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FIXED POINTS OF MULTI-VALUED SUBRAHMANYAN CONTRACTIONS

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Dedicated to the memory of Professor Ronald E. Bruck

Abstract. In this paper, we present some fixed point results for multi-valued Subrahmanyan contractions in complete metric spaces. The Ulam-Hyers stability property of the fixed point inclusion is also deduced. Our results generalize some existence and stability theorems given in the case of multi-valued graph contractions. Some open questions are pointed out.
Keywords. Complete metric space; Multi-valued Feng-Liu operators; Multi-valued Subrahmanyan contraction; Ulam-Hyers stability.

1. Introduction and Preliminaries

Let (X,d) be a metric space. We denote by $P(X)$ the family of all nonempty subsets of X and by $P_{cl}(X)$ the family of all nonempty closed subsets of X. We also use the following notations:
(1) the distance between a point $a \in X$ and a set $B \subset X$:

$$D(a,B) := \inf\{d(a,b) \mid b \in B\};$$

(2) the excess functional of A over B generated by d:

$$e(A,B) := \sup\{D(a,B) \mid a \in A\};$$

(3) the Hausdorff-Pompeiu functional generated by d:

$$H(A,B) = \max\{e(A,B), e(B,A)\}.$$

Let (X,d) be a metric space, and let $F : X \to P(X)$ be a multi-valued operator. A fixed point of F is an element $x^* \in X$ such that $x^* \in F(x^*)$. A strict fixed point of F is an element $x^* \in X$ such that $F(x^*) = \{x^*\}$. We denote by $Fix(F)$ the fixed point set of F and by $SFix(F)$ the set of all strict fixed points of F. We also denote by $Graph(F) := \{(x,y) \mid y \in F(x)\}$ the graph of F.

A multi-valued operator $F : X \to P(X)$ is said to be a multi-valued α-contraction if $\alpha \in [0,1[$ and the following relation holds

$$H(F(x),F(y)) \leq \alpha d(x,y), \text{ for all } (x,y) \in X \times X.$$

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If the above relation holds for every $(x,y) \in Graph(F)$, then F is called a multi-valued graph contraction with constant α.

In 1969, Nadler Jr. proved a fixed point theorem in complete metric spaces for multi-valued contractions with nonempty, closed, and bounded values, while Covitz and Nadler Jr. gave, one year later, a relaxed version of the result by removing the boundedness of the values.

Theorem 1.1. *(Covitz and Nadler [1]) Let (X,d) be a complete metric space. Suppose that $F : X \to P_{cl}(X)$ is a multi-valued α-contraction. Then $Fix(F) \neq \emptyset$.*

In the case of multi-valued graph contractions, the following result is known.

Theorem 1.2. *([2]) Let (X,d) be a complete metric space. Suppose that $F : X \to P(X)$ is a multi-valued graph contraction with closed graph. Then $Fix(F) \neq \emptyset$.*

The purpose of this paper is to give some fixed point results for multi-valued Subrahmanyan contractions in complete metric spaces. A local Ulam-Hyers stability property for the fixed point inclusion $x \in F(x)$ is also presented. Our results generalize some existence and stability theorems given in the case of multi-valued graph contractions. In particular, when the operator is single-valued, our existence result gives Theorem 3.3 in [3].

2. Main Results

Let (X,d) be a metric space, and let $F : X \to P(X)$ be a multi-valued operator. For arbitrary $(x_0,x_1) \in Graph(F)$, the sequence $(x_n)_{n\in\mathbb{N}}$ with the property $x_{n+1} \in F(x_n)$, $n \in \mathbb{N}$ is called the sequence of successive approximations for F staring from (x_0,x_1). Using the above notion, the concept of multi-valued weakly Picard operator is given below.

Definition 2.1. ([4]) Let (X,d) be a metric space. Then, $F : X \to P(X)$ is called a multivalued weakly Picard operator if, for each $x \in X$ and each $y \in F(x)$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that

(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$;
(iii) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Let us recall the following important notions.

Definition 2.2. Let (X,d) be a metric space, and let $F : X \to P(X)$ be a multivalued weakly Picard operator. Then we can define the multivalued operator $F^\infty : Graph(F) \to P(Fix(F))$ by the formula $F^\infty(x,y) = \{z \in Fix(F) \mid$ there exists a sequence of successive approximations of F starting from (x,y) that converges to $z\}$.

An important concept is given by the following definition.

Definition 2.3. Let (X,d) be a metric space, and let $F : X \to P(X)$ be a multivalued weakly Picard operator. Then, F satisfies the local retraction-displacement condition if there exists a selection f^∞ of F^∞ such that

$$d(x, f^\infty(x,y)) \leq C(x,y)d(x,y), \text{ for all } (x,y) \in Graph(F),$$

for some $C(x,y) > 0$.

For related notions, examples, and results, we refer to [4, 5, 6, 7, 8, 9, 10, 11].

We introduce now the concept of multi-valued Subrahmanyan contractions. For the single-valued case, see [3].

Definition 2.4. Let (X,d) be a metric space, and let $F : X \to P(X)$ be a multi-valued operator. We say that F is a multi-valued Subrahmanyan contraction if there exists $\psi : X \to [0,1[$ such that:

(i) $H(F(x),F(y)) \leq \psi(x)d(x,y)$, for all $(x,y) \in Graph(F)$;

(ii) $\psi(y) \leq \psi(x)$, for every $(x,y) \in Graph(F)$.

Remark 2.1. If $\alpha \in [0,1[$ and we consider $\psi(x) := \alpha$ for each $x \in X$, then we obtain the above mentioned concept of multi-valued graph contraction with constant α.

Our first main result is a fixed point theorem for a multi-valued Subrahmanyan contractions with closed graph.

Theorem 2.1. *Let (X,d) be a complete metric space, and let $F : X \to P(X)$ be a multi-valued Subrahmanyan contraction with closed graph. Then, the following conclusions hold:*

(a) $Fix(F) \neq \emptyset$;

(b) for every $(x_0,x_1) \in Graph(F)$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations for F starting (x_0,x_1) which converges to a fixed point $x^*(x_0,x_1)$ of F and the following apriori estimation holds

$$d(x_n,x^*(x_0,x_1)) \leq \frac{(q\psi(x_0))^n}{1-q\psi(x_0)}d(x_0,x_1), \text{ for every } n \in \mathbb{N},$$

where $q \in]1, \frac{1}{\psi(x_0)}[$;

(c) the following retraction-displacement type condition holds

$$d(x_0,x^*(x_0,x_1)) \leq \frac{1}{1-\psi(x_0)}d(x_0,x_1), \text{ for all } (x_0,x_1) \in Graph(F).$$

Proof. (a)-(b) Let $(x_0,x_1) \in Graph(F)$ be arbitrary. Let $1 < q < \frac{1}{\psi(x_0)}$. Then, for $x_1 \in F(x_0)$, there exists $x_2 \in F(x_1)$ such that

$$\begin{aligned} d(x_1,x_2) &\leq qH(F(x_0),F(x_1)) \\ &\leq q\psi(x_0)d(x_0,x_1). \end{aligned}$$

Similarly, for $x_2 \in F(x_1)$, there exists $x_3 \in F(x_2)$ such that

$$\begin{aligned} d(x_2,x_3) &\leq qH(F(x_1),F(x_2)) \\ &\leq q\psi(x_1)d(x_1,x_2) \\ &\leq (q\psi(x_0))^2 d(x_0,x_1). \end{aligned}$$

Inductively, we can construct a sequence of successive approximations for F starting from $(x_0,x_1) \in Graph(F)$ having the property

$$d(x_n,x_{n+1}) \leq (q\psi(x_0))^n d(x_0,x_1), n \in \mathbb{N}. \tag{2.1}$$

It follows from (2.1) that

$$\begin{aligned} d(x_n,x_{n+p}) &\le d(x_n,x_{n+1})+\cdots+d(x_{n+p-1},x_{n+p}) \\ &\le (q\psi(x_0))^n d(x_0,x_1)+\cdots+(q\psi(x_0))^{n+p-1} d(x_0,x_1) \\ &= (q\psi(x_0))^n \left(1+q\psi(x_0)+\cdots(q\psi(x_0))^{p-1}\right) d(x_0,x_1) \\ &= (q\psi(x_0))^n \frac{1-(q\psi(x_0))^p}{1-q\psi(x_0)} d(x_0,x_1) \\ &\le \frac{(q\psi(x_0))^n}{1-q\psi(x_0)} d(x_0,x_1). \end{aligned}$$

By the above relation we get that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d). By the completeness of the space, there exists $x^*(x_0,x_1)\in X$ such that $\{x_n\}_{n\in\mathbb{N}}$ is convergent to $x^*(x_0,x_1)\in X$. Since F has closed graph we immediately obtain that $x^*(x_0,x_1)$ is a fixed point for F. By the above proof, we also have that

$$d(x_n,x_{n+p}) \le \frac{(q\psi(x_0))^n}{1-q\psi(x_0)} d(x_0,x_1), \text{ for } n\in\mathbb{N} \text{ and } p\in\mathbb{N}^*. \tag{2.2}$$

Letting $p\to\infty$ in (2.2) we see that

$$d(x_n,x^*(x_0,x_1)) \le \frac{(q\psi(x_0))^n}{1-q\psi(x_0)} d(x_0,x_1). \tag{2.3}$$

(c) Taking $n=0$ in (2.3) yields that

$$d(x_0,x^*(x_0,x_1)) \le \frac{1}{1-q\psi(x_0)} d(x_0,x_1). \tag{2.4}$$

The conclusion follows by letting $q\searrow 1$. This completes the proof. □

Remark 2.2. By the above conclusions, we notice that, under the assumptions of Theorem 2.1, F is a multi-valued weakly Picard operator and satisfies the local retraction-displacement condition. Indeed, let us define the selection f^∞ of F^∞ by $f^\infty(x_0,x_1):=x^*(x_0,x_1)$ and denote $C(x_0)=\frac{1}{1-\psi(x_0)}$, for $(x_0,x_1)\in Graph(F)$. Then, by (2.4), we obtain

$$d(x_0,f^\infty(x_0,x_1)) \le C(x_0)d(x_0,x_1).$$

In general, the selection $f^\infty: Graph(F)\to Fix(F)$ of F^∞ is not continuous. For the continuity of the operator f^∞ in the single-valued case, see [12].

Remark 2.3. In particular, if F is a single-valued operator, Theorem 2.1 gives us the main result in [3].

The following Ulam-Hyers stability property can be obtained from the above results. For related definitions and results, see [13].

Definition 2.5. Let (X,d) be a metric space, and let $F:X\to P(X)$ be a multi-valued operator. Then, the fixed point inclusion

$$x\in F(x), x\in X, \tag{2.5}$$

is said to be local Ulam-Hyers stable if, for any $\varepsilon>0$ and any ε-solution z of the fixed point inclusion (2.5) (in the sense that $D(z,F(z))\le\varepsilon$), there exist $K:=K(z)>0$ and $x^*\in Fix(F)$ such that $d(z,x^*)\le K\varepsilon$.

Recall that if (X,d) is a metric space, then a set $Y \in P(X)$ is said to be proximinal if, for every $x \in X$, there exists $y \in Y$ such that $d(x,y) = D(x,Y)$.

Concerning the local Ulam-Hyers stability of the fixed point inclusion, we have the following result.

Theorem 2.2. *Let (X,d) be a complete metric space, and let $F : X \to P(X)$ be a multi-valued Subrahmanyan contraction with closed graph and proximinal values. Then, the fixed point inclusion (2.5) has the local Ulam-Hyers stablility property.*

Proof. Let $\varepsilon > 0$, and let z be an ε-solution of the fixed point inclusion. Then, by the proximinal property of the values of F, we have that there exists $w \in F(z)$ such that $d(z,w) = D(z,F(z)) \leq \varepsilon$. Thus, by Theorem 2.1 there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for F starting from $(z,w) \in Graph(F)$, which converges to a fixed point $x^* = x^*(z,w)$ of F and

$$d(z,x^*) \leq C(z)d(z,w) \leq C(z)\varepsilon.$$

□

A special fixed point for a multi-valued operator $F : X \to P(X)$ is the so-called strict fixed point or end-point for F. In this context, we have the following strict fixed point result for multi-valued Subrahmanyan contractions, which generalize one of the main theorem in [14].

Theorem 2.3. *Let (X,d) be a complete metric space, and let $F : X \to P(X)$ be a multi-valued Subrahmanyan contraction with closed graph. Suppose:*

(1) $F(F(x)) \subset F(x)$, for each $x \in X$;

(2) if $A \in P_{cl}(X)$ with $F(A) = A$, then A is a singleton.

Then, $Fix(F) = SFix(F) \neq \emptyset$

Proof. By Theorem 2.1, the existence of a fixed point is assured. Let $w \in Fix(F)$. By (1), we have that $F(w) \subset F(F(w)) \subset F(w)$. Thus $F(F(w)) = F(w)$ and so $F(w)$ is a fixed set for F. By (2), we have that $F(w)$ is a singleton. Hence, $F(w) = \{w\}$. Moreover, by the above considerations, we observe that $Fix(F) \subset SFix(F)$. Thus $Fix(F) = SFix(F) \neq \emptyset$. □

Remark 2.4. The condition $Fix(F) = SFix(F)$ appears often in convergence results for various iteration methods involving multi-valued operators.

A slight extension of the above results can be obtained if the metric condition on F is given in terms of the excess functional. For example, the following existence and approximation result takes place.

Theorem 2.4. *Let (X,d) be a complete metric space, and let $F : X \to P(X)$ be an operator with closed graph. We suppose that there exists $\psi : X \to [0,1[$ such that:*

(i) $e(F(x),F(y)) \leq \psi(x)d(x,y)$, for all $(x,y) \in Graph(F)$;

(ii) $\psi(y) \leq \psi(x)$, for every $(x,y) \in Graph(F)$.

Then, the following conclusions hold:

(a) $Fix(F) \neq \emptyset$;

(b) for every $(x_0,x_1) \in Graph(F)$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations for F starting (x_0,x_1) which converges to a fixed point $x^*(x_0,x_1)$ of F and the following apriori estimation holds

$$d(x_n,x^*(x_0,x_1)) \leq \frac{(q\psi(x_0))^n}{1-q\psi(x_0)}d(x_0,x_1), \text{ for every } n \in \mathbb{N},$$

where $q \in]1, \frac{1}{\psi(x_0)}[$.

(c) the following retraction-displacement type condition holds

$$d(x_0, x^*(x_0,x_1)) \leq \frac{1}{1-\psi(x_0)} d(x_0,x_1), \textit{ for all } (x_0,x_1) \in Graph(F).$$

Remark 2.5. It is an open question to obtain other stability properties (well-posedness in the sense of Reich and Zaslavski, see [9, 15, 16, 17]), Ostrowski stability property [9, 17]) for the fixed point inclusion $x \in F(x)$.

Finally, we also notice that a more general class of multi-valued operators can be defined by considering the Feng-Liu type approach (see [18]) in the multi-valued fixed point theory. More precisely, we define a multi-valued Feng-Liu-Subrahmanyan contraction as follows.

Definition 2.6. Let (X,d) be a metric space, $F : X \to P(X)$ a multi-valued operator, $b \in]0,1[$, and $x \in X$. Consider the set

$$I_b^x := \{y \in F(x) \mid bd(x,y) \leq D(x,F(x))\}.$$

Then, by definition, F is a multi-valued Feng-Liu-Subrahmanyan contraction if there exists $\psi : X \to [0,1[$ such that, for each $x \in X$, there is $y \in I_b^x$ for which the following conditions are satisfied:

(i) $D(y,F(y)) \leq \psi(x)d(x,y)$, for all $(x,y) \in Graph(F)$;

(ii) $\psi(y) \leq \psi(x)$, for every $(x,y) \in Graph(F)$.

It is obvious that any multi-valued Subrahmanyan contraction is a multi-valued Feng-Liu-Subrahmanyan contraction, but the reverse implication, in general, does not hold. It is an open question to present a fixed point theory (see [17] and [9]) for multi-valued Feng-Liu-Subrahmanyan contractions in complete metric spaces. Another open problem is to obtain topological properties of the fixed point set for multi-valued Subrahmanyan contractions, following the approach given in [12] for virtually nonexpansive operators. For the case of nonexpansive operators, we refer to the papers of Bruck [19, 20].

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