ d(S,T)\leq r\}\cap\mathscr{F}_n.$$

Therefore $\mathscr{A}\setminus\mathscr{F}_n$ is a porous set. Clearly,

$$\mathscr{A}\setminus\mathscr{F}=\mathscr{A}\setminus\cap_{n=1}^{\infty}\mathscr{F}_n=\cup_{n=1}^{\infty}(\mathscr{A}\setminus\mathscr{F}_n)$$

and so $\mathscr{A}\setminus\mathscr{F}$ is a σ-porous set. This completes the proof of this theorem.

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