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A TOTALLY RELAXED SELF-ADAPTIVE ALGORITHM FOR SOLVING VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS IN BANACH SPACES

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Abstract. Using the Halpern iterative method, we propose and study a totally relaxed iterative algorithm for approximating a common solution to variational inequality and fixed point problems in certain Banach space. Our algorithm uses a self-adaptive step size to avoid the dependence on the Lipschitz constant of the operator involved. Our method can also find fixed points of Bregman firmly nonexpansive mappings. We establish a strong convergence theorem and present some numerical experiments to illustrate the performance of our algorithm.

Keywords. Pseudomonotone operator; Reflexive Banach space; Self-adaptive step; Totally relaxed; Variational inequality.

1. Introduction

Let C be a nonempty, closed, and convex subset of a real Banach space E with dual E^*. We are concerned with the following Variational Inequality Problem (VIP):

Find a point $u \in C$ such that

$$\langle F(u), v-u \rangle \geq 0, \quad \forall\, v \in C, \tag{1.1}$$

where $F : C \to E^*$ is a given nonlinear mapping, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. We denote by $VIP(C,F)$ the solution set of the VIP (1.1). The VIP is now under the spotlight due to its applications in a large variety of fields, such as structural analysis, economics, and engineering sciences; see, e.g., [1, 2, 3, 4]). Recent progress on the theory and applications of the VIP can be found in [5, 6, 7, 8] and the references therein. Iterative approximations of solutions to the VIP under several monotonicity assumptions have been considered recently. These iterative algorithms date back to the celebrated gradient projection method [9]. The extragradient method due to Korpelevich (see [10]) was proposed for the VIP in finite dimensional spaces. It was further extended to Hilbert space by Nadezhkina and Takahashi (see [11]).

It is known that the extragradient method ([10]) requires calculating metric projections onto its feasible set twice per iteration. This fact was identified as a drawback of the method as

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projections onto general closed and convex subsets may be difficult to calculate especially when the structure of the set is not simple. To overcome this, Censor et al. [12] introduced the subgradient extragradient method in which the second projection onto its feasible set is replaced by a projection onto a well-constructed halfspace, which is easily defined. The subgradient extragradient was further improved by He et al. [13] (see also [14]) with the first projection replaced by a projection onto a certain half-space; see Fukushima [15]. The methods presented in [13] and [14] are now called the relaxed and totally relaxed methods, respectively. At this point, we note that reducing the number of projections in an iterative algorithm for finding a solution to the VIP plays a major role in attaining efficient execution and convergence rate of the method. For this reason, the Tseng extragradient method [16], initially introduced for solving variational inclusion problems, has been adopted for the solutions of VIPs. This method, which only uses one projection onto its feasible set, has been widely studied by authors; see, e.g., [17, 18] and the references therein. Recently, Oyewole et al. [17] introduced a totally relaxed, Tseng self-adaptive method for the solutions of VIPs in Banach spaces. They established a strong convergence theorem under some mild assumptions on the control parameters in the framework of 2-uniformly convex and uniformly smooth Banach spaces.

Recall that the Fixed point Problem (FPP) consists of finding a point $u \in C$ such that

$$u = Tu, \tag{1.2}$$

where $T : C \to C$ is a given nonlinear mapping. We denote by $F(T)$ the set of solutions to the FPP. The theory and solution methods regarding this problem when the underlying operator is Bregman firmly nonexpansive can be found in recent papers [19, 20, 21, 22].

In this paper, our aim is to find common solutions of the VIP (1.1) and the FPP (1.2), which consists of finding a point $u \in C$ such that $u \in VIP(C,F) \cap F(T)$. The study of such a common solution problem is important because it arises in many interesting applications and real life problems. We also recall that, to speed up the convergence of iterative schemes in optimization theory, many researchers often use inertial-type extrapolation [23, 24] by supplementing their methods with the term $\theta_n(x_n - x_{n-1})$. The parameter θ_n controls the momentum $x_n - x_{n-1}$ and is called the inertial parameter. For more information, see, for example, [17, 25]).

Motivated by the results of He et al. [14], Oyewole et al. [17] and the inertial technique [23, 24], in the present paper we propose an inertial, totally relaxed and self-adaptive Tseng method for approximating a common solution of fixed point and variational inequality problems. Assuming some mild conditions on the control parameters, we prove a strong convergence theorem in the framework of uniformly convex and uniformly smooth Banach spaces. A self-adaptive step size (see, for example, [17, 18]) is incorporated into the algorithm in order to avoid its dependence on the Lipschitz constant of the cost operator.

2. PRELIMINARIES

In this section, we recall some basic definitions and preliminary results which are useful in the convergence analysis of our algorithm. Let E be a Banach space, and let $f : E \to \mathbb{R} \cup \{\infty\}$ be a function. We denote by $\mathrm{dom} f$ the domain of f, that is, the set $\{x \in E : f(x) < +\infty\}$. We also denote the weak and strong convergence of a sequence $\{x_n\} \subset E$ to a point $x \in E$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Let E be a real Banach space. Given a function $h : E \to \mathbb{R}$,

(i) h is called Gâteaux differentiable at $x \in E$ if there exists an element of E, denoted by $h'(x)$ or $\nabla h(x)$, such that

$$\lim_{t \to 0^+} \frac{h(x+ty)-h(x)}{t} = \langle y, h'(x) \rangle, \quad y \in E,$$

where $h'(x)$ or $\nabla h(x)$ is called the Gâteaux differential or gradient of h at x. We say that h is Gâteaux differentiable on E if h is Gâteaux differentiable at every $x \in E$;

(ii) h is called weakly lower semicontinuous at $x \in E$ if $x_n \rightharpoonup x$ implies that $h(x) \leq \liminf_{n \to \infty} h(x_n)$. We say that h is weakly lower semicontinuous on E if h is weakly lower semicontinuous at every $x \in E$;

(iii) if h is a convex function, then it is said to be subdifferentiable at a point $x \in E$ if the set

$$\partial h(x) = \{w \in E^* : h(y) - h(x) \geq \langle w, y-x \rangle, \ y \in E\} \tag{2.1}$$

is nonempty. Each element in $\partial h(x)$ is called a subgradient of h at x or a subdifferential of h at x and the inequality in (2.1) is called the subdifferential inequality of h at x. The function h is subdifferentiable on E if h is subdifferentiable at each $x \in E$. It is known that if h is Gâteaux differentiable at x, then h is subdifferentiable at x and $\partial h(x) = \{h'(x)\}$, that is, $\partial h(x)$ is just a singleton set. For more details on Gâteaux differentiable functions on Banach spaces, we refer to [26].

Let E be a real Banach space with norm $\|\cdot\|$, and let E^* denote the dual space of E with dual norm $\|\cdot\|_*$. Let $1 \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of smoothness of E is given by

$$\rho_{E(t)} = \sup\left\{\frac{\|x-y\| + \|x+y\|}{2} - 1 : \|x\| = 1, \|y\| = t\right\}.$$

The space E is said to be uniformly smooth if $\lim_{t \to 0^+} \frac{\rho_E(t)}{t} = 0$ and q-uniformly smooth if there exists a constant $c_q > 0$ such that $\rho_E(t) \leq c_q t^q$ for any $t > 0$. The modulus of convexity of E, denoted δ_E, is defined by

$$\delta_E(\varepsilon) := \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \ \|x-y\| \geq \varepsilon\right\}.$$

E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for any $\varepsilon(0,2]$ and p-uniformly convex if there is a constant $\tau_p > 0$ such that $\delta_E(\varepsilon) \geq \tau_p \varepsilon^p$ for any $\varepsilon \in (0,2]$. For example, the L_p spaces are 2-uniformly convex for $1 \leq p \leq 2$ and p-uniformly convex if and only if its dual E^* is q-unifomly smooth (see [27]).

Let $p > 1$. The generalized duality mapping $J_E^p : E \to 2^{E^*}$ is defined as

$$J_E^p(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}.$$

J_E^p is one-to-one, single-valued, and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the generalized duality mapping of E^*. For more properties of duality mappings, see [28, 29, 30].

Let $f : E \to \mathbb{R}$ be a differentiable and convex function. Recall that the Bregman distance $\Delta_f : dom f \times int dom f \to [0, +\infty)$ is defined by

$$\Delta_f(x,y) := f(y) - f(x) - \langle \nabla f(x), y-x \rangle, \ \forall x, y \in E, \tag{2.2}$$

where $\nabla f(x)$ is the value of the gradient of f at x. It is known that the duality maping J_E^p is the gradient of the function $f_p(\cdot) = \frac{\|\cdot\|^p}{p}$, for $2 \le p < \infty$. Hence, if $f = f_p$ in (2.2), then the Bregman distance with respect to f_p becomes

$$\begin{aligned}\Delta_p(x,y) &= \frac{\|y\|^p}{p} - \frac{\|x\|^p}{p} - \langle J_E^p x, y-x\rangle \\ &= \frac{1}{p}(\|y\|^p - \|x\|^p) + \langle J_E^p x, x-y\rangle \\ &= \frac{1}{q}(\|x\|^p - \|y\|^p) - \langle J_E^p x - J_E^p y, y\rangle.\end{aligned}$$

Lemma 2.1. [31] *Let $x, y \in E$. If E is a q-uniformly smooth Banach space, then there exists $c_q > 0$ such that $\|x-y\|^q \le \|x\|^q - q\langle x, J_E^q y\rangle + c_q\|y\|^q$.*

Let $x,y,z \in E$. For $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\Delta_p(x,y) = \Delta_p(x,z) + \Delta_p(z,y) + \langle x-z, J_E^p z - J_E^p y\rangle, \tag{2.3}$$

$$\Delta_p(x,y) = -\Delta_p(y,x) + \langle y-x, J_E^p y - J_E^p x\rangle, \tag{2.4}$$

and

$$\Delta_p(x,y) = \frac{\|x\|^p}{p} + \frac{\|y\|^q}{p} - \langle x, J_E^p y\rangle.$$

Proposition 2.1. [32] *Let E be a p-uniformly convex Banach space. For $x, y \in E$, there exists a constant $\tau_p > 0$ such that $\tau_p\|x-y\|^p \le \Delta_p(x,y) \le \langle x-y, J_E^p x - J_E^p y\rangle$.*

Proposition 2.2. [33] *Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that $\Delta_p(x_n,y_n) \to 0$ as $n \to \infty$. If $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Let C be a nonempty, closed, and convex subset of a Banach space E. Then the metric projection $P_C x = \arg\min_{y\in C}\|x-y\|$, $\forall x \in E$ is the unique minimizer of the norm distance. It is characterized by the variational inequality $\langle J_E^p(x - P_C x), z - P_C x\rangle \le 0$, $\forall z \in C$. In the same vein, the Bregman projection $\Pi_C x = \arg\min_{y\in C}\Delta_p(x,y)$, $\forall x \in E$ can be defined as the unique minimizer of the Bregman distance [34, 35]. The Bregman projection is characterized by the following variational inequality:

$$\langle J_E^p x - J_E^p \Pi_C x, z - \Pi_C x\rangle \le 0, \ \ \forall\, z \in C. \tag{2.5}$$

It is known that it also satisfies $\Delta_p(\Pi_C x, z) \le \Delta_p(x,z) - \Delta_p(x, \Pi_C x)$, $\forall z \in E$. Let E be a strictly convex, smooth, and reflexive Banach space. We consider the function $V_p : E^* \times E \to [0,+\infty)$ defined by

$$V_p(x,x^*) = \frac{\|x^*\|^q}{q} - \langle x^*, x\rangle + \frac{\|x\|^p}{p} \ \ \forall\, x \in E,\ x^* \in E^*,$$

with $\frac{1}{q} + \frac{1}{p} = 1$. The mapping V_p is nonnegative and $V_p(x,x^*) = \Delta_p(x, J_{E^*}^q(x^*))$ for all $x \in E$ and $x^* \in E^*$. It follows (see [36]) from the subdifferential inequality for $f(x) = \frac{\|x\|^q}{q}$, $x \in E^*$, that

$$\langle J_{E^*}^q x, y\rangle \le \frac{\|x+y\|^q}{q} - \frac{\|x\|^q}{q},\ \forall\, x,y \in E^*$$

from which we infer that $V_p(x,x^*)+\langle y^*, J^q_{E^*}(x^*)-x\rangle \le V_p(x,x^*+y^*)$ for all $x\in E$ and $x^*,y^*\in E^*$. Furthermore, the mapping V_p is convex in the second variable. Thus, for all $z\in E$, $\{x_i\}_{i=1}^N$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$,

$$\Delta_p\left(z, J^q_{E^*}\left(\sum_{i=1}^N t_i J^p_E x_i\right)\right) = V_p\left(z,\left(\sum_{i=1}^N t_i J^p_E x_i\right)\right) \le \sum_{i=1}^N t_i \Delta_p(z,x_i).$$

Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and let $T : C\to C$ be a mapping. A point $x^*\in C$ is called an asymptotic fixed point of T if there exists a sequence $\{x_n\}\subset C$ such that $x_n \rightharpoonup x^*$ and $\|x_n - Tx_n\|\to 0$ as $n\to\infty$. We denote by $\hat{F}(T)$ the asymptotic fixed point set of T. A mapping $T : C\to C$ is said to be Bregman firmly nonexpansive if

$$\langle J^p_E Tx - J^p_E Ty, Tx-Ty\rangle \le \langle J^p_E Tx - J^p_E Ty, x-y\rangle$$

for all $x,y\in C$. It is clear from the definition of the Bregman distance that the above definition is equivalent to

$$\Delta_p(Tx,Ty)+\Delta_p(Ty,Tx)+\Delta_p(Tx,x)+\Delta_p(Ty,y)\le \Delta_p(Ty,x)+\Delta_p(Tx,y).$$

We know from [37, 38] that, for every Bregman firmly nonexpansive mapping, $\hat{F}(T)=F(T)$.

Definition 2.1. (see [39, 40]). Let $F : C\to E^*$ be an operator. The Minty Variational Inequalities (MVI) consists of finding a point $x^*\in C$ such that $\langle F(y), y-x^*\rangle \ge 0$, $\forall y\in C$. We denote by $M(C,F)$ the set of solutions of the MVI. Some existence results for the MVI were presented in [39]. The assumption that $M(C,F)\ne\emptyset$ was used for solving VIP(1.1) in finite dimensional spaces (see, for example, [41]). It is not difficult to prove that pseudomonotonicity implies that $M(C,F)\ne\emptyset$, but that the converse is not true in general.

Lemma 2.2. [40] *Consider the VIP (1.1). Suppose that the mapping $h : [0,1]\to E^*$ defined by $h(t) = F(tx+(1-t)y)$ for $x,y\in C$ and $t\in[0,1]$ is continuous for each x and y (that is, F is hemicontinuous). Then $M(C,F)\subset VIP(C,F)$. Moreover, if F is pseudomonotone, then $VIP(C,F)$ is closed, convex, and $M(C,F) = VIP(C,F)$.*

Lemma 2.3. [42] *Let $\{a_n\}$ be a sequence of positive real numbers, $\{b_n\}\subset\mathbb{R}$, and $\{c_n\}\subset(0,1)$ such that $\sum_{n=1}^\infty c_n=\infty$ and $a_{n+1}\le(1-c_n)a_n+c_nb_n$, $\forall n\in\mathbb{N}$. If $\limsup_{k\to\infty} b_{n_k}\le 0$ for each subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1}-a_{n_k})\ge 0$, then $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

In this section, we introduce our proposed method and present its convergence analysis. First, we make the following assumptions.

Assumption 3.1.

(A1) The feasible set C is nonempty, closed and convex.
(A2) The mapping $F : E\to E^*$ is pseudomonotone on E, L-Lipschitz continuous on E, and weakly sequentially continuous on C. (However, our proposed method does not require L to be known).
(A3) The solution set $\Gamma = VIP(C,F)\cap F(T)$ is nonempty.
(A4) The feasible set C is defined by $C := \cap_{i=1}^m C^i$, where $C^i := \{z\in E : h_i(z)\le 0\}$.

In addition, we assume that $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n = o(\alpha_n)$, (that is, $\lim_{n\to\infty}\frac{\varepsilon_n}{\alpha_n} = 0$), where $\{\alpha_n\} \subset (0,1)$ satisfies

(A5) $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\alpha_n = 0$.

We study the convergence of the following Tseng-type extragradient method.

Algorithm 3.1. Tseng-type extragradient method for VIP

Initialization: Choose $\mu \in \left(0, \frac{q\tau_p\lambda_{n+1}^q}{\lambda_n^q c_q}\right)^{\frac{1}{q}}$ and $\theta > 0$. Select initial points $w, x_0,\ x_1 \in C$, $\lambda_0 > 0$ and set the counter $n := 1$. For $i = 1,2\cdots,m$ and given the current iterate w_n, construct the family of half spaces $C_n^i := \{z \in E : h_i(w_n) + \langle h_i'(w_n), z - w_n\rangle \le 0\}$ and set $C_n = \cap_{i=1}^m C_n^i$.

Iterative step: Calculate x_{n+1} and λ_{n+1} as follows:

Step 1: Given $x_{n-1},\ x_n$ and λ_n, for each $n \ge 1$, choose θ_n such that $\theta_n \in [0,\bar{\theta}_n]$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \ne x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2: Compute

$$\begin{cases} w_n = J_{E^*}^q(J_E^p x_n + \theta_n(J_E^p x_{n-1} - J_E^p x_n)), \\ y_n = \Pi_{C_n} J_{E^*}^q(J_E^p w_n - \lambda_n F(w_n)), \\ \lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \dfrac{\mu\|y_n - w_n\|}{\|F(y_n) - F(w_n)\|}\right\}, & \text{if } F(y_n) \ne F(w_n), \\ \lambda_n, & \text{otherwise,} \end{cases} \\ z_n = J_{E^*}^q(J_E^p y_n - \lambda_n(F(y_n) - F(w_n))), \\ x_{n+1} = J_{E^*}^q(\alpha_n J_E^p w + (1-\alpha_n)J_E^p T z_n). \end{cases} \tag{3.2}$$

Stopping criterion: If $x_{n+1} = w_n = y_n$ and $z_n = Tz_n$ for some $n \ge 1$, then stop. Otherwise, set $n := n+1$ and return to **Iterative step**.

Lemma 3.1. [43] *The sequence $\{\lambda_n\}$ generated by (3.1) is a monotonically decreasing sequence and $\lim_{n\to\infty}\lambda_n = \lambda \ge \min\left\{\frac{\mu}{L}, \lambda_0\right\}$.*

Proof. This is a consequence of the fact that F is assumed to be L-Lipschitz. □

Remark 3.1. From the definition of C and C_n, it is not difficult to see that $C \subset C_n$. Indeed, for each $i = 1,2,\cdots,m$, and $x \in C^i$, the subdifferential inequality yields

$$h_i(w_n) + \langle h_i'(w_n), x - w_n\rangle \le h_i(x) \le 0.$$

It follows from the definition of C_n^i that $x \in C_n^i$. Hence $C^i \subset C_n^i$ for all i and hence $C \subset C_n$ for all $n \ge 1$.

Proposition 3.1. *Assume that the sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded. Then, for any $u \in \Gamma$, $\lim_{n\to\infty}\theta_n(\Delta_p(u,x_{n-1}) - \Delta_p(u,x_n)) = 0$.*

Proof. It follows from (3.1) that $\theta_n\|x_n - x_{n-1}\| \le \varepsilon_n$ for each $n \ge 1$. From the fact that $\lim_{n\to\infty}\frac{\varepsilon_n}{\alpha_n} = 0$, this implies that

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le \lim_{n\to\infty}\frac{\varepsilon_n}{\alpha_n} = 0. \tag{3.3}$$

Now, for some $M > 0$, we have

$$\begin{aligned}\Delta_p(u,x_{n-1}) - \Delta_p(u,x_n) &= \frac{\|u\|^p}{p} + \frac{\|x_{n-1}\|^p}{q} - \langle u, J_E^p x_{n-1}\rangle - \left(\frac{\|u\|^p}{p} + \frac{\|x_n\|^p}{q} - \langle u, J_E^p x_n\rangle\right)\\ &= \frac{1}{q}(\|x_{n-1}\|^p - \|x_n\|^p) + \langle u, J_E^p x_n - J_E^p x_{n-1}\rangle\\ &\le \frac{1}{q}(M\|x_n - x_{n-1}\|) + \|u\| \cdot \|J_E^p x_n - J_E^p x_{n-1}\|.\end{aligned}$$

Since J_E^p is uniformly continuous on bounded subsets of E, it follows from (3.3) that

$$\lim_{n\to\infty} \alpha_n \cdot \frac{\theta_n}{\alpha_n}\|J_E^p x_n - J_E^p x_{n-1}\| = \lim_{n\to\infty} \alpha_n \cdot \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$$

Thus $\lim_{n\to\infty} \theta_n(\Delta_p(u,x_{n-1}) - \Delta_p(u,x_n)) = 0$, as asserted. □

Lemma 3.2. *Suppose that Assumption 3.1 holds and that $\{w_n\}$ is a sequence generated by Algorithm 3.1. Let $\{w_{n_k}\}$ be a subsequence of $\{w_n\}$ which converges weakly to $v \in E$. If $\lim_{k\to\infty}\|w_{n_k} - y_{n_k}\| = 0$, then $v \in VIP(C,F)$.*

Proof. Using the definition of $\{y_n\}$ and (2.5), we have $\langle J_E^p w_{n_k} - \lambda_{n_k} F(w_{n_k}) - J_E^p y_{n_k}, y - y_{n_k}\rangle \le 0$, $\forall y \in C_{n_k}$, which is equivalent to

$$\frac{1}{\lambda_{n_k}}\langle J_E^p w_{n_k} - J_E^p y_{n_k}, y - y_{n_k}\rangle \le \langle F(w_{n_k}), y - y_{n_k}\rangle,\ \forall\, y \in C_{n_k}.$$

It follows that

$$\frac{1}{\lambda_{n_k}}\langle J_E^p w_{n_k} - J_E^p y_{n_k}, y - y_{n_k}\rangle + \langle F(w_{n_k}), y_{n_k} - w_{n_k}\rangle \le \langle F(w_{n_k}), y - w_{n_k}\rangle,\ \forall\, y \in C_{n_k}. \tag{3.4}$$

Since $\|w_{n_k} - y_{n_k}\| \to 0$ as $k \to \infty$ and J_E^p is norm-to-norm uniformly continuous on bounded subsets of E, we obtain $\|J_E^p w_{n_k} - J_E^p y_{n_k}\| \to 0$. Taking the limit as $k \to \infty$ in (3.4), we obtain $\liminf_{k\to\infty}\langle F(w_{n_k}), y - w_{n_k}\rangle \ge 0$, $\forall y \in C_{n_k}$. Using this and the facts that $w_{n_k} \in C$ and $C \subset C_{n_k}$, we find that $\liminf_{k\to\infty}\langle F(w_{n_k}), y - w_{n_k}\rangle \ge 0$, $\forall y \in C$.

Next, we show that $v \in C$. Indeed, it follows from $y_{n_k} \in C_{n_k}$ that $h_i(w_{n_k}) + \langle h_i'(w_{n_k}), y_{n_k} - w_{n_k}\rangle \le 0$, which implies that

$$h_i(w_{n_k}) \le \langle h_i'(w_{n_k}), w_{n_k} - y_{n_k}\rangle \le \|h_i'(w_{n_k})\| \cdot \|w_{n_k} - y_{n_k}\|.$$

Since h_i' is Lipschitz continuous and $\{w_{n_k}\}$ is bounded, it follows that $\{h_i'(w_{n_k})\}$ is bounded too. Thus there exists $K_i > 0$ such that $\|h_i'(w_{n_k})\| \le K_i$ for each i. Hence, $h_i(w_{n_k}) \le K \cdot \|w_{n_k} - y_{n_k}\|$, where $K = \max_{1\le i\le m}\{K_i\}$. Using the weak continuity of h_i, we have

$$h_i(v) \le \liminf_{k\to\infty} h_i(w_{n_k}) \le \lim_{k\to\infty} K \cdot \|w_{n_k} - y_{n_k}\| = 0.$$

Thus $v \in C$.

Now choose a sequence $\{\varepsilon_k\}$ of positive numbers such that $\{\varepsilon_k\}$ is decreasing and $\varepsilon_k \to 0$ as $k \to \infty$. For each $k \ge 1$, denote by N_{n_k} the smallest positive integer such that

$$\langle F(w_{n_k}), y - w_{n_k}\rangle + \varepsilon_k \ge 0,\ \forall\, k \in N_{n_k}. \tag{3.5}$$

Observe that $\{N_{n_k}\}$ is increasing because $\{\varepsilon_k\}$ is decreasing. Let the point $A_{N_{n_k}} \in E$ satisfy $\langle A_{N_{n_k}}, F(w_{N_{n_k}}\rangle = 1$. Then we can write (3.5), for each $k \geq 1$, as follows $\langle F(w_{N_{n_k}}), y + \varepsilon_k A_{N_{n_k}} - w_{N_{n_k}}\rangle \geq 0$. Now, since the operator F is pseudomonotone, we have

$$\langle F(y + \varepsilon_k A_{N_{n_k}}), y + \varepsilon_k A_{N_{n_k}} - w_{N_{n_k}}\rangle \geq 0.$$

Therefore,

$$\langle F(y), y - w_{N_{n_k}}\rangle \geq \langle F(y) - F(y + \varepsilon_k A_{N_{n_k}}), y + \varepsilon_k A_{N_{n_k}} - w_{N_{n_k}}\rangle - \varepsilon_k \langle F(y), A_{N_{n_k}}\rangle. \tag{3.6}$$

Next, we show that $\lim_{k\to\infty} \varepsilon_k A_{N_{n_k}} = 0$. Since $w_{n_k} \rightharpoonup v \in E$ and F is weakly sequentially continuous on C, we have that $\{F(w_{N_{n_k}})\}$ converges weakly to $F(v)$. We may suppose that $F(v) \neq 0$ otherwise $v \in VIP(C,F)$. Since the norm $\|\cdot\|$ is sequentially weakly lower semicontinuous, we have $\|F(v)\| \leq \liminf_{k\to\infty} \|F(w_{n_k})\|$. Since $w_{N_{n_k}} \subset w_{n_k}$ and $\varepsilon_k \to 0$ as $k \to \infty$, we have

$$0 \leq \limsup_{k\to\infty} \|\varepsilon_k A_{N_{n_k}}\| = \limsup_{k\to\infty} \frac{\varepsilon_k}{\|F(w_{n_k})\|} \leq \frac{\limsup \varepsilon_k}{\liminf_{k\to\infty} \|F(w_{n_k})\|} = 0$$

and $\lim_{k\to\infty} \varepsilon_k A_{N_{n_k}} = 0$. It follows from (3.6) that $\liminf_{k\to\infty}\langle F(y), y - w_{N_{n_k}}\rangle \geq 0$. Hence, for all $y \in C$,

$$\langle F(y), y - v\rangle = \lim_{k\to\infty} \langle F(y), y - w_{N_{n_k}}\rangle = \liminf_{k\to\infty} \langle F(y), y - w_{N_{n_k}}\rangle \geq 0.$$

Therefore, using Lemma 2.2, we conclude that $v \in VIP(C,F)$, as asserted. □

Lemma 3.3. *Let the sequence $\{x_n\}$ be iteratively defined by Algorithm 3.1. Then*

$$\Delta_p(u, z_n) \leq \Delta(p, w_n) - \left(1 - \frac{c_q \lambda_n^q \mu^q}{q \tau_p \lambda_{n+1}^q}\right) \Delta(y_n, w_n).$$

Proof. Fix a point $u \in VIP(C,F)$. Then it follows from (3.2) that

$$\begin{aligned} \Delta_p(u, z_n) &= \Delta_p(u, J_{E^*}^q(J_E^p y_n - \lambda_n(F(y_n) - F(w_n)))) \\ &= \frac{1}{p}\|u\|^p - \langle u, J_E^p y_n - \lambda_n(F(y_n) - F(w_n))\rangle + \frac{1}{q}\|J_E^p y_n - \lambda_n(F(y_n) - F(w_n))\|^q. \end{aligned} \tag{3.7}$$

Using Lemma 2.1, we now obtain

$$\|J_E^p y_n - \lambda_n(F(y_n) - F(w_n))\|^q \leq \|J_E^p y_n\|^q - q\lambda_n \langle y_n, F(y_n) - F(w_n)\rangle + c_q \lambda_n^q \|F(y_n) - F(w_n)\|^q.$$

Combining this with (3.7), we have

$$\begin{aligned} \Delta_p(u, z_n) &= \frac{1}{p}\|u\|^p - \langle u, J_E^p y_n - \lambda_n(F(y_n) - F(w_n))\rangle + \frac{1}{q}\|y_n\|^q - \lambda_n \langle y_n, F(y_n) - F(w_n)\rangle \\ &\quad + \frac{c_q \lambda_n^q}{q}\|F(y_n) - F(w_n)\|^q \\ &= \Delta_p(u, y_n) + \lambda_n \langle u - y_n, F(y_n) - F(w_n)\rangle + \frac{c_q \lambda_n^q}{q}\|F(y_n) - F(w_n)\|^q. \end{aligned}$$

From (2.3), we have

$$\begin{aligned} \Delta_p(u, z_n) = \Delta_p(u, w_n) &+ \Delta_p(w_n, y_n) + \langle u - w_n, J_E^p w_n - J_E^p y_n\rangle \\ &+ \lambda_n \langle u - y_n, F(y_n) - F(w_n)\rangle + \frac{c_q \lambda_n^q}{q}\|F(y_n) - F(w_n)\|^q. \end{aligned}$$

Next, it follows from (2.4) that

$$\begin{aligned}&\Delta_p(u,z_n)\\&=\Delta_p(u,w_n)-\Delta_p(y_n,w_n)+\langle y_n-w_n,J_E^py_n-J_E^pw_n\rangle+\langle u-w_n,J_E^pw_n-J_E^py_n\rangle\\&\quad+\lambda_n\langle u-y_n,F(y_n)-F(w_n)\rangle+\frac{c_q\lambda_n^q}{q}\|F(y_n)-F(w_n)\|^q\\&=\Delta_p(u,w_n)-\Delta_p(y_n,w_n)+\langle u-y_n,J_E^pw_n-J_E^py_n\rangle+\lambda_n\langle u-y_n,F(y_n)-F(w_n)\rangle\\&\quad+\frac{c_q\lambda_n^q}{q}\|F(y_n)-F(w_n)\|^q.\end{aligned}\tag{3.8}$$

From the definition of y_n and (2.5), we obtain that $\langle u-y_n,J_E^pw_n-\lambda_nF(w_n)-J_E^py_n\rangle\le 0$, which implies that $\langle u-y_n,J_E^pw_n-J_E^py_n\rangle\le\lambda_n\langle u-y_n,F(w_n)\rangle$. Combining the last inequality with (3.8), we obtain

$$\begin{aligned}\Delta_p(u,z_n)&\le\Delta_p(u,w_n)-\Delta_p(y_n,w_n)+\lambda_n\langle u-y_n,F(w_n)\rangle\\&\quad+\lambda_n\langle u-y_n,F(y_n)-F(w_n)\rangle+\frac{c_q\lambda_n^q}{q}\|F(y_n)-F(w_n)\|^q\\&=\Delta_p(u,w_n)-\Delta_p(y_n,w_n)+\lambda_n\langle u-y_n,F(y_n)\rangle+\frac{c_q\lambda_n^q}{q}\|F(y_n)-F(w_n)\|^q.\end{aligned}\tag{3.9}$$

Since $u\in VIP(C,F)$, we have $\langle F(u),y_n-u\rangle\ge 0$. The pseudomonotonicity of F implies that $\langle F(y_n),y_n-u\rangle\ge 0$. Thus we infer from (3.9) that

$$\Delta_p(u,z_n)\le\Delta_p(u,w_n)-\Delta_p(y_n,w_n)+\frac{c_q\lambda_n^q}{q}\|F(y_n)-F(w_n)\|^q.$$

It follows from the definition of λ_{n+1} that

$$\Delta_p(u,z_n)\le\Delta_p(u,w_n)-\Delta_p(y_n,w_n)+\frac{c_q\mu^q\lambda_n^q}{q\lambda_{n+1}^q}\|y_n-w_n\|^q.$$

Using Proposition 2.1, we obtain

$$\begin{aligned}\Delta_p(u,z_n)&=\Delta_p(u,w_n)-\Delta_p(y_n,w_n)+\frac{c_q\mu^q\lambda_n^q}{q\tau_p\lambda_{n+1}^q}\Delta_p(y_n,w_n)\\&=\Delta_p(u,w_n)-\left(1-\frac{c_q\mu^q\lambda_n^q}{q\tau_p\lambda_{n+1}^q}\right)\Delta_p(y_n,w_n).\end{aligned}\tag{3.10}$$

This completes the proof. □

Lemma 3.4. *The sequence $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{Tz_n\}$ are all bounded.*

Proof. Fix a point $u\in\Gamma$. It is not difficult to infer from (3.1) that

$$\Delta_p(u,w_n)\le(1-\theta_n)\Delta_p(u,x_n)+\theta_n\Delta_p(u,x_{n-1}).\tag{3.11}$$

Once again, using (3.1), (3.10), and (3.11), we find that

$$\begin{aligned}
\Delta_p(u,x_{n+1}) &\le \alpha_n\Delta_p(u,w)+(1-\alpha_n)\Delta_p(u,Tz_n)\\
&\le \alpha_n\Delta_p(u,w)+(1-\alpha_n)\Delta_p(u,z_n)\\
&\le \alpha_n\Delta_p(u,w)+(1-\alpha_n)[(1-\theta_n)\Delta_p(u,x_n)+\theta_n\Delta_p(u,x_{n-1})]\\
&\le \max\left\{\Delta_p(u,w),\max\{\Delta_p(u,x_n),\Delta_p(u,x_{n-1})\}\right\}\\
&\le \vdots\\
&\le \max\left\{\Delta_p(u,w),\max\{\Delta_p(u,x_1),\Delta_p(u,x_0)\}\right\},\ n\ge 1.
\end{aligned}$$

Thus $\{x_n\}$ is bounded. As a consequence of this, the sequences $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{Tz_n\}$ are bounded too, as asserted. □

Theorem 3.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then $\{x_n\}$ converges strongly to the point $u\in\Gamma$, which is nearest to w.*

Proof. Let $u=\Pi_\Gamma w$ and $\eta_n=\frac{c_q\mu^q\lambda_n^q}{q\tau_p\lambda_{n+1}^q}$. In view of (3.1), (3.10), and (3.11), we obtain that

$$\begin{aligned}
&\Delta_p(u,x_{n+1})\\
&\le V_p(u,\alpha_nJ_E^pw+(1-\alpha_n)J_E^pTz_n-\alpha_n(J_E^pw-J_E^pu))+\alpha_n\langle J_E^pw-J_E^pu,x_{n+1}-u\rangle\\
&\le (1-\alpha_n)V_p(u,J_E^pTz_n)+\alpha_n\langle J_E^pw-J_E^pu,x_{n+1}-u\rangle\\
&\le (1-\alpha_n)\Delta_p(u,z_n)-(1-\alpha_n)\Delta_p(z_n,Tz_n)+\alpha_n\langle J_E^pw-J_E^pu,x_{n+1}-u\rangle\\
&\le (1-\alpha_n)\Delta(u,w_n)-(1-\alpha_n)(1-\eta_n)\Delta_p(y_n,w_n)-(1-\alpha_n)\Delta_p(z_n,Tz_n)\\
&\quad+\alpha_n\langle J_E^pw-J_E^pu,x_{n+1}-u\rangle\\
&\le (1-\alpha_n)\Delta_p(u,x_n)-(1-\alpha_n)(1-\eta_n)\Delta_p(y_n,w_n)-(1-\alpha_n)\Delta_p(z_n,Tz_n)+\alpha_nb_n, \qquad (3.12)
\end{aligned}$$

where

$$b_n:=\left(\frac{\theta_n}{\alpha_n}(\Delta_p(u,x_{n-1})-\Delta_p(u,x_n))+\langle J_E^pw-J_E^pu,x_{n+1}-u\rangle\right).$$

It follows from (3.12) that

$$(1-\alpha_n)(1-\eta_n)\Delta_p(y_n,w_n)\le\Delta_p(u,x_n)-\Delta_p(u,x_{n+1})+\alpha_nM_1, \tag{3.13}$$

where $M_1=\sup_{n\in\mathbb{N}}b_n$. It follows from (3.12) that

$$(1-\alpha_n)\Delta_p(z_n,Tz_n)\le\Delta_p(u,x_n)-\Delta_p(u,x_{n+1})+\alpha_nM_1. \tag{3.14}$$

We now proceed to show that $\{x_n\}$ converges strongly to u. Set $a_n:=\Delta_p(u,x_n)$. It is not difficult to see that (3.12) yields the inequality $a_{n+1}\le(1-\alpha_n)a_n+\alpha_nb_n$. To complete this proof, we only need to apply Lemma 2.3. Indeed, it suffices to show that $\lim_{k\to\infty}b_{n_k}\le 0$ whenever a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfies $\limsup_{k\to\infty}(a_{n_k+1}-a_{n_k})\ge 0$. From (A5) and (3.13), we obtain that

$$\begin{aligned}
\limsup_{k\to\infty}(1-\alpha_{n_k})(1-\eta_n)\Delta_p(y_{n_k},w_{n_k})&\le\limsup_{k\to\infty}(a_{n_k+1}-a_{n_k})+M_1\lim_{k\to\infty}\alpha_{n_k}\\
&=-\liminf_{k\to\infty}(a_{n_k+1}-a_{n_k})\le 0,
\end{aligned}$$

which implies that $\Delta_p(y_{n_k}, w_{n_k}) \to 0$ as $k \to \infty$. It follows from Proposition 2.2 that

$$\lim_{k\to\infty} \|y_{n_k} - w_{n_k}\| = 0. \tag{3.15}$$

Using similar arguments, we infer from (A5) and (3.14) that

$$\begin{aligned}\limsup_{k\to\infty}(1-\alpha_{n_k})\Delta_p(z_{n_k}, Tz_{n_k}) &\leq \limsup_{k\to\infty}(a_{n_k+1} - a_{n_k}) + M_1 \lim_{k\to\infty} \alpha_{n_k} \\ &= -\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \leq 0,\end{aligned}$$

from which it follows that $\Delta_p(z_{n_k}, Tz_{n_k}) \to 0$ as $k \to \infty$. Once again calling on Proposition 2.2, we see that

$$\lim_{k\to\infty} \|z_{n_k} - Tz_{n_k}\| = 0. \tag{3.16}$$

Using (3.1), we see that $\|J_E^p w_{n_k} - J_E^p x_{n_k}\| = \theta_{n_k}\|J_E^p x_{n_k} - J_E^p x_{n_k-1}\| \to 0$ as $k \to \infty$. Since $J_{E^*}^q$ is norm-to-norm uniformly continuous on bounded subsets of E^*, it follows that $\lim_{k\to\infty}\|w_{n_k} - x_{n_k}\| = 0$. Next, using this and (3.15), we obtain

$$\|y_{n_k} - x_{n_k}\| \leq \|y_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{3.17}$$

Furthermore, it follows from (3.1) that

$$\|J_E^p z_{n_k} - J_E^p y_{n_k}\| = \|J_E^p y_{n_k} - \lambda_{n_k}(F(y_{n_k}) - F(w_{n_k})) - J_E^p y_{n_k}\| \leq \frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}}\|y_{n_k} - w_{n_k}\|.$$

Thus it follows from (3.15) that $\lim_{k\to\infty}\|J_E^p z_{n_k} - J_E^p y_{n_k}\| = 0$. Now the norm-to-norm uniform continuity of $J_{E^*}^q$ on bounded subsets of E^* yields $\|z_{n_k} - y_{n_k}\| \to 0$ as $k \to \infty$. Using (3.1) once again, we have

$$\Delta_p(x_{n_k+1}, z_{n_k}) \leq \alpha_{n_k}\Delta_p(x_{n_k}, z_{n_k}) + (1-\alpha_{n_k})\Delta_p(z_{n_k}, Tz_{n_k}).$$

By (A5) and Proposition 2.2, we obtain $\lim_{k\to\infty}\|x_{n_k+1} - z_{n_k}\| = 0$. Therefore,

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - z_{n_k}\| + \|z_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{3.18}$$

Next, we show that $\lim_{k\to\infty} b_{n_k} \leq 0$. Indeed, it is sufficient to show that

$$\limsup_{k\to\infty}\langle J_E^p w - J_E^p u, x_{n_k+1} - u\rangle \leq 0.$$

Let $\{x_{n_{k_j}}\}$ be a subsequence of the sequence $\{x_{n_k}\}$ such that

$$\limsup_{k\to\infty}\langle J_E^p w - J_E^p u, x_{n_k+1} - u\rangle = \lim_{j\to\infty} J_E^p w - J_E^p u, x_{n_{k_j}+1} - u\rangle.$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup v \in E$. It follows from (3.17) that $y_{n_{k_j}} \rightharpoonup v$. It also follows from previous arguments that $\{w_{n_{k_j}}\}$ and $\{z_{n_{k_j}}\}$ converge weakly to v. Thus it follows from (3.15) and Lemma 3.2 that $v \in VIP(C,F)$. From (3.16) and $\hat{F}(T) = F(T)$, we obtain $v \in F(T)$. Therefore, $v \in \Gamma$. Since $u = \Pi_\Gamma w$, we conclude by (2.5) and (3.18) that

$$\begin{aligned}
&\limsup_{k\to\infty}\langle J_E^p w - J_E^p u, x_{n_k+1} - u\rangle \\
&\leq \limsup_{k\to\infty}\langle J_E^p w - J_E^p u, x_{n_k} - u\rangle + \limsup_{k\to\infty}\langle J_E^p w - J_E^p u, x_{n_k+1} - x_{n_k}\rangle \\
&= \lim_{j\to\infty}\langle J_E^p w - J_E^p u, x_{n_{k_j}} - u\rangle = \langle J_E^p w - J_E^p u, v - u\rangle \leq 0.
\end{aligned}$$

Using Lemma 2.3, we now conclude that $\Delta_p(u, x_n) \to 0$ as $n \to \infty$. Proposition 2.2 now yields that $\|x_n - u\| \to 0$ as $n \to \infty$, that is, $\{x_n\}$ converges strongly to u. The proof is now complete. □

We now present some consequences of our main result. If $E = H$ is a real Hilbert space, we have the following method for approximating a common solution of a variational inequality and a fixed point problem.

Algorithm 3.2. Totally relaxed Tseng extragradient method

Initialization: Choose $\mu \in (0,1)$ and $\theta > 0$. Select initial points w, x_0, $x_1 \in C$, and $\lambda_0 > 0$, and set the counter $n := 1$. For $i = 1, 2\cdots, m$ and given the current iterate w_n, construct the family of half-spaces $C_n^i := \{z \in E : h_i(w_n) + \langle h_i'(w_n), z - w_n\rangle \leq 0\}$ and set $C_n = \cap_{i=1}^m C_n^i$.

Iterative step: Calculate x_{n+1} and λ_{n+1} as follows:

Step 1: Given x_{n-1}, x_n and λ_n, for each $n \geq 1$, choose θ_n such that $\theta_n \in [0, \bar{\theta}_n]$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2: Compute

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_{C_n}(w_n - \lambda_n F(w_n)), \\ \alpha_{n+1} = \begin{cases} \min\left\{\lambda_n, \dfrac{\mu\|y_n - w_n\|}{\|F(y_n) - F(w_n)\|}\right\}, & \text{if } F(y_n) \neq F(w_n), \\ \lambda_n, & \text{otherwise,} \end{cases} \\ z_n = y_n - \lambda_n(F(y_n) - F(w_n))), \\ x_{n+1} = \alpha_n w + (1 - \alpha_n)T z_n. \end{cases}$$

Stopping criterion: If $x_{n+1} = w_n = y_n$ and $z_n = Tz_n$ for some $n \geq 1$, then stop. Otherwise, set $n := n+1$ and return to **Iterative step**.

Here P_C is the metric projection of H onto C and $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n = \circ(\alpha_n)$. This means that $\lim_{n\to\infty} \dfrac{\varepsilon_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0,1)$ satisfies

(i) $\sum_{n=1}^\infty \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$.

Next, If E is a 2-uniformly convex and uniformly smooth Banach space, we have the following method for approximating a common solution of a variational inequality and a fixed point problem.

Algorithm 3.3. Totally relaxed Tseng extragradient method

Initialization: Choose $\mu \in \left(0, \dfrac{1}{\kappa\sqrt{2c_2}}\right)$ and $\theta > 0$. Select initial points $w, x_0,\ x_1 \in C$, $\lambda_0 > 0$ and set the counter $n := 1$. For $i = 1, 2 \cdots, m$ and given the current iterate w_n, construct the family of half-spaces $C_n^i := \{z \in E : h_i(w_n) + \langle h_i'(w_n), z - w_n\rangle \leq 0\}$ and set $C_n = \cap_{i=1}^m C_n^i$.
Iterative step: Calculate x_{n+1} and λ_{n+1} as follows:

Step 1: Given x_{n-1}, x_n and λ_n, for each $n \geq 1$, choose θ_n such that $\theta_n \in [0, \bar{\theta}_n]$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2: Compute

$$\begin{cases} w_n = J^{-1}(Jx_n + \theta_n(Jx_n - Jx_{n-1})), \\ y_n = Proj_{C_n}(Jw_n - \lambda_n F(w_n)), \\ \lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \dfrac{\mu\|y_n - w_n\|}{\|F(y_n) - F(w_n)\|}\right\}, & \text{if } F(y_n) \neq F(w_n), \\ \lambda_n, & \text{otherwise,} \end{cases} \\ z_n = J^{-1}(Jy_n - \lambda_n(F(y_n) - F(w_n))), \\ x_{n+1} = J^{-1}(\alpha_n Jw + (1-\alpha_n)JTz_n). \end{cases}$$

Stopping criterion: If $x_{n+1} = w_n = y_n$ and $z_n = Tz_n$ for some $n \geq 1$, then stop. Otherwise set $n := n+1$ and return to **Iterative step**.

Here $Proj$ is the generalized projection, κ is the 2-uniformly smoothness constant, c_2 is the 2-uniform convexity constant, J is the duality mapping, and $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n = \circ(\alpha_n)$. This means that $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$, where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$.

4. Numerical Examples

In this section, we report on some numerical experiments, which show the applicability of the main result. The computer programming for the following examples was carried out on a personal Dell latitude E7450 laptop with specifications 8/256 of RAM and ROM, respectively, and processor speed 2.0 Ghz. In the given examples, we make a comparison of our main result with a variant where the set C_n replaced by the feasible set $C := \{x \in E : \|x\| \leq 1\}$. In the final example, we also make a comparison with the case that the Lipschitz constant is used instead of adaptive step sizes.

Example 4.1. Consider the following fractional programming problem which was first used in [44].

$$\min g(x) = \frac{x^T Px + a^T x + a_0}{b^T x + b_0}$$
$$\text{subject to } x \in X = \{x \in \mathbb{R}^4 : b^T x + b_0 > 0\},$$

where

$$P = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, a_0 = -2, and\ b_0 = 4.$$

Since P is symmetric and positive definite, and g is pseudoconvex on X, we have that

$$F(x) = \nabla g(x) = \frac{(b^T x + b_0)(2Px + a) - b(x^T Px + a^T x + a_0)}{(b^T x + b_0)^2}$$

is pseudomonotone. Now, let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $T(x) := P_C$, where P_C is the metric projection onto $C := \{x \in \mathbb{R}^4 : 1 \leq x_i \leq 10\} \subset X$. For this example, choose the sequences $\alpha_n = \frac{1}{2n+1}$, $\varepsilon_n = \frac{1}{n^{1.1}}$, $\lambda_0 = 0.25$, and $\mu = \theta = 0.5$. Let $\|x_{n+1} - x_n\| \leq 10^{-5}$ be the stopping criterion. For several values of the initial points x_0 and x_1, the results of our experiments are given in Figure 1.

Case 1 $w = 1$, $x_0 = [10, 10, 10, 10]^T$ and $x_1 = [20, 20, 20, 20]^T$;
Case 2 $w = 0.5$, $x_0 = [10, 20, 30, 40]^T$ and $x_1 = [20, 10, 20, 10]^T$;
Case 3 $w = 0.5$, $x_0 = [-10, 10, 10, -10]^T$ and $x_1 = [5, -5, -5, 5]^T$;
Case 4 $w = 0.25$, $x_0 = [1, 2, 3, 4]^T$ and $x_1 = [4, 3, 2, 1]^T$.

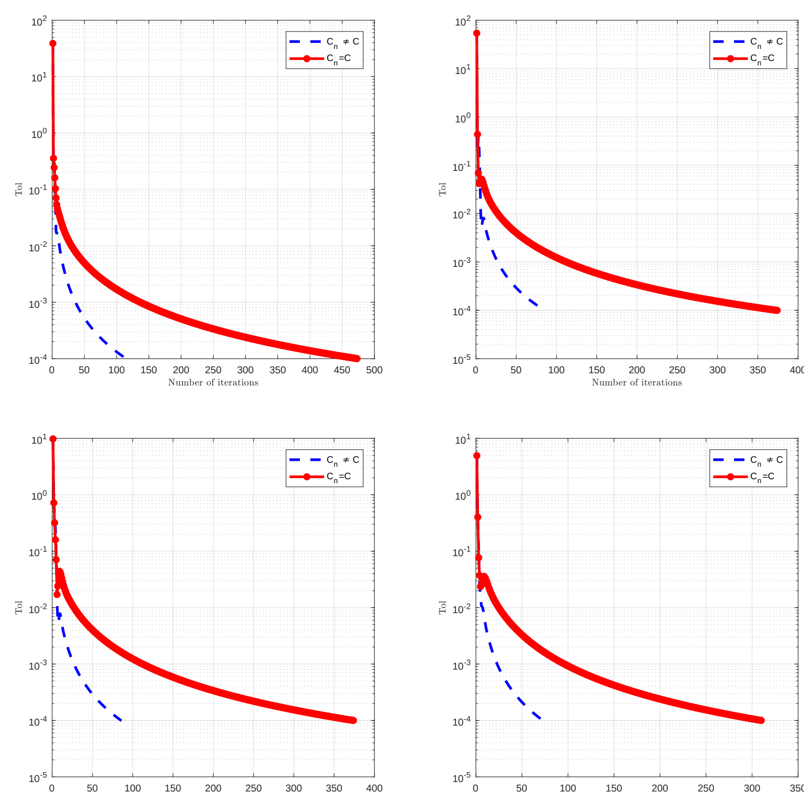

FIGURE 1. Example 4.1. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4.

Example 4.2. This example was given in [45] with

$$F(x) = \begin{pmatrix} 0.5x_1x_2 - 2x_2 - 10^7 \\ -4x_1 + 0.1x_2^2 - 10^7 \end{pmatrix}$$

and $C := \{x \in \mathbb{R}^2 : (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1\}$. It was shown that F is not monotone but pseudomonotone and Lipschitz continuous with $L = 5$ (see [45, Example 6.8]). We choose the parameters $\alpha_n = \frac{1}{50n+13}$, $\varepsilon = \frac{1}{n^2}$, $\theta = \mu = 0.5$ and $\lambda_0 = 0.75$. Using $\|x_{n+1} - x_n\| \leq 10^{-4}$ as our stopping criterion, we consider the following cases for w and the initial points x_0 and x_1.

(i) $w = 2.5$, $x_0 = [20, 20]^T$, and $x_1 = [10, 10]^T$;
(ii) $w = 2.5$, $x_0 = [50, 50]^T$, and $x_1 = [10, 10]^T$;
(iii) $w = 2.5$, $x_0 = [30, 40]^T$, and $x_1 = [10, 20]^T$;
(iv) $w = 1.2$, $x_0 = [5, 5]^T$, and $x_1 = [-10, 20]^T$.

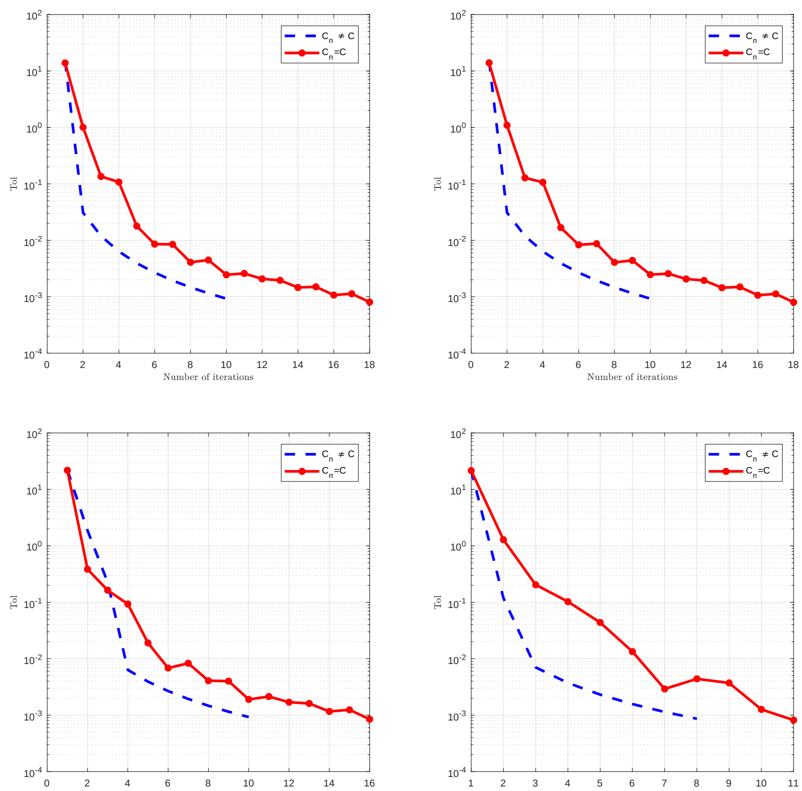

FIGURE 2. Example 4.2. Top left: (i), Top right: (ii), Bottom left: (iii), Bottom right: (iv).

Example 4.3. Let $E = L^2([0, \pi])$ with inner product $\langle x, y \rangle := \int_0^\pi x(t)y(t)dt$, $\forall x, y \in E$, $t \in [0, \pi]$ and norm

$$\|x\| := \left(\int_0^\pi \|x(t)\|^2 dt, \right)^{\frac{1}{2}}.$$

Define the Volterra operator $F : L^2([0, 1]) \to L^2([0, 1])$ by $F(x(s)) := \int_0^t x(s)ds$, $\forall\, x \in L^2([0, 1])$, $t \in [0, 1]$. Then F is monotone and linear with a Lipschitz constant $L = \frac{2}{\pi}$ (see [46, Exercise 20.12]).

Define the feasible set C by $C := \{x \in E : \|x\| \leq 1\}$, and take $T = P_C$. For this example, we choose the parameters $\alpha_n = \frac{1}{2n+1}$, $\varepsilon = \frac{10^{50}}{n^2}$, $w = 0.1$, $\theta = \mu = 0.5$, and $\lambda_0 = 0.5$. In practice, we choose $\|x_{n+1} - x_n\| \leq 10^{-4}$ as our stopping criterion. This example is carried out with the following initial values of x_0 and x_1.

(I) $x_0 = t^{\frac{2}{3}}$ and $x_1 = t^2 + 3t + 11$;
(II) $x_0 = t + 1$ and $x_1 = t^3 + 2t$;
(III) $x_0 = 2t + 5$ and $x_1 = \frac{t^2}{2} + 1$;
(IV) $x_0 = e^{-2t}$ and $x_1 = \frac{1}{2}\cos t$.

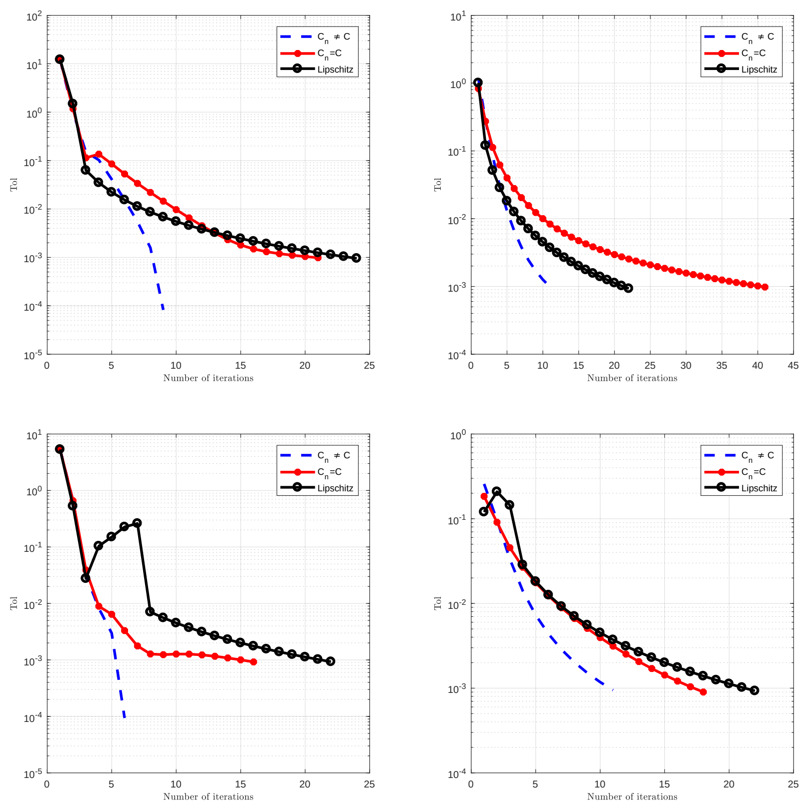

FIGURE 3. Example 4.3. Top left: (I), Top right: (II), Bottom left: (III), Bottom right: (IV).

5. CONCLUSION

In the framework of p-uniformly convex and smooth Banach spaces, we proposed and studied a totally relaxed self-adaptive inertial algorithm for finding a common solution to variational inequality and fixed point problems. The Halpern method was considered for achieving strong convergence and the Tseng method was incorporated for reducing the number of projections in our algorithm. A strong convergence theorem was established, and numerical examples were presented to illustrate the performance of our algorithm.

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