

R.E. BRUCK, PROOF MINING AND A RATE OF ASYMPTOTIC REGULARITY FOR ERGODIC AVERAGES IN BANACH SPACES

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Dedicated to the Memory of Professor Ronald E. Bruck

Abstract. We analyze a proof of Bruck to obtain an explicit rate of asymptotic regularity for Cesàro means in uniformly convex Banach spaces. Our rate only depends on a norm bound and a modulus η of uniform convexity. One ingredient for the proof by Bruck is a result of Pisier, which demonstrates that every uniformly convex (in fact, every uniformly nonsquare) Banach space has some Rademacher type $q > 1$ with a suitable constant C_q . We explicitly determine q and C_q , which only depend on the single value $\eta(1)$ of our modulus. Beyond these specific results, we summarize how work of Bruck has inspired developments in the proof mining program, which applies tools from logic to obtain results in various areas of mathematics.

Keywords. Asymptotic regularity; Ergodic averages; Proof mining; Rademacher type; Uniformly non-square Banach spaces.

1. INTRODUCTION

Proof mining is the project of applying proof-theoretic transformations to obtain new quantitative and qualitative information from given proofs in areas of core mathematics such as nonlinear analysis, convex optimization and geodesic geometry (see, e.g., [1]). Bruck, who himself did fundamental work on quantitative issues of metric fixed point theory ([2]), was a major source of inspiration in this program both by providing in his research deep results which naturally asked for a more finitary quantitative treatment as well as by introducing fundamental new notions which were particularly suited for such a proof-theoretic enterprise.

In Section 2, we give a short survey on the important role which results of Bruck have had in the development of proof mining. The most recent proof-theoretic analysis of a work of Bruck was carried out in [3] which gives an explicit rate of metastability (in the sense of T. Tao) for a strong nonlinear ergodic theorem in uniformly convex Banach spaces due to [4] which in turn is crucially based on Bruck's seminal work on the nonlinear mean ergodic theorem [5] and the convex approximation property [6]. In [6], the asymptotic regularity of Cesàro means in uniformly convex Banach spaces was established using that uniformly convex Banach spaces

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Received March 18, 2022; Accepted July 9, 2022.

X , which are thus B-convex, have a nontrivial Rademacher type $q > 1$. This latter fact was first established by Pisier in [7].

In Section 3, we extract from Pisier's proof explicit lower estimates > 1 for q and an upper estimate for the relevant constant C_q witnessing that the space has Rademacher type q in terms of a constant $\delta > 0$ which witnesses that X is uniformly nonsquare. If X is uniformly convex and η is some modulus of uniform convexity, then δ can be taken as $\delta := \eta(1)$.

In Section 4, we then give an explicit rate of asymptotic regularity in terms of η, q, C_q and so - by Section 3 - in terms of η alone. Note that we do not use here the optimal modulus δ_X of uniform convexity but any function η witnessing the $\forall \varepsilon \exists \delta$ -definition of uniform convexity. In Remark 4.2, we comment on an alternative way to obtain a rate of asymptotic regularity, which combines our work with a result of Zhu, Huang and Li [8].

2. BRUCK AND PROOF MINING

The work of Bruck contains many important results which he proved by *prima facie* noneffective means but which do ask - e.g. by their general logical form - for additional computational information. One such example is his convergence result on an iteration scheme (for suitable sequences $(\lambda_n), (\theta_n)$ in $[0, 1]$)

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1)$$

for demicontinuous (single valued) pseudo-contractions $T : C \rightarrow C$ on nonempty closed and convex subsets C of a Hilbert space X ([9, Theorem 4]). While the Mann iteration of T is not even asymptotically regular for pseudo-contractions (see [10]) and the Ishikawa iteration is asymptotically regular but strongly convergent only for compact C , Bruck's hybrid scheme converges strongly towards a fixed point of T if T has a fixed point, which e.g. is the case when C additionally is bounded (Bruck's scheme was also studied in Banach spaces; see, e.g., [11, 12, 13]). By a proof-theoretic analysis of Bruck's noneffective proof, Körnlein extracted in [14, 15] an explicit effective rate of metastability in the sense of T. Tao for this convergence result. This work is based in turn on [16], where the special case of Lipschitzian pseudo-contractions had been treated. In this situation, one has the asymptotic regularity of the iteration, i.e.,

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0,$$

even in general Banach spaces and with easier conditions on the scalars involved by a result from [17] from which a polynomial rate of asymptotic regularity was extracted in [16].

Another inspiration for proof mining came from Bruck's nonconstructive proof of the existence of sunny nonexpansive retractions [18] onto fixed point sets $\text{Fix}(T)$ of nonexpansive mappings $T : C \rightarrow C$ with $C \subseteq X$, where X is reflexive, C is bounded, closed, and convex (actually Bruck's conditions are still weaker), and T satisfies the so-called conditional fixed point property CFP. These conditions always hold, for example, for uniformly smooth spaces. The now common terminology 'sunny nonexpansive retraction' is due to Reich [19]. Bruck [18] refers to these mappings as 'nonexpansive ray retractions'. In [20], Bruck demonstrated that $\text{Fix}(T)$ is a nonexpansive retract whenever the underlying space X is a reflexive strictly convex Banach space making use of Zorn's lemma. If X is uniformly smooth, then there is even a sunny nonexpansive retraction and there can - already in smooth spaces - only be at most one such sunny nonexpansive retraction [21]. The first more constructive way of approaching this unique

sunny nonexpansive retraction is Reich's [13] fundamental result that in uniformly smooth Banach spaces the sequence (x_n) defined by x_n being the fixed point of the strict contraction

$$T_n : C \rightarrow C \quad \text{with} \quad T_n(x) := \left(1 - \frac{1}{n}\right) T(x) + \frac{1}{n}u \quad \text{for} \quad u \in C$$

strongly converges to the sunny nonexpansive retraction onto $\text{Fix}(T)$ applied to u . Reich's result actually is more general and, in particular, even applies to the aforementioned pseudo-contractions (for more information on (sunny) nonexpansive retractions and retracts, see [22]). However, the convergence proof is highly noneffective and, in fact, one can show that even for simple situations (like $X := \mathbb{R}$) there in general is no computable rate of convergence. In [23], a completely constructive proof for the metastable reformulation of the convergence of (x_n) (which – noneffectively – trivially is equivalent to the latter) is given together with an explicit rate of metastability (for X being both uniformly smooth and uniformly convex). A different aspect by which Bruck's research is particularly attractive in the course of the proof mining paradigm is his formulation of important classes of mappings such as the firmly nonexpansive mappings [18] and – together with Reich – the averaged (first defined in [24]) and strongly nonexpansive (SNE) mappings [25] as well as the strongly quasi-nonexpansive mappings [26]. All these classes beautifully fit the requirements for proof-theoretically well-behaved classes of nonlinear mappings and play a crucial rule in papers using the proof mining methodology, such as [27, 28, 29]. For example, the condition of being firmly nonexpansive is purely universal and hence 'tame' in the context of the functional interpretations used in proof mining. The same is true for averaged mappings once one has an averaging constant $\alpha \in (0, 1)$ together with a witness $N \in \mathbb{N}$ such that $\alpha \in [\frac{1}{N}, 1 - \frac{1}{N}]$ given. In this case, extracted bounds will additionally depend on N (but not on α). Being SNE in its original formulation seemingly has a very high logical complexity, but, in fact, is equivalent to the existence of a number-theoretic function ω satisfying a purely universal condition (see [28]):

$$\forall c, k \in \mathbb{N} \forall x, y \in C \quad \left(\|x - y\| \leq c \wedge \|x - y\| - \|Tx - Ty\| < 2^{-\omega(k)} \rightarrow \|(x - y) - (Tx - Ty)\| \leq 2^{-k} \right).$$

Then extractable bounds will depend additionally on ω . Moreover, from any proof of the property SNE for a class of mappings which satisfy the logical conditions in the metamathematical bound extraction theorems in proof mining, one can extract such an SNE-modulus. This e.g. has been done for the class of firmly nonexpansive mappings in uniformly convex Banach spaces in [28] and for the averaged mappings in Hilbert space in [30]. The important property of SNE-mappings being closed under composition results in a simple computation of an SNE-modulus for the compositions of SNE-mappings in terms of SNE-moduli for the individual mappings. All this plays a crucial role in the extraction of a polynomial rate of asymptotic regularity in [31] for Bauschke's solution of the zero displacement conjecture (see also the recent generalization of [31] in [30]) as well as the quantitative analysis of proximal point type algorithms in [32, 33, 34]. In the latter papers, it was crucially used that all firmly nonexpansive mappings in uniformly convex Banach spaces (and so all resolvents) have a common SNE-modulus and that all averaged mappings in Hilbert spaces with some control on the averaging constants have a common SNE-modulus as well. Also the property of being strongly quasi-nonexpansive is logically very well-behaved (when localized to some fixed point p of T) and gives rise to a corresponding modulus which witnesses this property quantitatively and which is used in the

extraction of explicit rates of asymptotic regularity and metastability of algorithms which compute common fixed points of such mappings in geodesic settings, such as $\text{CAT}(\kappa)$ -spaces with $\kappa > 0$ (see [28]).

As mentioned in the introduction, in the course of extracting explicit rates for a strong nonlinear ergodic theorem due to [4], which was carried out in [3], we recently, in particular, analyzed two proofs from [5] and [6], respectively. The first proof concerns the existence of a convex continuous and unbounded function $\gamma: [0, \infty) \rightarrow [0, \infty)$ such that $\gamma(0) = 0$ and for all $x_1, x_2 \in C$ and $\lambda \in [0, 1]$

$$\gamma(\|T(\lambda x_1 + (1 - \lambda)x_2) - (\lambda T x_1 + (1 - \lambda)T x_2)\|) \leq \|x_1 - x_2\| - \|T x_1 - T x_2\|$$

and the second one proves the convex approximation property of B -convex Banach spaces using the fact that such spaces have Rademacher type $q > 1$ by a result due to Pisier [7]. The latter property was used in [6] to show that, in uniformly convex Banach spaces, the Cesàro means

$$x_n := \frac{1}{n} \sum_{i=0}^{n-1} T^i x$$

of a nonexpansive map T satisfy the asymptotic regularity property $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$. In the remaining sections, we provide an explicit rate of convergence that only depends on some norm bound b and a given modulus of uniform convexity η .

3. ON THE RADEMACHER TYPE OF UNIFORMLY NONSQUARE BANACH SPACES

In this section, we analyze a proof of Pisier [7] to extract quantitative information on the Rademacher types of uniformly nonsquare Banach spaces. We then derive information on a probabilistic characterization of B -convexity, which is also due to Pisier [7]. This supplements the quantitative analysis of a proof by Bruck [6] that was carried out in [3].

Consider a sequence (ε_i) of independent random variables that take values ± 1 with probability $1/2$. These are conveniently realized by Rademacher functions r_i on $[0, 1]$ (see, e.g., [35, Theorem 2.b.3]). For $1 \leq q < \infty$ and points x_1, \dots, x_n in a given Banach space X , we consider the expected value

$$\mathbb{E} \left(\left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^q \right) = \frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^q = \int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^q dt.$$

We now recall the definition of Rademacher type (see [36, Section 9.2] or [37, Definition 1.e.12]):

Definition 3.1. A Banach space $(X, \|\cdot\|)$ is said to have Rademacher type $q \in [1, 2]$ with constant C_q if all finite sequences (x_1, \dots, x_n) in X validate

$$\mathbb{E} \left(\left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^q \right)^{1/q} \leq C_q \cdot \left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q}. \quad (*)$$

Let us also recall the notion of a uniformly nonsquare Banach space, which is due to James [38] and can be equivalently expressed as follows.

Definition 3.2. A Banach space $(X, \|\cdot\|)$ is uniformly nonsquare if there exists a $\delta \in (0, 1]$ such that

$$\min \left\{ \frac{\|x - y\|}{2}, \frac{\|x + y\|}{2} \right\} \leq (1 - \delta) \cdot \max\{\|x\|, \|y\|\}$$

holds for all $x, y \in X$.

We can now state our goal more precisely. Given $\delta > 0$, we want to determine q and C_q such that any Banach space that is uniformly nonsquare for δ has Rademacher type q with constant C_q . Let us first give a quantitative version of the result that uniformly convex entails uniformly nonsquare:

Lemma 3.1. *Consider a uniformly convex Banach space $(X, \|\cdot\|)$ with a given modulus of convexity $\eta : (0, 2] \rightarrow (0, 1]$, which means that all $\varepsilon \in (0, 2]$ and $x, y \in X$ validate*

$$\|x\|, \|y\| \leq 1 \text{ and } \|x - y\| \geq \varepsilon \quad \Rightarrow \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \eta(\varepsilon).$$

Then X is uniformly nonsquare, where we may take $\delta := \eta(1)$ in Definition 3.2.

Also note that if Definition 3.2 holds for δ , then $1 - \delta$ is an upper bound for $\lambda_2(X)$ from [7, p. VII.1].

Proof. (See also the proof of [39, Theorem 2.2.5]). Put $\lambda := 1 - \eta(1) < 1$. Since we may assume that X is non-trivial, let $e \in X$ be such that $\|e\| = 1$. Then the fact that η is a modulus of convexity (applied to $x := 0$, $y := e$ and $\varepsilon := 1$) yields $\eta(1) \leq \frac{1}{2}$ and thus $\lambda \in [\frac{1}{2}, 1)$. For any $x, y \in X$, we show

$$\min\{\|x - y\|, \|x + y\|\} \leq 2\lambda \cdot \max\{\|x\|, \|y\|\}.$$

Without loss of generality, we assume $\|x\| \geq \|y\|$. If we have $\|x\| = 0$, then the claim holds trivially. We may thus assume $\|x\| > 0$, so that we can set $\tilde{x} := x/\|x\|$ and $\tilde{y} := y/\|x\|$ to get $\|\tilde{x}\| = 1$ and $\|\tilde{y}\| \leq 1$. Due to $\lambda \geq 1/2$, the claim holds if we have $\|x - y\| \leq \|x\|$. In the remaining case we have $\|\tilde{x} - \tilde{y}\| > 1$, so that uniform convexity yields $\|\tilde{x} + \tilde{y}\|/2 \leq 1 - \eta(1) = \lambda$ and hence

$$\|x + y\| = \|\tilde{x} + \tilde{y}\| \cdot \|x\| \leq 2\lambda \cdot \|x\|,$$

as required. \square

In order to connect with an equivalent definition of Rademacher type, we will use the following.

Proposition 3.1 (Kahane-Kintchine inequality). *For $q > 1$, there exists a constant K_q such that all finite sequences (x_1, \dots, x_n) in any Banach space X validate*

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t)x_i \right\| dt \leq \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t)x_i \right\|^q dt \right)^{1/q} \leq K_q \cdot \int_0^1 \left\| \sum_{i=1}^n r_i(t)x_i \right\| dt.$$

Indeed, we may take $K_q := ((2q - 1)/(q - 1))^{q-1}$.

Proof. The result coincides with Theorem 1.e.13 of [37] (where K_q is given in the proof). \square

Pisier in [7] used a different definition of Rademacher type q , which demands a constant c_q with

$$\mathbb{E} \left(\left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2 \right)^{1/2} \leq c_q \cdot \left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q}. \quad (**)$$

We have the following relation with Definition 3.1 ($q \in [1, 2]$).

Lemma 3.2. (1) If $(*)$ holds for C_q , then $(**)$ holds for $c_q := K_2 C_q$.
 (2) If $(**)$ holds for c_q , then $(*)$ holds for $C_q := c_q$.

Proof. 1) Given $(*)$, we obtain from the Kahane-Kintchine inequality

$$\begin{aligned} \mathbb{E} \left(\left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2 \right)^{1/2} &= \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^2 dt \right)^{1/2} \leq K_2 \cdot \int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\| dt \\ &\leq K_2 \cdot \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^q dt \right)^{1/q} = K_2 \cdot \mathbb{E} \left(\left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^q \right)^{1/q} \\ &\leq K_2 \cdot C_q \cdot \left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q}. \end{aligned}$$

2) Given $(**)$, we have

$$\begin{aligned} \mathbb{E} \left(\left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^q \right) &= \int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^q dt \leq \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^2 dt \right)^{q/2} \\ &= \left(\mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2 \right)^{q/2} \leq c_q^q \cdot \sum_{i=1}^n \|x_i\|^q, \end{aligned}$$

as

$$\left(\int_0^1 |f(t)|^q dt \right)^{1/q} \leq \left(\int_0^1 |f(t)|^2 dt \right)^{1/2}$$

holds for square-integrable f and $q \in [1, 2]$. □

By analysing the proof of [7, Corollary 1], we obtain the following numerical estimate.

Theorem 3.1. Let $(X, \|\cdot\|)$ be a uniformly nonsquare Banach space with $\delta \in (0, 1)$ witnessing this property. Define $\lambda := 1 - \delta$. Assume that $\zeta \in (0, 1)$ is so small and that $p' \in [2, \infty)$ is so large that

$$\frac{1 - \zeta}{1 + 2\sqrt{2\zeta}} \geq \frac{1}{2} \sqrt{2\lambda^2 + 2} \quad \text{and} \quad \frac{1}{2^{1/p'}} \geq 1 - \zeta.$$

Take p with $1 = \frac{1}{p} + \frac{1}{p'}$. Then, for any $q \in (1, p)$, the space X has Rademacher type q with constant

$$C_q = 3 \cdot \frac{2^{1/q}}{2^{(1/q) - (1/p)} - 1}.$$

Proof. For $n \in \mathbb{N}$, Pisier [7] defined $\mu_n(X)$ as the least real $\mu \geq 0$ such that

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^2 dt \right)^{1/2} \leq \mu \cdot n \cdot \max_{1 \leq i \leq n} \|x_i\|$$

holds for any finite sequence (x_1, \dots, x_n) in X . Similarly, he defined $v_n(X)$ as the least $v \geq 0$ with

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t)x_i \right\|^2 dt \right)^{1/2} \leq v \cdot \sqrt{n} \cdot \left(\sum_{i=1}^n \|x_i\|^2 \right)^{1/2}.$$

The λ of the present theorem is an upper bound for $\lambda_2(X)$ from [7], as noted after Lemma 3.1 above. Together with inequality (2) on page VII.10 of [7] (which needs to be corrected with the missing factor $1/n$), we have

$$\mu_2(X) \leq \frac{1}{2} \left[\frac{4\lambda^2 + 4}{2} \right]^{1/2} = \frac{1}{2} \sqrt{2\lambda^2 + 2} \in (0, 1).$$

For ζ and p' as in the present theorem, the argument on pages VII.10-11 of [7] does now yield

$$v_2(X) \leq 1 - \zeta \leq \frac{1}{2^{1/p'}}.$$

By Lemma 4 of [7], it follows that X has Rademacher type q with a suitable constant C_q , for any q as in the present theorem. To determine C_q , we work out the proof of the cited lemma (which Pisier describes as analogous to the one of Lemma 2 from [7]). Assume that we have $v_N(X) \leq 1/N^{1/p'}$ for some integer $N \geq 2$ and real $p' \geq 2$ (we only need $N = 2$ but state the original more general result). Given $q \in (1, p)$ with $1 = (1/p) + (1/p')$, we shall establish (**) for suitable c_q . Due to Lemma 3.2, it will follow that the Rademacher property (*) holds for $C_q := c_q$. Aiming at (**), we consider an arbitrary sequence (x_1, \dots, x_n) in X . For $k \in \mathbb{N}$, we put

$$A(k) := \left\{ j \in \{1, \dots, n\} : \left(\frac{\sum_{i=1}^n \|x_i\|^q}{N^{k+1}} \right)^{1/q} < \|x_j\| \leq \left(\frac{\sum_{i=1}^n \|x_i\|^q}{N^k} \right)^{1/q} \right\}.$$

Write $|A(k)|$ for the cardinality of $A(k)$. We pick a bijection $f : \{1, \dots, |A(k)|\} \rightarrow A(k)$ and compute

$$\begin{aligned} \left(\int_0^1 \left\| \sum_{i \in A(k)} r_i(t)x_i \right\|^2 dt \right)^{1/2} &= \left(\int_0^1 \left\| \sum_{i=1}^{|A(k)|} r_i(t)x_{f(i)} \right\|^2 dt \right)^{1/2} \\ &\leq \mu_{|A(k)|}(X) \cdot |A(k)| \cdot \max_{i \in A(k)} \|x_i\|. \end{aligned}$$

Here the inequality holds by the definition of $\mu_n(X)$. The equality relies on the fact that the Rademacher functions represent independent copies of the same random variable, which allows us to omit the index shift from r_i to $r_{f(i)}$. For each $i \in \{1, \dots, n\}$ with $x_i \neq 0$, we have $i \in A(k)$ for a unique integer $k \geq 0$ (so that almost all $A(k)$ are empty). Using the Kahane-Kintchine

inequality, we have

$$\begin{aligned}
\mathbb{E} \left(\left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2 \right)^{1/2} &= \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^2 dt \right)^{1/2} \leq K_2 \cdot \int_0^1 \sum_{k=0}^{\infty} \left\| \sum_{i \in A(k)} r_i(t) x_i \right\| dt \\
&\leq K_2 \cdot \sum_{k=0}^{\infty} \left(\int_0^1 \left\| \sum_{i \in A(k)} r_i(t) x_i \right\|^2 dt \right)^{1/2} \\
&\leq K_2 \cdot \left(\sum_{k=0}^{\infty} \frac{\mu_{|A(k)|}(X) \cdot |A(k)|}{N^{k/q}} \right) \cdot \left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q},
\end{aligned}$$

which is already close to (**). We have $|A(k)| \leq N^{k+1}$, as on page VII.5 of [7]. By Proposition 3 and Lemma 3 from the same reference, this yields the first inequality in

$$\mu_{|A(k)|}(X) \cdot |A(k)| \leq v_N(X)^{k+1} \cdot N^{k+1} \leq \frac{N^{k+1}}{N^{(k+1)/p'}} = N^{(k+1)/p}.$$

The second inequality and equality rely on $v_N(X) \leq 1/N^{1/p'}$ and $1 = (1/p) + (1/p')$. One can conclude

$$K_2 \cdot \sum_{k=0}^{\infty} \frac{\mu_{|A(k)|}(X) \cdot |A(k)|}{N^{k/q}} \leq K_2 \cdot \sum_{k=0}^{\infty} \frac{N^{1/p}}{N^{k \cdot (1/q - 1/p)}} = K_2 \cdot \frac{N^{1/q}}{N^{(1/q) - (1/p)} - 1} =: c_{N,q}.$$

We have thus established (**) with $c_{N,q}$ at the place of c_q . Under the assumptions of the theorem, we get the Rademacher property (*) for $C_q := c_{2,q}$, as seen above. Now we only need to note that $K_2 = 3$. \square

In the next section, we need the following consequence of the Rademacher property.

Proposition 3.2 ([36, Proposition 9.11]). *Assume that $(X, \|\cdot\|)$ is a Banach space of Rademacher type $q \in [1, 2]$ with constant C_q . Then, for every finite sequence X_1, \dots, X_n of independent mean zero Radon random variables in $L_q(X)$, one has*

$$\mathbb{E} \left(\left\| \sum_{i=1}^n X_i \right\|^q \right)^{1/q} \leq 2C_q \cdot \left(\sum_{i=1}^n \mathbb{E}(\|X_i\|^q) \right)^{1/q}. \quad (+)$$

Together with the previous results, the proposition yields c and q as in condition (9) of [3].

Remark 3.1. We only need the special case of Proposition 3.2 in which each of the X_i assumes value $(x_j - x)/q$ with probability λ_j , for a given convex combination $x = \sum \lambda_j x_j$. In this case, one can deduce the proposition by elementary manipulations of finite sums.

4. A RATE OF ASYMPTOTIC REGULARITY FOR ERGODIC AVERAGES IN UNIFORMLY CONVEX BANACH SPACES

For Hilbert spaces, there is an easy quadratic rate of asymptotic regularity for the sequence of Cesàro means, which is given (see, e.g., [40]), which was further improved and simplified in

[41] and [42]) by

$$\left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i x - T \left(\frac{1}{n} \sum_{i=0}^{n-1} T^i x \right) \right\| \leq \frac{1}{\sqrt{n}} \cdot \text{diam}(C) \quad \text{for } n \in \mathbb{N} \setminus \{0\}.$$

The asymptotic regularity for the Cesàro means in uniformly convex Banach spaces was first proved in [6] (see, however, also the earlier paper [43] of Reich). Although this proof by Bruck is essentially constructive, the concrete rate of convergence hidden in the proof is left implicit. In this section, we extract the explicit rate that is specified in the following theorem. Let us emphasize that q and C_q can be chosen according to Theorem 3.1, where we may take $\delta = \eta(1)$ due to Lemma 3.1. Thus our rate depends on η and b only, where η is a modulus of uniform convexity of the space in question and $b > 0$ is such that $C \subseteq B_{b/2}(0) := \{x \in X : \|x\| \leq b/2\}$. We point out that the superscript of ξ denotes iterations, which are explained by $\xi^0(t) := t$ and $\xi^{p+1}(t) := \xi(\xi^p(t))$.

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a Banach space that is uniformly convex with modulus η . Consider a nonexpansive map $T : C \rightarrow C$ on a nonempty subset $C \subseteq B_{b/2}(0)$ that is closed and convex. Assume that X is of Rademacher type $q \in (1, 2]$ with constant C_q . Given $\varepsilon > 0$, pick $\tilde{p} \in \mathbb{N}$ so large that we have $2C_q \cdot \tilde{p}^{(1-q)/q} \leq \varepsilon/(9b)$. Consider $p \in \mathbb{N}$ with $p \geq 2b/\delta^2$ for*

$$\delta := \xi^{\tilde{p}}\left(\frac{\varepsilon}{9}\right) \quad \text{with} \quad \xi(t) := \frac{t}{12} \cdot \eta\left(\min\left\{2, \frac{t}{b}\right\}\right).$$

For any $\alpha < \xi^{p-1}(\delta^2/2)$ with $0 < \alpha < \varepsilon/3$ and arbitrary $x \in C$, we then have

$$\left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i x - T \left(\frac{1}{n} \sum_{i=0}^{n-1} T^i x \right) \right\| \leq \varepsilon \quad \text{for all } n \geq \frac{b}{\alpha}.$$

The proof of the theorem will occupy us for the rest of this section. As noted before, we use a modulus $\eta : (0, 2] \rightarrow (0, 1]$ that satisfies the condition in Lemma 3.1 but need not be optimal. We set

$$\eta_1(0) := 0 \quad \text{and} \quad \eta_1(\varepsilon) := \sup \{ \eta(\varepsilon') \mid 0 < \varepsilon' \leq \min\{2, \varepsilon\} \}$$

to obtain a function $\eta_1 : [0, \infty) \rightarrow [0, 1]$ that is increasing. Now let $\tilde{\eta} : [0, \infty) \rightarrow [0, \infty)$ be given by

$$\tilde{\eta}(\varepsilon) := \frac{1}{2} \cdot \int_0^\varepsilon \eta_1(t) dt.$$

The point is that this makes $\tilde{\eta}$ convex. Given that $\varepsilon \in (0, 2]$ entails $0 < \tilde{\eta}(\varepsilon) \leq \eta_1(\varepsilon)$, we see that $\tilde{\eta}$ is still a modulus of uniform convexity for X . We now consider the function $\gamma : [0, \infty) \rightarrow [0, \infty)$ with

$$\gamma(\varepsilon) := \frac{b}{2} \cdot \tilde{\eta}\left(\frac{4\varepsilon}{b}\right).$$

Let us note that γ is continuous and strictly increasing with image $[0, \infty)$. We thus get a continuous and strictly increasing inverse $\gamma^{-1} : [0, \infty) \rightarrow [0, \infty)$. Furthermore, the function γ is convex. Our functions $\tilde{\eta}$ and γ coincide with η and γ_2 from Remark 2.3 and Definition 2.1 of [3], respectively (given that $\eta_1(t) \leq 1$ entails $\tilde{\eta}(\varepsilon) \leq \varepsilon/2$ and hence $\gamma(\varepsilon) \leq \varepsilon$). By [3, Lemma 2.2] (essentially a result of Bruck [5]), we learn that T is of type (γ) , i.e.,

$$\gamma(\|T(\lambda x_1 + (1-\lambda)x_2) - (\lambda T x_1 + (1-\lambda)T x_2)\|) \leq \|x_1 - x_2\| - \|T x_1 - T x_2\|$$

holds for any $x_1, x_2 \in C$ and all $\lambda \in [0, 1]$.

Lemma 4.1. For $\tilde{q} : [0, \infty) \rightarrow [0, \infty)$ with $\tilde{q}(\varepsilon) := \gamma^{-1}(3\varepsilon) + \varepsilon$ and for ξ as in Theorem 4.1, we have

$$t \in [0, \xi^p(\varepsilon)) \Rightarrow \tilde{q}^p(t) \in [0, \varepsilon).$$

Proof. We use induction to reduce to the case of $p = 1$. The claim is immediate for $p = 0$. In the induction step, the case of $p = 1$ (with $\xi^p(\varepsilon)$ at the place of ε) and the induction hypothesis yield

$$t \in [0, \xi^{p+1}(\varepsilon)) = [0, \xi(\xi^p(\varepsilon))) \Rightarrow \tilde{q}(t) \in [0, \xi^p(\varepsilon)) \Rightarrow \tilde{q}^{p+1}(t) = \tilde{q}^p(\tilde{q}(t)) \in [0, \varepsilon).$$

To establish the result for $p = 1$, we first observe that

$$\tilde{\eta}(\varepsilon) \geq \frac{1}{2} \cdot \frac{\varepsilon}{2} \cdot \eta_1\left(\frac{\varepsilon}{2}\right) \geq \frac{\varepsilon}{4} \cdot \eta\left(\min\left\{2, \frac{\varepsilon}{2}\right\}\right)$$

as η_1 is increasing. With $2\varepsilon/b$ at the place of ε , we have

$$\xi(\varepsilon) = \frac{\varepsilon}{12} \cdot \eta\left(\min\left\{2, \frac{\varepsilon}{b}\right\}\right) \leq \frac{b}{6} \cdot \tilde{\eta}\left(\frac{2\varepsilon}{b}\right) = \frac{1}{3} \cdot \gamma\left(\frac{\varepsilon}{2}\right).$$

Since \tilde{q} is strictly increasing, it follows that $t < \xi(\varepsilon)$ entails

$$\tilde{q}(t) < \tilde{q}\left(\frac{1}{3} \cdot \gamma\left(\frac{\varepsilon}{2}\right)\right) = \gamma^{-1}\left(\gamma\left(\frac{\varepsilon}{2}\right)\right) + \frac{1}{3} \cdot \gamma\left(\frac{\varepsilon}{2}\right) < \varepsilon.$$

Here the last inequality holds because we have $\gamma(t) \leq t$, as observed above. \square

The following is a final preparation for the proof of our main theorem.

Lemma 4.2. For $n \in \mathbb{N} \setminus \{0\}$, we consider $q_n : [0, \infty) \rightarrow [0, \infty)$ with $q_n(\varepsilon) := \gamma^{-1}(2\varepsilon + (b/n)) + \varepsilon$. Given $\varepsilon > 0$, we have

$$n \geq \frac{b}{\varepsilon} \Rightarrow q_n^p(\varepsilon) \leq \tilde{q}^p(\varepsilon) \text{ for all } p \in \mathbb{N}.$$

Proof. Since $n \geq b/\varepsilon$ entails $2\varepsilon + (b/n) \leq 3\varepsilon$, the result for $p = 1$ follows from the fact that γ^{-1} is increasing. We derive the general case by induction on p . As before, the case of $p = 0$ is immediate. In the induction step, the induction hypothesis and the fact that q_n is increasing yield $q_n^{p+1}(\varepsilon) = q_n(q_n^p(\varepsilon)) \leq q_n(\tilde{q}^p(\varepsilon))$. Since \tilde{q} is increasing with $\varepsilon \leq \tilde{q}(\varepsilon)$, we get $\varepsilon \leq \tilde{q}^p(\varepsilon)$ by an auxiliary induction on p . Given $n \geq b/\varepsilon$, we thus have $n \geq b/\tilde{q}^p(\varepsilon)$. Now the result for $p = 1$ (with $\tilde{q}^p(\varepsilon)$ at the place of ε) yields

$$q_n(\tilde{q}^p(\varepsilon)) \leq \tilde{q}(\tilde{q}^p(\varepsilon)) = \tilde{q}^{p+1}(\varepsilon),$$

as needed to complete the induction step. \square

We now have all ingredients to show the result that was stated above:

Proof of Theorem 4.1. Let $F_\delta(T) = \{x \in C : \|x - Tx\| \leq \delta\}$ be the set of δ -approximate fixed points of T . For $S \subseteq X$, we write

$$\text{co}_p(S) := \left\{ \sum_{i=1}^n \lambda_i x_i \mid x_i \in S \text{ and } \lambda_i \geq 0 \text{ with } \sum_{i=1}^n \lambda_i = 1 \text{ for } n \leq p \right\}$$

for the set of convex combinations of at most p elements. By $\text{co}(S) := \bigcup_{p \in \mathbb{N}} \text{co}_p(S)$ we denote the convex hull. For arbitrary $\varepsilon > 0$, we will show that the δ that is specified in Theorem 4.1 validates

$$\text{co}(F_\delta(T)) \subseteq F_{\varepsilon/3}(T). \quad (++)$$

Before we prove this, we show how to deduce the theorem. First note that the closure of $\text{co}(F_\delta(T))$ will still be contained in $F_{\varepsilon/3}(T)$, as the latter is closed. As in Theorem 4.1, we assume $p \geq 2b/\delta^2$ and $\alpha < \xi^{p-1}(\delta^2/2)$. For \tilde{q} and q_n as above, Lemmas 4.1 and 4.2 yield

$$q_n^{p-1}(\alpha) \leq \tilde{q}^{p-1}(\alpha) < \frac{\delta^2}{2} \quad \text{for all } n \geq \frac{b}{\alpha}.$$

For $x_n := T^n x$, we clearly have $\|x_{n+1} - Tx_n\| = 0 \leq \alpha$. We have established all properties that are used in the proof of Theorem 1.3 from [6], which shows that any $x \in C$ validates

$$\frac{1}{n} \cdot \sum_{i=0}^{n-1} T^i x \in F_\varepsilon(T) \quad \text{for all } n \geq \frac{b}{\alpha}.$$

This coincides with the conclusion of the desired Theorem 4.1. It remains to show that the δ from the theorem satisfies $(++)$. Due to the assumption that X has Rademacher type q with constant C_q , we can apply Proposition 3.2. As noted above, this yields condition (9) of [3], with X and $2C_q$ at the place of X^2 and c . Essentially by [3, Lemma 2.6] (based on the proof of [6, Theorem 1.1]), we have $\text{co}(M) \subseteq \text{co}_{\tilde{p}}(M) + B_{\varepsilon/9}$ for any $M \subseteq B_{b/2}$, provided that we have $2C_q \cdot \tilde{p}^{(1-q)/q} \leq \varepsilon/(9b)$ as in Theorem 4.1. The function \tilde{q} is continuous and strictly increasing with image $[0, \infty)$, as the same holds for γ^{-1} . Let $\sigma = \tilde{q}^{-1} : [0, \infty) \rightarrow [0, \infty)$ be its inverse. As in the proof of Theorem 1.2 in [6], one can show

$$\delta \leq \sigma^{\tilde{p}}(\varepsilon/9) \quad \Rightarrow \quad \text{co}(F_\delta(T)) \subseteq F_{\varepsilon/3}(T).$$

To obtain $(++)$ for $\delta = \xi^{\tilde{p}}(\varepsilon/9)$ as in Theorem 4.1, we show $\xi^i(t) \leq \sigma^i(t)$ by induction on $i \in \mathbb{N}$. Let us first note that Lemma 4.1 yields $\tilde{q}(\xi(s)) \leq s$, as \tilde{q} is continuous. Since $\sigma = \tilde{q}^{-1}$ is increasing, we can conclude $\xi(s) \leq \sigma(s)$. Given $s := \xi^i(t) \leq \sigma^i(t)$, we thus obtain

$$\xi^{i+1}(t) = \xi(\xi^i(t)) \leq \sigma(\xi^i(t)) \leq \sigma(\sigma^i(t)) = \sigma^{i+1}(t),$$

as needed for the induction step. This completes the proof. \square

As promised in the introduction, we now discuss an alternative rate of the asymptotic regularity.

Remark 4.1. As shown by Bruck (see [6, Theorem 2.1]), the above function γ can be transformed into a continuous, strictly increasing, and convex $\tilde{\gamma} : [0, \infty) \rightarrow [0, \infty)$ with $\tilde{\gamma}(0) = 0$ such that

$$\tilde{\gamma} \left(\left\| T \left(\sum_{i=1}^n \lambda_i x_i \right) - \sum_{i=1}^n \lambda_i T x_i \right\| \right) \leq \max_{1 \leq i, j \leq n} (\|x_i - x_j\| - \|T x_i - T x_j\|)$$

holds for any convex combination $\sum \lambda_i x_i$. By a result of Zhu, Huang and Li, the conclusion of our Theorem 4.1 holds for all n above a certain n_0 that depends on such a $\tilde{\gamma}$ (see Lemma 3.5 of [8], which works in a more general semigroup setting). In Section 2 of [3], we have proved how $\tilde{\gamma}$ can be expressed in terms of b, c, q , and η , assuming that X^2 has Rademacher type q with constant c (but note that the $\tilde{\gamma}$ in [3] is not made convex). Now Section 3 of the present paper demonstrates how to express c and q in terms of η . This makes it possible to express the rate of

Zhu, Huang and Li in terms of η and b . A very rough comparison suggests that our rate from Theorem 4.1 is better, as it involves fewer iterations of η when q is close to 1. However, we have not established a precise comparison between the two rates.

To conclude, we observe how certain assumptions can be weakened.

Remark 4.2. First, note that the proof above involves $x_n = T^n x$ with $\|x_{n+1} - Tx_n\| = 0$. The argument does also go through for different x_n that satisfy $\|x_{n+1} - Tx_n\| \leq \alpha$ for all n . Secondly, the assumption that $C \subseteq B_{b/2}(0)$ is bounded can be secured when $T : C' \rightarrow C'$ is defined on an unbounded set C' (still assumed to be closed and convex) and has a fixed point $f = T(f)$, as pointed out in [3]. Indeed, for given $x \in C'$, we can then consider

$$C := C' \cap B_{\|x-f\|}(f) \subseteq B_{\|x-f\|+\|f\|}(0),$$

which is closed and convex with $T(C) \subseteq C \ni x$. In fact, it is enough to have a $d > 0$ such that T has arbitrarily good approximate fixed points in $B_d(0)$, i.e., such that $B_d(0) \cap F_\varepsilon(T) \neq \emptyset$ holds for all $\varepsilon > 0$. Indeed, for any $f \in F_\varepsilon(T)$, we inductively obtain

$$\|T^n f - f\| \leq \|T^n f - T^{n-1} f\| + \|T^{n-1} f - f\| \leq \|Tf - f\| + (n-1) \cdot \varepsilon \leq n \cdot \varepsilon.$$

Given $n \in \mathbb{N} \setminus \{0\}$ and for arbitrary $\varepsilon > 0$, we now pick $f \in B_d(0) \cap F_{\varepsilon/n}(0)$ to obtain

$$\|T^n x\| \leq \|T^n x - T^n f\| + \|T^n f - f\| + \|f\| \leq \|x - f\| + \varepsilon + \|f\| \leq \|x\| + 2 \cdot d + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we may now omit it, for each $n \in \mathbb{N}$. This allows us to consider

$$R := \limsup_{n \rightarrow \infty} \|x - T^n x\| \leq 2 \cdot (\|x\| + d).$$

As in the proof of [44, Theorem 1] (for the simple case where (a_{nk}) is the identity matrix), the set

$$C := \left\{ y \in C' \mid \limsup_{n \rightarrow \infty} \|y - T^n x\| \leq R \right\} \subseteq B_{R+\|x\|+2d}(0)$$

is closed and convex with $T(C) \subseteq C \ni x$. Note that the Browder-Göhde-Kirk fixed point theorem will now yield an actual fixed point in C . The fact that approximate fixed points can play the role of actual ones is also guaranteed by logical metatheorems from proof mining (see Corollaries 5.2 and 6.8 as well as the comment following Remark 5.9 of [45]).

Acknowledgment

Both authors were supported by the ‘Deutsche Forschungsgemeinschaft’ (DFG, German Research Foundation) – Projects 460597863 and DFG KO 1737/6-2.

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