Appl. Set-Valued Anal. Optim. 5 (2023), No. 1, pp. 1-18
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.5.2023.1.01

CHARACTERIZATION OF SOLUTION SETS OF GEODESIC CONVEX SEMI-INFINITE PROGRAMMING ON RIEMANNIAN MANIFOLDS

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Abstract. This paper is concerned with the problem of geodesic convex semi-infinite programming on Riemannian manifolds. First, we establish Karush-Kuhn-Tucker necessary optimality conditions for optimal solutions under the Guignard constraint qualification. Then, some characterizations of the solution sets of convex smooth semi-infinite programming on Riemannian manifolds are given.
Keywords. Geodesic convexity; Guignard constraint qualification; Karush-Kuhn-Tucker optimality conditions; Riemannian manifolds; Semi-infinite programming.

2020 Mathematics Subject Classification. 90C34, 90C46.

1. INTRODUCTION

Semi-infinite programming can be viewed as an extension of optimization problems, where the finite number of constraints is replaced by the infinite many constraints. Arising from various practical problems in engineering, economic, and information technology, semi-infinite programming has recently been investigated in numerous papers, see e.g., [1, 2, 3, 4, 5] and the references therein. Besides, numerious concepts, techniques, and algorithms in optimization in Euclidean spaces were generalized to Riemannian manifolds to deal with some realistic optimization problems. Jongen et al [6] derived the Fritz John type first order necessary optimality condition without assuming any constraint qualifying or reduction strategy, and described geometrical characteristics of the feasible set with examples. Vázquez et al. [7] discussed generalized semi-infinite programming and its application. They investigated the geometric and topological properties of the feasible set, as well as its differences from the ordinary semi-infinite with examples. Constraint qualifiers were developed by using the first-order approximations of the feasible set. Then, using the directional differentiability properties of the optimal value function of the so-called lower level problem, necessary and sufficient first- and second-order optimality conditions were derived.

Standard semi-infinite optimization (SIP) problems were often solved by using discretization methods. Due to the independence of the infinite index set, transferring these methods to the situation of general semi-infinite optimization (GSIP) problems is problematic. For solving

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Received June 21, 2022; Accepted July 25, 2022.

the GSIP through numerical approaches, the GSIP is generally converted to the SIP. A GSIP problem can be converted into a SIP problem under the right conditions. However, it has the potential to disrupt convexity at a lower level, which is critical for numerical approaches. Stein [8] focused on the structural features of generic semi-infinite programming, formulating optimality conditions, and presenting a conceptually novel solution approach. Stein and Still [9] introduced a new numerical solution method for solving semi-inifinte optimization problems with convex lower level problems. They demonstrated the convergence features of the method and provided numerical examples from design centering and robust optimization, where they solved so-called generalised semi-infinite optimization problems. When the bi-level structure of general semi-infinite programmes is properly exploited, it is explained that it can be solved efficiently under specific conditions. The spherical convexity of quadratic functions on spherically convex sets was studied in [10]. The paper [11] presented a version of the Newton method for finding a singularity of a special class of locally Lipschitz continuous vector fields. Schwientek et al. [12] introduced the transformation based discretization method for solving the GSIP and illustrated it with several examples which include a problem of volume-maximal inscription of multiple variable bodies into a larger fixes body.

It is known that the establishing optimality conditions and the characterization of solution sets of optimization problems play an important roles in optimization. This gives the test for a feasible point to be an optimal solution of optimization problems, which is a necessary condition in building algorithms to solve the optimization problems. Although the algorithms to solve the optimization problems on Riemannian manifolds have recently been investigated extensively, the establishing optimality conditions and the characterization of solution sets of optimization problems have not yet been considered comprehensively. The concepts and techniques of optimization on the sphere were discussed in [13]. The paper [14] established optimality conditions for the nonlinear programming problems on Riemannian manifolds. Intrinsic formulation of KKT conditions and constraint qualifications for constrained optimizations on smooth manifolds were discussed in [15]. Necessary and sufficient optimality conditions for vector equilibrium problems on Hadamard manifolds were provided in [16]. The characterization of solution sets of convex smooth unconstrained optimization were extended from Euclidean spaces in [17] to Riemannian manifolds in [18]. The optimality conditions and duality for semi-infinite programming on Hadamard manifolds were discussed in [19]. However, the characterization of solution sets of convex constrained optimizations have not been yet considered in [13, 18, 19]. Moreover, to the best of our knowledge, there is no paper dealing with optimality conditions and the characterization of solution sets of semi-infinite programming on Riemannian manifolds.

The above observations motivate us to establish Karush-Kuhn-Tucker optimality conditions and characterize the solution sets of semi-infinite programming on Riemannian manifolds in this paper. The outline of the paper is as follows. Section 2 recalls the basic notions of Riemannian manifolds. Then, Karush-Kuhn-Tucker necessary optimality conditions for optimal solutions is examined. Section 3 concentrates on characterizing the solution sets of the semi-infinite programming on Riemannian manifolds. Some detailed examples are provided to illustrate the outcomes of the paper.

2. Preliminaries and Auxiliary Results

In this paper, the notation $\langle\cdot,\cdot\rangle$ is utilized to denote the inner product in the p-dimensional Euclidean space $\mathscr{E}$. Let $\mathscr{E}_1$ be a p_1-dimensional linear subspace of $\mathscr{E}$ with $p_1 \leq p$. Then, $\mathscr{E}_1$ is also an Euclidean space with $\langle\cdot,\cdot\rangle_{\mathscr{E}_1} = \langle\cdot,\cdot\rangle$, and $\mathscr{E}_1$ is equipped with a topology induced from that of $\mathscr{E}$. Moreover, there is a natural isometry identifying $\mathscr{E}_1$ and $\mathbb{R}^{p_1}$. For a given $\bar{x} \in \mathscr{E}_1$, $\mathscr{U}(\bar{x})$ is the system of the open neighborhoods of $\bar{x}$ in $\mathscr{E}_1$. For $A \subseteq \mathscr{E}_1$, int$A$, cl$A$, aff$A$, span$A$, and co$A$ stand for its interior, closure, affine hull, linear hull, and convex hull of A in $\mathscr{E}_1$, respectively (shortly, resp). The cone and the convex cone (containing the origin) generated by A in $\mathscr{E}_1$ are indicated resp by cone$A$, pos$A$. The negative polar cone and the strictly negative polar cone of A in $\mathscr{E}_1$ are defined resp by

$$A^- := \{x^* \in \mathscr{E}_1 \mid \langle x^*, x\rangle \leq 0,\ \forall x \in A\},$$

and

$$A^s := \{x^* \in \mathscr{E}_1 \mid \langle x^*, x\rangle < 0,\ \forall x \in A\}.$$

It should be noted that if $A_1 \subset A_2$, then $A_2^- \subset A_1^-$. Moreover, we can check that $A^- = (\text{cl cone}A)^-$; see also [20]. If $\langle x^*, x\rangle \geq 0$ for all $x^* \in A^*$, where A^* is a subset of the dual space of $\mathscr{E}_1$, we write $\langle A^*, x\rangle \geq 0$. The map φ from an Euclidean space $\mathscr{E}$ to another Euclidean space $\mathscr{E}'$ is said to be of class $C^1(C^\infty$, resp) if φ is continuously differentiable (infinitely continuously differentiable, resp) on $\mathscr{E}$. Recall the following definition of the Riemannian manifolds; see [21, 22, 23, 24, 25, 26] for more details.

Let $\mathscr{E}$ be an Euclidean space and $M^n \subset \mathscr{E}$ be a smooth manifold. The tangent space to M^n at $\bar{x}$ is denoted by $T_{\bar{x}}M^n$. It should be noted that $T_{\bar{x}}M^n$ is a n-dimensional vector subspace of $\mathscr{E}$. The tangent bundle of a manifold M^n is the disjoint union of the tangent spaces of M^n. If $F \equiv f : M^n \to \mathbb{R}$ is a smooth map, then the differential of f at $\bar{x}$ is a linear mapping $df(\bar{x}) : T_{\bar{x}}M^n \to \mathbb{R}$, defined by

$$df(\bar{x})[v] := \frac{d}{dt} f(\gamma(t))\bigg|_{t=0}, [v] \in T_{\bar{x}}M^n.$$

One calls a smooth manifold M^n a Riemannian manifold with respect to $\mathscr{G}$ if its tangent spaces are endowed with a smoothly varying inner product $\mathscr{G}$, defined by $\mathscr{G}(v_1, v_2) = \langle v_1, v_2\rangle_x$, $\forall v_1, v_2 \in T_xM^n$. The length of a piecewise smooth curve on a Riemannian manifold $\gamma_{x,y} : [0,1] \to M^n$, joining x to y with $\gamma_{x,y}(0) = x, \gamma_{x,y}(1) = y$, is defined by $L(\gamma_{x,y}) = \int_0^1 \|\gamma'_{x,y}(t)\|dt$. The the Riemannian distance $d(x,y)$ is defined by minimizing this length over the set of all such curves joining x to y, which induces the original topology on M^n. The covariant derivative associated with a Riemannian manifold M^n is called Levi-Civita connection ${}^{LC}\nabla$. A geodesic is a smooth curve γ satisfying the equation ${}^{LC}\nabla_{\gamma'(t)}\gamma'(t) = 0$. A geodesic joining x to y in M^n is said to be minimal if its length equals $d(x,y)$. A Hadamard manifold M^n is a complete simply connected Riemannian manifold of nonpositive sectional curvature.

The exponential map $\exp_{\bar{x}} : T_{\bar{x}}M^n \to M^n$ is defined by $\exp_{\bar{x}}(v) = \gamma_{\bar{x},v}(1)$, where $\gamma_{\bar{x},v} : [0,1] \to M^n$ is the geodesic starting at $\bar{x}$ with velocity v, i.e., $\gamma_{\bar{x},v}(0) = \bar{x}$ and $\gamma'_{\bar{x},v}(0) = v$. It is easy to see that $\exp_{\bar{x}}(tv) = \gamma_{\bar{x},v}(t)$ for all $t \in [0,1]$ and $\exp_{\bar{x}}(0_{\bar{x}}) = \gamma_{\bar{x},v}(0) = \bar{x}$. The exponential map is a retraction; see [21, 22, 27].

Proposition 2.1 (Hopf-Rinow Theorem)**.** *If M^n is a complete and connected Riemannian manifold, then any two points in M^n can be joined by a minimizing geodesic segment.*

Proposition 2.2. [23] *If M^n is a Hadamard manifold, then* $\exp_{\bar{x}} : T_{\bar{x}}M^n \to M^n$ *is a diffeomorphism with the inverse map* $\exp_{\bar{x}}^{-1} : M^n \to T_{\bar{x}}M^n$ *satisfying* $\exp_{\bar{x}}^{-1}(\bar{x}) = 0_{\bar{x}}$. *Moreover, for any $x \in M^n$, there exists a unique minimal geodesic $\gamma_{\bar{x},x} : [0,1] \to M^n$ satisfying* $\gamma_{\bar{x},x}(t) = \exp_{\bar{x}}(t\exp_{\bar{x}}^{-1}(x))$.

Definition 2.1. Let A be a given nonempty subset of the Riemannian manifold M^n and $\bar{x} \in \mathrm{cl}A$. The contingent cone (see [20, 28]) of A at $\bar{x}$ is

$$\mathscr{T}(A,\bar{x}) := \{v \in T_{\bar{x}}M^n \mid \exists \tau_k \downarrow 0, \exists v_k \in T_{\bar{x}}M^n, v_k \to v,\ \forall k \in \mathbb{N}, \exp_{\bar{x}}(\tau_k v_k) \in A\}.$$

If M^n is a Hadamard manifold, then there exists $x_k \in M^n$ such that $x_k = \exp_{\bar{x}}(v_k)$, and we can replace v_k by $\exp_{\bar{x}}^{-1}(x_k)(= v_k)$ in the above definition.

The Riemannian gradient of $f : M^n \to \mathbb{R}$ at $\bar{x}$ is a vector in $T_{\bar{x}}M^n$, denoted by $\mathrm{grad}f(\bar{x})$, uniquely defined by

$$\langle \mathrm{grad}f(\bar{x}), v\rangle_{\bar{x}} = df(\bar{x})[v],\ \forall v \in T_{\bar{x}}M^n.$$

It is worth to mention that $\mathrm{grad}f(\bar{x}) \in T_{\bar{x}}M^n$ and $df(\bar{x}) : T_{\bar{x}}M^n \to \mathbb{R}$.

Definition 2.2. [21, 22, 27] Let M^n be a complete Riemannian manifold, $f : M^n \to \mathbb{R}$, $\bar{x} \in M^n$, and $v \in T_{\bar{x}}M^n$.

(i) Assume that $\gamma : (-\varepsilon,\varepsilon) \to M^n$ is a smooth curve satisfying $\gamma(0) = \bar{x}, \gamma'(0) = v$ and $f \circ \gamma : (-\varepsilon,\varepsilon) \to \mathbb{R}$ is smooth. Then, we have the following Taylor expansion

$$f(\gamma(t)) = f(\bar{x}) + t\langle \mathrm{grad}f(\bar{x}), v\rangle_{\bar{x}} + o(t),$$

where $o(t)/t \to 0$ when $t \to 0$.

(ii) If the curve is obtained by $\gamma(t) = \exp_{\bar{x}}(tv)$, then

$$f(\exp_{\bar{x}}(tv)) = f(\bar{x}) + t\langle \mathrm{grad}f(\bar{x}), v\rangle_{\bar{x}} + o(t).$$

(iii) If M^n is a Hadamard manifold, then $\gamma(t) = \gamma_{\bar{x},x}(t) = \exp_{\bar{x}}(t\exp_{\bar{x}}^{-1}(x))$, where $x = \exp_{\bar{x}}(v)$, and

$$f(\gamma_{\bar{x},x}(t)) = f(\bar{x}) + t\langle \mathrm{grad}f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} + o(t).$$

Definition 2.3. [25, 26] Let M^n be a Riemannian manifold.

(i) A subset S of M^n is said to be geodesic convex if, for any pair of distinct points $x, y \in S$, there is a unique minimizing geodesic $\gamma_{x,y} : [0,1] \to M^n$ joining x to y such that $\gamma_{x,y}(t) \in S$, $\forall t \in [0,1]$. A singleton is geodesic convex.

(ii) Let S be a geodesic convex of M^n and $f : S \to \mathbb{R}$. The function f is said to be geodesic convex at $\bar{x}$ if, for any point $x \in S$ and for any geodesic $\gamma_{\bar{x},x} : [0,1] \to M^n$ joining $\bar{x}$ to x,

$$f(\gamma_{\bar{x},x}(t)) \le tf(\bar{x}) + (1-t)f(x), \forall t \in [0,1].$$

We say that f is geodesic convex on S if f is geodesic convex at any point of S. If M^n is a Hadamard manifold, then f is geodesic convex at $\bar{x}$ if and only if

$$f(\exp_{\bar{x}}(t\exp_{\bar{x}}^{-1}(x))) \le tf(\bar{x}) + (1-t)f(x), x \in S, \forall t \in [0,1].$$

Proposition 2.3. [21, 22, 25, 26] *Let S be a geodesic convex of a complete Riemannian manifold M^n, $\bar{x} \in S$, and $f : S \to \mathbb{R}$ be a smooth function in a neighborhood of $\bar{x}$. Denote by $\Gamma_{\bar{x},x}$ the set of all geodesics joining the points $\bar{x}$ and $x \in S$.*

(i) f *is geodesic convex at* $\bar{x}$ *if and only if*

$$f(\gamma_{\bar{x},x}(t)) - f(\bar{x}) \geq \langle \operatorname{grad} f(\bar{x}), \gamma'_{\bar{x},x}(0)\rangle_{\bar{x}}, \forall t \in [0,1], \forall \gamma_{\bar{x},x} \in \Gamma_{\bar{x},x}.$$

(ii) Let $v \in T_{\bar{x}}M^n$ *such that* $\exp_{\bar{x}}(tv) \in S$*. If* f *is geodesic convex at* $\bar{x}$*, then*

$$f(\exp_{\bar{x}}(tv)) - f(\bar{x}) \geq \langle \operatorname{grad} f(\bar{x}), v\rangle_{\bar{x}}, \forall t \in [0,1].$$

(iii) Suppose further that the inverse map $\exp_{\bar{x}}^{-1} : M^n \to T_{\bar{x}}M^n$ *exists. If* f *is geodesic convex at* $\bar{x}$*, then*

$$f(\exp_{\bar{x}}(t\exp_{\bar{x}}^{-1}(x))) - f(\bar{x}) \geq \langle \operatorname{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}}, \forall t \in [0,1],$$

Especially, if f *is geodesic convex at* $\bar{x}$*, then*

$$f(x) - f(\bar{x}) \geq \langle \operatorname{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}}, \forall x \in S.$$

Proposition 2.4. [25, 26] *Let* $S \subset \mathscr{M}^n$ *be an open geodesic convex set, and* $f : S \to \mathbb{R}$ *be a twice continuously differentiable function. Then,* f *is geodesic convex on* S *iff the following geodesic Hessian (or second covariant derivative) is positive semidefinite at all the points of each geodesic convex coordinate neighbourhood of* S*:*

$$H_u^g f(x(u)) = J(x(u))^T H_x f(x(u)) J(x(u)) + \nabla_x f(x(u))(Hx(u) - Jx(u)\Gamma(u)),$$

where matrix $\Gamma(u)$ *is the second Christoffel symbols with respect to the Riemannian metric of* $\mathscr{M}^n$ *representation.*

Proposition 2.5. [25, 26] *Let* $S \subset \mathbb{R}^n$ *be an open geodesic convex set, and let* $f : S \to \mathbb{R}$ *be a twice continuously differentiable function. Then,* f *is geodesic convex on* S *iff the following matrix is positive semidefinite at all the points of each geodesic convex coordinate neighbourhood* $x(u)$*, with* $x : U \subset \mathbb{R}^k \to \mathbb{R}^n$*, of* S*:* $H^g f(x) = Hf(x) + \nabla f(x)\Gamma$*, where matrix* Γ *is the second Christoffel symbols with respect to the Riemannian metric of* $\mathbb{R}^n$*.*

In some cases, it is difficult to check the geodesic convexity assumptions via the definitions. Then, Proposition 2.4 and Proposition 2.5 could be utilized to verify the geodesic convexity as in the following example.

Example 2.1. Let $\mathscr{E} = \mathbb{R}^3$ and $M^2 = \mathbb{S}^2 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. Then, M^2 is a Riemannian manifold with the usual metric $\langle u, v\rangle_{\bar{x}} = \langle u, v\rangle, \forall u, v \in T_{\bar{x}}M^2$, where $= \langle ., .\rangle$ is the standard inner product on $\mathbb{R}^3$. Let $S = \{x \in M^2 \mid x_3 \leq -\frac{\sqrt{3}}{2}\}$, $\bar{x} = (0,0,-1)$, and $f_i : M^2 \to \mathbb{R} (i = 1, ..., 3)$ be defined by $f_1(x) = d(\bar{x}, x) = \arccos(-x_3)$, $f_2(x) = -x_3^2 (= x_1^2 + x_2^2 - 1)$, and $f_3(x) = x_3, f_4(x) = x_1^2$. Then, we can check that S is a geodesic convex subset of M^2. Now, we check that f_1, f_2 are geodesic convex on S. By the similar approach to the approach of Example 6.5.1 in [25], we introduce the coordinate representation of S as follows

$$x(u) = (x_1(u_1,u_2), x_2(u_1,u_2), x_3(u_1,u_2)) = (u_1, u_2, -\sqrt{1 - u_1^2 - u_2^2}),$$

$$(u_1,u_2) \in U = \left\{ u \in \mathbb{R}^2 \mid -\frac{1}{2} \leq u_1 \leq \frac{1}{2}, -\sqrt{\frac{1}{4} - u_1^2} \leq u_2 \leq \sqrt{\frac{1}{4} - u_1^2} \right\}.$$

Then, by some calculations,

$$Jx(u)=\begin{bmatrix} 1 & 0\\ 0 & 1\\ \frac{u_1}{\sqrt{1-u_1^2-u_2^2}} & \frac{u_2}{\sqrt{1-u_1^2-u_2^2}}\end{bmatrix}, H_{x_1}(u)=H_{x_2}(u)=\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix}$$

$$H_{x_3}(u)=\begin{bmatrix} \frac{1-u_2^2}{(1-u_1^2-u_2^2)^{3/2}} & \frac{u_1u_2}{(1-u_1^2-u_2^2)^{3/2}}\\ \frac{u_1u_2}{(1-u_1^2-u_2^2)^{3/2}} & \frac{1-u_1^2}{(1-u_1^2-u_2^2)^{3/2}}\end{bmatrix}, f_1(x(u))=\arccos(-x_3(u))=-\arccos(\sqrt{1-u_1^2-u_2^2}),$$

$$\nabla_x f_1(x(u))=\begin{bmatrix} 0\\ 0\\ \frac{1}{\sqrt{1-x_3^2}}\end{bmatrix}, H_x f_1(x(u))=\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \frac{x_3}{(1-x_3^2)^{3/2}}\end{bmatrix},$$

$$\mathscr{G}=Jx(u)^TJx(u)=\begin{bmatrix} \frac{1-u_2^2}{1-u_1^2-u_2^2} & \frac{u_1u_2}{1-u_1^2-u_2^2}\\ \frac{u_1u_2}{1-u_1^2-u_2^2} & \frac{1-u_1^2}{1-u_1^2-u_2^2}\end{bmatrix}, \mathscr{G}^{-1}=\begin{bmatrix} 1-u_1^2 & -u_1u_2\\ -u_1u_2 & 1-u_2^2\end{bmatrix},$$

$$\Gamma^{u_1}=\begin{bmatrix} \frac{u_1(1-u_2^2)}{1-u_1^2-u_2^2} & \frac{u_1^2u_2}{1-u_1^2-u_2^2}\\ \frac{u_1^2u_2}{1-u_1^2-u_2^2} & \frac{u_1(1-u_1^2)}{1-u_1^2-u_2^2}\end{bmatrix}, \Gamma^{u_2}=\begin{bmatrix} \frac{u_2(1-u_2^2)}{1-u_1^2-u_2^2} & \frac{u_1u_2^2}{1-u_1^2-u_2^2}\\ \frac{u_1u_2^2}{1-u_1^2-u_2^2} & \frac{u_2(1-u_1^2)}{1-u_1^2-u_2^2}\end{bmatrix}.$$

Hence,

$$\begin{aligned}
& H_u^g f_1(x(u))\\
= {} & Jx(u)^T H_x f_1(x(u))Jx(u)+\nabla_x f_1(x(u))(Hx(u)-Jx(u)\Gamma(u))\\
= {} & Jx(u)^T H_x f_1(x(u))Jx(u)+\sum_{i=1}^{3}\frac{\partial}{\partial x_i}f_1(x(u))(Hx_i(u)-Jx_i(u)\Gamma(u))\\
= {} & \begin{bmatrix} 1 & 0 & \frac{u_1}{\sqrt{1-u_1^2-u_2^2}}\\ 0 & 1 & \frac{u_2}{\sqrt{1-u_1^2-u_2^2}}\end{bmatrix}.\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \frac{x_3}{(1-x_3^2)^{3/2}}\end{bmatrix}.\begin{bmatrix} 1 & 0\\ 0 & 1\\ \frac{u_1}{\sqrt{1-u_1^2-u_2^2}} & \frac{u_2}{\sqrt{1-u_1^2-u_2^2}}\end{bmatrix}\\
+ {} & 0.\left(\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix}-(1.\Gamma^{u_1}+0.\Gamma^{u_2})\right)+0.\left(\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix}-(0.\Gamma^{u_1}+1.\Gamma^{u_2})\right)\\
+ {} & \frac{x_3}{(1-x_3^2)^{3/2}}.\left(\begin{bmatrix} \frac{1-u_2^2}{(1-u_1^2-u_2^2)^{3/2}} & \frac{u_1u_2}{(1-u_1^2-u_2^2)^{3/2}}\\ \frac{u_1u_2}{(1-u_1^2-u_2^2)^{3/2}} & \frac{1-u_1^2}{(1-u_1^2-u_2^2)^{3/2}}\end{bmatrix}-\left(\frac{u_1}{\sqrt{1-u_1^2-u_2^2}}.\Gamma^{u_1}+\frac{u_2}{\sqrt{1-u_1^2-u_2^2}}.\Gamma^{u_2}\right)\right)\\
= {} & \frac{\sqrt{1-u_1^2-u_2^2}}{(u_1^2+u_2^2)^{3/2}}\begin{bmatrix} u_2^2 & -u_1u_2\\ -u_1u_2 & u_1^2\end{bmatrix}
\end{aligned}$$

is a positive semidefinite matrix. This together with Theorem 6.3.1 in [25] entails that f_1 is geodesic convex; see another approach in Example 3 in [13]. Similarly, we deduce from

$$H_u^g f_2(x(u))=\frac{2}{1-u_1^2-u_2^2}\begin{bmatrix} u_2^4+(u_1^2-2)u_2^2-2u_1^2+1 & -u_1u_2(u_1^2+u_2^2)\\ -u_1u_2(u_1^2+u_2^2) & u_1^4+(u_2^2-2)u_1^2-2u_2^2+1\end{bmatrix},$$

$$\varphi(u_2^2)=u_2^4+(u_1^2-2)u_2^2-2u_1^2+1=u_2^4+u_1^2u_2^2+1-2(u_1^2+u_2^2)$$

$$\geq u_4^2+u_1^2u_2^2+1-2.\frac{1}{4}\geq 0(\text{since } u_1^2+u_2^2\leq \frac{1}{4}),$$
$$\det(H_u^g f_2(x(u)))=8\left(\frac{1}{2}-u_1^2-u_2^2\right)\geq 0$$
that f_2 are geodesic convex; see also Remark 3 in [10]. Similarly, since
$$H_u^g f_3(x(u))=\frac{1}{\sqrt{1-u_1^2-u_2^2}}\begin{bmatrix} 1-u_2^2 & u_1u_2 \\ u_1u_2 & 1-u_1^2\end{bmatrix},$$
$$H_u^g f_4(x(u))=\frac{2}{1-u_1^2-u_2^2}\begin{bmatrix} u_1^2(u_2^2-2)+1-u_2^2 & -u_1^3u_2 \\ -u_1^3u_2 & u_1^2(u_1^2-1)\end{bmatrix},$$
f_3 is geodesic convex and f_4 is not geodesic convex. By calculating directly, we also can check that f_4 is not geodesic convex as follows. For $x=(\frac{1}{10},\frac{3\sqrt{2}}{10},-\frac{9}{10}),x'=(\frac{1}{10},-\frac{3\sqrt{2}}{10},-\frac{9}{10})\in S$, one has
$$\langle x,x'\rangle=\frac{16}{25},\sqrt{1-\langle x,x'\rangle^2}=\frac{3\sqrt{41}}{25},d(x,x')=\arccos\left(\frac{16}{25}\right).$$
Hence, the geodesic [13, 26] joining from x to x' in S is
$$\gamma_{x,x'}(t)=\left(\cos\left(t\arccos\left(\frac{16}{25}\right)\right)-\frac{16}{3\sqrt{41}}\sin\left(t\arccos\left(\frac{16}{25}\right)\right)\right)x$$
$$+\frac{25\sin\left(t\arccos\left(\frac{16}{25}\right)\right)}{3\sqrt{41}}x',t\in[0,1].$$
Thus, for $t=\frac{1}{2}$,
$$f_4\left(\gamma_{x,x'}(\frac{1}{2})\right)=\left(\frac{1}{10}\cos\left(\frac{1}{2}\arccos\left(\frac{16}{25}\right)\right)+\frac{3}{10\sqrt{41}}\sin\left(\frac{1}{2}\arccos\left(\frac{16}{25}\right)\right)\right)^2\approx 0.0122$$
$$>0.01=\frac{1}{2}.\left(\frac{1}{10}\right)^2+\frac{1}{2}.\left(\frac{1}{10}\right)^2=\frac{1}{2}f_4(x)+\frac{1}{2}f_4(x'),$$
leading that f_4 is not geodesic convex on S.

As there exists an isometry identifying a n-dimensional Euclidean space $\mathscr{E}$ and $\mathbb{R}^n$, the following lemmas utilized frequently in the proof of the optimality conditions of semi-infinite programming can be checked similarly to the results in [29, 30, 31].

Lemma 2.1. *Let $\{C_t|t\in\Gamma\}$ be an arbitrary collection of nonempty convex sets in $\mathscr{E}$ and $K:=\text{pos}\left(\bigcup_{t\in\Gamma}C_t\right)$. Then, every nonzero vector of K can be expressed as a non-negative linear combination of n or fewer linear independent vectors, each belonging to a different C_t.*

Lemma 2.2. *Suppose that S,P are arbitrary (possibly infinite) index sets, $a_s=a(s)=(a_1(s),...,a_n(s))$ maps S onto $\mathscr{E}$, and so does a_p. Suppose that the set $\text{co}\{a_s,s\in S\}+\text{pos}\{a_p,p\in P\}$ is closed. Then the following statements are equivalent:*
$$I:\quad \begin{cases}\langle a_s,x\rangle<0,s\in S,S\neq\emptyset\\ \langle a_p,x\rangle\leq 0,p\in P\end{cases}\quad \textit{has no solution } x\in\mathscr{E};$$
$$II:\qquad 0\in\text{co}\{a_s,s\in S\}+\text{pos}\{a_p,p\in P\}.$$

Lemma 2.3. *If X is a nonempty compact subset of $\mathscr{E}$, then*

(i) co$X$ *is a compact set;*
(ii) *If* $0 \notin$ co$X$*, then* pos$X$ *is a closed cone.*

In this paper, we consider the following constrained semi-infinite programming:

(P) : min $f(x)$
s.t. $g_\ell(x) \leq 0, \ \ell \in L$,

where $M^n \subset \mathscr{E}$ is a complete Riemannian manifold fulfilling that $\exp_x^{-1} : M^n \to T_xM^n$ exists for all $x \in M^n$, and $f, g_\ell : M^n \to \mathbb{R}$ are smooth geodesic convex functions on M^n. The index set L is an arbitrary nonempty set, not necessary finite. A point $\bar{x}$ is a locally optimal solution to (P) if there exists a (topological) neighborhood U of $\bar{x}$ such that $f(x) \geq f(\bar{x})$, $\forall x \in \Omega \cap U$, where $\Omega := \{x \in M^n \mid g_\ell(x) \leq 0, \ \ell \in L\}$ is the feasible solution set of (P). If $U = M^n$, the word "locally" is omitted. The solution set of problem (P) is defined by

$$S := \{x \in M^n \mid f(x) \leq f(x'), \forall x' \in \Omega\}.$$

For a given $\bar{x} \in M^n$, define $L(\bar{x}) := \{\ell \in L | g_\ell(\bar{x}) = 0\}$. The set of active constraint multipliers at $\bar{x} \in \Omega$ is

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}_+^{|L|} | \lambda_\ell g_\ell(\bar{x}) = 0, \forall \ell \in L\}.$$

Notice that $\lambda \in \Lambda(\bar{x})$ if there exists a finite index set $J \subset L(\bar{x})$ such that $\lambda_\ell > 0$ for all $\ell \in J$ and $\lambda_\ell = 0$ for all $\ell \in L \setminus J$.

In the sequel, we always use the notion co$A$, pos$A$, A^-, A^s for a given subset A of the tangent space $T_xM^n := \mathscr{E}_1$, which is a linear subspace of $\mathscr{E}$. The linearizing cone to Ω at $\bar{x}$ is the set defined by

$$\mathscr{L}(\bar{x}) := \{d \in T_{\bar{x}}M^n \mid \langle \mathrm{grad} g_\ell(\bar{x}), d\rangle_{\bar{x}} \leq 0, \forall \ell \in L(\bar{x})\}.$$

We can check that $\mathscr{L}(\bar{x}) = \left(\bigcup_{\ell \in L(\bar{x})} \mathrm{grad} g_\ell(\bar{x}) \right)^-$.

Definition 2.4. (i) [19] The (ACQ) holds at $\bar{x} \in \Omega$ if $\mathscr{L}(\bar{x}) \subseteq \mathscr{T}(\Omega, \bar{x})$, and the set $\Delta :=$ pos $\bigcup_{\ell \in L(\bar{x})} \mathrm{grad} g_\ell(\bar{x})$ is closed.

(ii) The (GCQ) holds at $\bar{x} \in \Omega$ if $\mathscr{L}(\bar{x}) \subseteq \mathrm{clcone}\mathscr{T}(\Omega, \bar{x})$, and the set $\Delta :=$ pos $\bigcup_{\ell \in L(\bar{x})} \mathrm{grad} g_\ell(\bar{x})$ is closed.

It is easy to see that $(ACQ) \Rightarrow (GCQ)$. Moreover, $\mathscr{L}(\bar{x}) \subseteq \mathrm{clcone}\mathscr{T}(\Omega, \bar{x})$ implies that $(\mathrm{clcone}\mathscr{T}(\Omega, \bar{x}))^- \subseteq \mathscr{L}(\bar{x})^-$.

Proposition 2.6. *Suppose that $\bar{x}$ is a locally optimal solution of (P) and (GCQ) holds at $\bar{x}$. Then, there exists $\lambda \in \Lambda(\bar{x})$ such that* $\mathrm{grad} f(\bar{x}) + \sum_{\ell \in L} \lambda_\ell \mathrm{grad} g_\ell(\bar{x}) = 0.$

Proof. We first justify that

$$-\mathrm{grad} f(\bar{x}) \in (\mathscr{T}(\Omega, \bar{x}))^- = (\mathrm{clcone}\mathscr{T}(\Omega, \bar{x}))^-. \tag{2.1}$$

$*$ *Case 1.* $\mathrm{grad} f(\bar{x}) = 0$. Then, $-\mathrm{grad} f(\bar{x}) = 0 \in (\mathscr{T}(\Omega, \bar{x}))^-$. Hence, (2.1) holds for the *Case 1*.

$*$ *Case 2.* $\operatorname{grad} f(\bar{x}) \neq 0$. Then, for an arbitrary $v \in \mathscr{T}(\Omega, \bar{x})$, there exist $\tau_k \downarrow 0$ and $v_k \to v$ such that $\exp_{\bar{x}}(\tau_k v_k) \in \Omega$ for all k. We derive from the Taylor expansion of f at $\bar{x}$ that

$$f(\exp_{\bar{x}}(\tau_k v_k)) = f(\bar{x}) + \tau_k \langle \operatorname{grad} f(\bar{x})), v_k \rangle_{\bar{x}} + o(\tau_k).$$

We deduce from the fact $\bar{x}$ is a locally optimal solution of (P) that

$$0 \leq \frac{f(\exp_{\bar{x}}(\tau_k v_k)) - f(\bar{x})}{\tau_k} = \langle \operatorname{grad} f(\bar{x}), v_k \rangle_{\bar{x}} + \frac{o(\tau_k)}{\tau_k}.$$

Letting $k \to \infty$, one has $0 \leq \langle \operatorname{grad} f(\bar{x}), v \rangle_{\bar{x}}$, $\forall v \in \mathscr{T}(\Omega, \bar{x})$, or equivalently, $\langle -\operatorname{grad} f(\bar{x}), v \rangle_{\bar{x}} \leq 0$, $\forall v \in \mathscr{T}(\Omega, \bar{x})$, i.e., $-\operatorname{grad} f(\bar{x}) \in (\mathscr{T}(\Omega, \bar{x}))^-$. Hence, claim (2.1) holds for *Case 2*, leading that (2.1) satisfies both two possibilities.

We derive from (2.1) and (GCQ) that $-\operatorname{grad} f(\bar{x}) \in (\operatorname{clcone} \mathscr{T}(\Omega, \bar{x}))^- \subseteq \mathscr{L}(\bar{x})^-$, i.e.,

$$\langle -\operatorname{grad} f(\bar{x}), v \rangle_{\bar{x}} \leq 0, \forall v \in \mathscr{L}(\bar{x}). \tag{2.2}$$

Now we prove that there is no $v \in T_{\bar{x}} M^n$ fulfilling

$$\begin{cases} \langle \operatorname{grad} f(\bar{x})), v \rangle_{\bar{x}} < 0, \\ \langle \operatorname{grad} g_\ell(\bar{x})), v \rangle_{\bar{x}} \leq 0, \quad \forall \ell \in L(\bar{x}). \end{cases} \tag{2.3}$$

Suppose to the contrary that there is $\bar{v} \in T_{\bar{x}} M^n$ such that

$$\begin{cases} \langle \operatorname{grad} f(\bar{x})), \bar{v} \rangle_{\bar{x}} < 0, \\ \langle \operatorname{grad} g_\ell(\bar{x})), \bar{v} \rangle_{\bar{x}} \leq 0, \quad \forall \ell \in L(\bar{x}). \end{cases}$$

This leads that

$$\begin{cases} \langle -\operatorname{grad} f(\bar{x})), \bar{v} \rangle_{\bar{x}} > 0, \\ \bar{v} \in \mathscr{L}(\bar{x}), \end{cases}$$

contradicting (2.2), which in turn implies that (2.3) holds.

Moreover, we deduce from Lemma 2.3 that $\operatorname{cograd} f(\bar{x})$ is a compact set. Hence, $\operatorname{cograd} f(\bar{x}) + \Delta$ is closed. Combining this and (2.3), one deduces from Lemma 2.2 that

$$0 \in \operatorname{cograd} f(\bar{x}) + \operatorname{pos} \bigcup_{\ell \in L(\bar{x})} \operatorname{grad} g_\ell(\bar{x}).$$

In view of Lemma 2.1, there exists $\lambda \in \Lambda(\bar{x})$ such that

$$\operatorname{grad} f(\bar{x}) + \sum_{\ell \in L} \lambda_\ell \operatorname{grad} g_\ell(\bar{x}) = 0.$$

This completes the proof. □

Remark 2.1. The KKT necessary optimality conditions for semi-infinite on Hadarmard manifolds under (ACQ) was established in [19]. That results is generalized for semi-infinite on Riemannian manifolds under (GCQ) in Proposition 2.6. It should be noted that the Hadamard manifolds has sectional curvature is less than or equal zero, while the sectional curvature of Riemann manifolds could be is greater than zero. Moreover, the (GCQ) could be applied in some cases that (ACQ) could not be applied as in the following example.

Example 2.2. Let $M^2 = \{x \in \mathbb{R}^2 \mid x_1, x_2 > 0\}$. Then, M^2 is a Riemannian manifold (see, e.g., [25]) with the metric

$$\langle u, v \rangle_x = \langle \mathscr{G}(x) u, v \rangle, \forall u, v \in T_x M^2 = \mathbb{R}^2,$$

where $\mathscr{G}(x) = \begin{bmatrix} \frac{1}{x_1^2} & 0 \\ 0 & \frac{1}{x_2^2} \end{bmatrix}$, and $\langle .,. \rangle$ is standard inner product on $\mathbb{R}^2$. M^2 is also a Hadamard manifold as its sectional curvature is equal to zero. The Riemannian distance function is

$$d(x,y) = \left\| \left(\ln \frac{x_1}{y_1}, \ln \frac{x_2}{y_2} \right) \right\|, \forall x, y \in M^2.$$

For $x \in M^2$, the exponent map $\exp_x : T_x M^2 \to M^2$ is defined by

$$\exp_x(v) = \left(x_1 e^{\frac{v_1}{x_1}}, x_2 e^{\frac{v_2}{x_2}} \right), v \in T_x M^2.$$

Furthermore, $\exp_x^{-1} : M^2 \to T_x M^2$ is $\exp_x^{-1}(y) = \left(x_1 \ln \frac{y_1}{x_1}, x_2 \ln \frac{y_2}{x_2} \right)$.

Consider the problem

$$\begin{aligned} &(P) : \min f(x) = 2\sqrt{x_1} + \ln x_2 \\ &\text{s.t.}\ \ g_\ell(x) = 1 - (1-\ell)\ln x_1 - \ell \ln x_2 \le 0, \ \ \ell \in L_1 = [0,1], \\ &\qquad g_2(x) = \ln x_1 . \ln x_2 - \ln x_1 - \ln x_2 + 1 \le 0, \end{aligned}$$

where $f, g_\ell : M^2 \to \mathbb{R}$ and $L = [0,1] \cup \{2\}$. Then,

$$\Omega = \{x \in M^2 \mid \ln x_1 \ge 1, \ln x_2 \ge 1, (\ln x_1 - 1)(\ln x_2 - 1) \le 0\}.$$

Setting $t_1 = \ln x_1 - 1, t_2 = \ln x_2 - 1$, one has

$$\begin{aligned} \Omega &= \{t \in \mathbb{R}^2_+ \mid t_1 \ge 0, t_2 \ge 0, t_1 t_2 \le 0\} \\ &= \{t \in \mathbb{R}^2_+ \mid t_1 = 0, t_2 \ge 0\} \cup \{t \in \mathbb{R}^2_+ \mid t_1 \ge 0, t_2 = 0\} \\ &= \{x \in M^2 \mid x_1 = e, x_2 \ge e\} \cup \{x \in M^2 \mid x_1 \ge e, x_2 = e\}. \end{aligned}$$

It is easy to check that $S = \{(e,e)\}$. By employing some formulas in [25] to calculate, we obtain

$$\begin{aligned} \operatorname{grad} f(x) &= \mathscr{G}(x)^{-1} \begin{bmatrix} \frac{1}{\sqrt{x_1}} \\ \frac{1}{x_2} \end{bmatrix} = \begin{bmatrix} x_1\sqrt{x_1} \\ x_2 \end{bmatrix} = (x_1\sqrt{x_1}, x_2), \\ \operatorname{grad} g_\ell(x) &= \mathscr{G}(x)^{-1} \begin{bmatrix} -\frac{1-\ell}{x_1} \\ -\frac{\ell}{x_2} \end{bmatrix} = (-(1-\ell)x_1, -\ell x_2), \ell \in L_1, \\ \operatorname{grad} g_2(x) &= \mathscr{G}(x)^{-1} \begin{bmatrix} \frac{\ln x_2 - 1}{x_1} \\ \frac{\ln x_1 - 1}{x_2} \end{bmatrix} = (x_1(\ln x_2 - 1), x_2(\ln(x_1 - 1)). \end{aligned}$$

Taking $\bar{x} = (e,e) \in S$, one has $L(\bar{x}) = L$. Let us take $v \in \mathscr{T}(\Omega, \bar{x})$. Then, there exist $\tau_k \downarrow 0$ and $v_k \in T_{\bar{x}} M^2 = \mathbb{R}^2$ with $v^k = (v_1^k, v_2^k) \to v = (v_1, v_2)$ such that

$$\exp_{\bar{x}}(\tau_k v^k) = \left(e.e^{\frac{\tau_k v_1^k}{e}}, e.e^{\frac{\tau_k v_2^k}{e}} \right) = \left(e^{\frac{\tau_k v_1^k}{e}+1}, e^{\frac{\tau_k v_2^k}{e}+1} \right) \in \Omega, \forall k,$$

which leads to

$$\left\{ \begin{array}{l} e.e^{\frac{\tau_k v_1^k}{e}} = e \\ e.e^{\frac{\tau_k v_2^k}{e}} \ge e, \end{array} \right. \forall k \text{ or } \left\{ \begin{array}{l} e.e^{\frac{\tau_k v_1^k}{e}} \ge e \\ e.e^{\frac{\tau_k v_2^k}{e}} = e, \end{array} \right. \forall k.$$

Consequently,

$$\left\{ \begin{array}{l} \frac{\tau_k v_1^k}{e} = 0 \\ \frac{\tau_k v_2^k}{e} \ge 0, \end{array} \right. \forall k \text{ or } \left\{ \begin{array}{l} \frac{\tau_k v_1^k}{e} \ge 0 \\ \frac{\tau_k v_2^k}{e} = 0, \end{array} \right. \forall k,$$

or equivalently,

$$\begin{cases} v_1^k = 0 \\ v_2^k \geq 0, \end{cases} \forall k \text{ or } \begin{cases} v_1^k \geq 0 \\ v_2^k = 0, \end{cases} \forall k.$$

Letting k to infinity, one has that

$$\mathscr{T}(\Omega,\bar{x}) \subseteq \Gamma := \{x \in \mathbb{R}^2 \mid v_1 = 0, v_2 \geq 0\} \cup \{x \in \mathbb{R}^2 \mid v_1 \geq 0, v_2 = 0\}.$$

Moreover, it is easy to check that $\Gamma \subseteq \mathscr{T}(\Omega,\bar{x})$. Thus, $\mathscr{T}(\Omega,\bar{x}) = \Gamma$. Further,

$$\operatorname{grad} f(\bar{x}) = (e\sqrt{e}, e),$$

$$\operatorname{grad} g_\ell(\bar{x}) = (-(1-\ell)e, -\ell e), \forall \ell \in L_1, \operatorname{grad} g_2(\bar{x}) = \{(0,0)\},$$

$$\bigcup_{\ell \in L(\bar{x})} \operatorname{grad} g_\ell(\bar{x}) = \left\{x \in T_{\bar{x}}M^2 \mid x_1 + x_2 = -e, x_1 \leq 0, x_2 \leq 0\right\} \cup \{(0,0)\},$$

$$\left(\bigcup_{\ell \in L(\bar{x})} \operatorname{grad} g_\ell(\bar{x})\right)^- = \mathbb{R}^2_+ \not\subseteq \mathscr{T}(\Omega,\bar{x}), \left(\bigcup_{\ell \in L(\bar{x})} \operatorname{grad} g_\ell(\bar{x})\right)^- = \mathbb{R}^2_+ \subseteq \operatorname{clcone} \mathscr{T}(\Omega,\bar{x}),$$

$$\operatorname{pos} \bigcup_{\ell \in L(\bar{x})} \operatorname{grad} g_\ell(\bar{x}) = \left\{x \in T_{\bar{x}}M^2 \mid x_1 \leq 0, x_2 \leq 0\right\}$$

is closed, i.e., (ACQ) does not hold at $\bar{x}$. Hence, the result in [19] could not be applied. However, (GCQ) holds at $\bar{x}$, and thus, all the assumptions in Proposition 2.6 hold. Now, let $\bar{\lambda} : L \to \mathbb{R}$ be defined by

$$\bar{\lambda}(\ell) = \begin{cases} 1+\sqrt{e}, & \text{if } \ell = \frac{1}{1+\sqrt{e}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\bar{\lambda} \in \Lambda(\bar{x})$ and

$$\operatorname{grad} f(x) + \sum_{\ell \in L} \bar{\lambda}_\ell \operatorname{grad} g_\ell(x) = (e\sqrt{e}, e) + (1+\sqrt{e})\left(-\left(1 - \frac{1}{1+\sqrt{e}}\right)e, -\frac{1}{1+\sqrt{e}}e\right) = (0,0),$$

i.e., the conclusion of Proposition 2.6 is satisfied.

3. Characterization of Solution Sets

The characterization of solution sets of constrained optimizations in Euclidean spaces and their applications in giving the important information of the optimal solutions, such as the existence, the error bound, and the properties of solution methods were considered in [17, 32, 33, 34]. In this section, we give the characterization of solution sets of semi-infinite programming on Riemannian manifolds, which extends some above-mentioned results when L is a finite set and the objective function and constraint functions are smooth. For $\bar{x} \in S$ and the Lagrange multiplier $\bar{\lambda} \in \Lambda(\bar{x})$ with respect to $\bar{x}$, define

$$\widetilde{L}(\bar{x}) := \{\ell \in L(\bar{x}) \mid \bar{\lambda}_\ell > 0\},$$

$$\Omega_1 := \{x \in M^n \mid g_\ell(x) = 0, \forall \ell \in \widetilde{L}(\bar{x}), g_\ell(x) \leq 0, \forall \ell \in L \setminus \widetilde{L}(\bar{x})\},$$

and

$$\Omega_2 := \{x \in M^n \mid g_\ell(x) \leq 0, \forall \ell \in L \setminus \widetilde{L}(\bar{x})\}.$$

It is easy to see that $\widetilde{L}(\bar{x}) \subseteq L(\bar{x}) \subseteq L$ and $\Omega_1 \subseteq \Omega \subseteq \Omega_2$.

Lemma 3.1. *Suppose that $S \neq \emptyset$, i.e., there exists some $\bar{x} \in S$. Then, the following statements are true.*

(i) [25] *S is a geodesic convex set.*
(ii) $S = \{x \in M^n \mid f(x) = f(\bar{x})\}$.
(iii) S is a geodesic convex subset of Ω.

Proof. (i) See Lema 6.1.1 in [25].
(ii) ($\subseteq$) Let $x \in S$. Then, $x \in \Omega$. Moreover, since $x, \bar{x} \in S$, we have $f(x) \leq f(\bar{x})$ and $f(\bar{x}) \leq f(x)$, which leads $f(x) = f(\bar{x})$. Hence, $x \in \{x \in \Omega \mid f(x) = f(\bar{x})\}$, i.e., $S \subseteq \{x \in \Omega \mid f(x) = f(\bar{x})\}$.
($\supseteq$) Let $x \in \{x \in \Omega \mid f(x) = f(\bar{x})\}$. Then, $x \in \Omega$ and $f(x) = f(\bar{x}) \leq f(x'), \forall x' \in \Omega$. Hence, $x \in S$.
(iii) Let $\bar{x}, \bar{x}' \in S$. Then, for any $x \in \Omega$, one has $f(\bar{x}) \leq f(x), f(\bar{x}') \leq f(x)$. Since Ω is a geodesic convex set, there exists a unique minimizing geodesic $\gamma_{\bar{x},\bar{x}'}(t)$ joining $\bar{x}$ to $\bar{x}'$ containing in Ω. Hence, we deduce from the geodesic convexity of f that

$$f(\gamma_{\bar{x},\bar{x}'}(t)) \leq tf(\bar{x}) + (1-t)f(\bar{x}') \leq tf(x) + (1-t)f(x) = f(x), \forall t \in [0,1], \forall x \in \Omega.$$

Thus $\gamma_{\bar{x},\bar{x}'}(t) \in S$ for all $t \in [0,1]$, which demonstrates that S is geodesic convex. □

Example 3.1. Let $\mathscr{E} = \mathbb{R}^3$ and $M^2 = \mathbb{S}^2 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. Then, M^2 is a Riemannian manifold with the usual metric $\langle u, v\rangle_{\bar{x}} = \langle u, v\rangle, \forall u, v \in T_{\bar{x}}M^2$, where $\langle ., . \rangle$ is standard inner product on $\mathbb{R}^3$. Consider the problem

$$(P): \min f(x) = -x_3^2$$
$$\text{s.t.} \quad g_\ell(x) = x_3 - \ell \leq 0, \;\; \ell \in L = [-\tfrac{\sqrt{3}}{2}, 0],$$

where $f, g_\ell : \mathbb{S}^2 \to \mathbb{R}$. Then, $\Omega = \{x \in \mathbb{S}^2 \mid x_3 \leq -\frac{\sqrt{3}}{2}\}$ and $S = \{x \in \Omega \mid x_3 = -1\} = \{(0,0,-1)\}$. So, it follows from Definition 2.2 (i) that the conclusion of Lemma 3.1 holds.

Proposition 3.1. *Suppose that $\bar{x} \in S$, (GCQ) holds at $\bar{x}$ and there exists $\bar{\lambda} \in \Lambda(\bar{x})$ such that*

$$\text{grad} f(\bar{x}) + \sum_{\ell \in L} \bar{\lambda}_\ell \text{grad} g_\ell(\bar{x}) = 0. \tag{3.1}$$

Suppose that $f, g_\ell (\ell \in L)$ are geodesic convex at $\bar{x}$ on Ω. Then, for any $x \in S$, $\sum_{\ell \in L(\bar{x})} \bar{\lambda}_\ell g_\ell(x) = 0$ and $\mathscr{L}(., \bar{\lambda}) = f(.) + \sum_{\ell \in L(\bar{x})} \bar{\lambda}_\ell g_\ell(.)$ is constant on S.

Proof. Since $\bar{\lambda}_\ell g_\ell(\bar{x}) = 0$, $\forall \ell \in L$ and $g_\ell(\bar{x}) < 0, \forall \ell \notin L(\bar{x})$, we have $\bar{\lambda}_\ell = 0, \forall \ell \notin L(\bar{x})$. Hence,

$$\bar{\lambda}_\ell = 0, \forall \ell \notin \widetilde{L}(\bar{x}). \tag{3.2}$$

It is straightforward that, for each $x \in S$,

$$\sum_{\ell \in L(\bar{x})} \bar{\lambda}_\ell g_\ell(x) = 0 \Leftrightarrow \sum_{\ell \in \widetilde{L}(\bar{x})} \bar{\lambda}_\ell g_\ell(x) = 0 \Leftrightarrow g_\ell(x) = 0, \forall \ell \in \widetilde{L}(\bar{x}). \tag{3.3}$$

If $\widetilde{L}(\bar{x}) = \emptyset$, the proof is trivial. If otherwise, suppose to the contrary that there exist $\bar{\ell} \in \widetilde{L}(\bar{x})$ and $x \in S$ such that $g_{\bar{\ell}}(x) < 0 = g_{\bar{\ell}}(\bar{x})$. This together with the geodesic convexity of $g_{\bar{\ell}}$ at $\bar{x}$ implies that

$$\langle \text{grad} g_{\bar{\ell}}(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} \leq g_{\bar{\ell}}(x) - g_{\bar{\ell}}(\bar{x}) < 0. \tag{3.4}$$

For each $\ell \in \widetilde{L}(\bar{x}) \setminus \{\bar{\ell}\}$, we have $g_\ell(x) \le 0 = g_\ell(\bar{x})$, which in turn along with the geodesic convexity of $g_\ell(\ell \in L)$ at $\bar{x}$ derives

$$\langle \text{grad} g_\ell(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} \le g_\ell(x) - g_\ell(\bar{x}) \le 0, \forall \ell \in \widetilde{L}(\bar{x}) \setminus \{\bar{\ell}\}.$$

Combining this, (3.1), (3.2), and (3.4), one arrives at

$$\langle \text{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} = -\sum_{\ell \in L} \bar{\lambda}_\ell \langle \text{grad} g_\ell(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} = -\sum_{\ell \in \widetilde{L}(\bar{x})} \bar{\lambda}_\ell \langle \text{grad} g_\ell(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} > 0. \tag{3.5}$$

However, since $f(x) = f(\bar{x})$, we deduce from the fact f is geodesic convex at $\bar{x}$ that

$$\langle \text{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} \le f(x) - f(\bar{x}) \le 0,$$

contradicting (3.5). Hence, (3.3) holds. It follows from (3.2) and (3.3) that

$$\mathscr{L}(x, \bar{\lambda}) = f(x) + \sum_{\ell \in \widetilde{L}(\bar{x})} \bar{\lambda}_\ell g_\ell(x) = f(x) = f(\bar{x}), \ \forall x \in S,$$

i.e., $\mathscr{L}(., \bar{\lambda}) = f(.) + \sum_{\ell \in L(\bar{x})} \bar{\lambda}_t g_\ell(.)$ is constant on S. ☐

Now, we give some characterization of the solution sets of geodesic convex SIP on Riemannian manifolds. In our opinion, the solution set S_5 is more interesting since it give more information to verify that a feasible point is a solution.

Proposition 3.2. *Suppose that $\bar{x} \in S$, (GCQ) holds at $\bar{x}$, and there exists $\bar{\lambda} \in \Lambda(\bar{x})$ such that*

$$\text{grad} f(\bar{x}) + \sum_{\ell \in L} \bar{\lambda}_\ell \text{grad} g_\ell(\bar{x}) = 0. \tag{3.6}$$

If $f, g_\ell(\ell \in L)$ are geodesic convex at $\bar{x}$ on Ω, then the solution sets is characterized by

$$S = S_1 = S_2 = S_3 = S_4 = S_5,$$

where

(i) $S_1 = \{x \in \Omega_1 \mid \langle \text{grad} f(x), \exp_x^{-1}(\bar{x})\rangle_x = 0\}$,
(ii) $S_2 = \{x \in \Omega_1 \mid \langle \text{grad} f(x), \exp_x^{-1}(\bar{x})\rangle_x \ge 0\}$,
(iii) $S_3 = \{x \in \Omega_1 \mid \langle \text{grad} f(x), \exp_x^{-1}(\bar{x})\rangle_x = \langle \text{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}}\}$,
(iv) $S_4 = \{x \in \Omega_1 \mid \langle \text{grad} f(x), \exp_x^{-1}(\bar{x})\rangle_x \ge \langle \text{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}}\}$,
(v) $S_5 = \{x \in \Omega_1 \mid \langle \text{grad} f(x), \exp_x^{-1}(\bar{x})\rangle_x = \langle \text{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} = 0\}$.

Proof. It is immediate that $S_5 \subseteq S_1 \subseteq S_2$ and $S_5 \subseteq S_3 \subseteq S_4$. Hence, we only need to prove that $S \subseteq S_5$, $S_2 \subseteq S$ and $S_4 \subseteq S$.
(a) $(S \subseteq S_5)$ Let $x \in S$. Since $\bar{x} \in S$, we implies from Lemma 3.1 (ii) that $f(x) = f(\bar{x})$. This together with the geodesic convexity of f at $\bar{x}$ leads that

$$\langle \text{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} \le f(x) - f(\bar{x}) = 0. \tag{3.7}$$

In view of Proposition 3.1, we deduce from $x \in S$ that

$$g_\ell(x) = 0, \forall \ell \in \widetilde{L}(\bar{x}) \text{ and } \bar{\lambda}_\ell = 0, \forall \ell \notin \widetilde{L}(\bar{x}).$$

It follows from the above inequalities, (3.6), and the geodesic convexity of $g_\ell(\ell \in L)$ at $\bar{x}$ that

$$\begin{aligned}\langle \mathrm{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} &= -\sum_{\ell\in L}\bar{\lambda}_\ell\langle \mathrm{grad} g_\ell(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}}\\ &= -\sum_{\ell\in\widetilde{L}(\bar{x})}\bar{\lambda}_\ell\langle \mathrm{grad} g_\ell(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}}\\ &\geq -\sum_{\ell\in\widetilde{L}(\bar{x})}\bar{\lambda}_\ell(g_\ell(x)-g_\ell(\bar{x}))=0. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), one obtains

$$\langle \mathrm{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} = 0. \tag{3.9}$$

By invoking Lemma 3.1, we deduce from $x, \bar{x} \in S$ that, for any $\gamma_{x,\bar{x}}$,

$$f(\gamma_{x,\bar{x}}(t)) = f(x) = f(\bar{x}), \forall t \in [0,1].$$

Granting this and taking into account the Taylor expansion of f at x give us

$$0 = \frac{f(\gamma_{x,\bar{x}}(t)) - f(x)}{t} = \langle \mathrm{grad} f(x), \exp_x^{-1}(\bar{x})\rangle_x + \frac{o(t)}{t}.$$

Letting t to zero, we have

$$\langle \mathrm{grad} f(x), \exp_x^{-1}(\bar{x})\rangle_x = 0,$$

which in turn along with (3.9) concludes that $x \in S_5$.
(b) $(S_2 \subseteq S)$ Let $x \in S_2$. Then, $x \in \Omega_1 \subseteq \Omega$ and

$$\langle \mathrm{grad} f(x), \exp_x^{-1}(\bar{x})\rangle_{\bar{x}} \geq 0.$$

This together with the geodesic convexity of f at x tells us that $f(\bar{x}) - f(x) \geq 0$, leading that

$$f(x) \leq f(\bar{x}) \leq f(x'), \forall x' \in \Omega.$$

The proof is complete.
(c) Let $x \in S_4$. Then, $x \in \Omega_1 \subseteq \Omega$ and

$$\langle \mathrm{grad} f(x), \exp_x^{-1}(\bar{x})\rangle_x \geq \langle \mathrm{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}}.$$

By analyzing similarly as the proof in (a), one has

$$\langle \mathrm{grad} f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} = 0,$$

which implies that

$$\langle \mathrm{grad} f(x), \exp_x^{-1}(\bar{x})\rangle_x \geq 0.$$

Again from the geodesic convexity of f on Ω, we arrive at $f(x) \leq f(\bar{x})$. Thus $x \in S$. □

In our opinion, the solution set S_5 is more interesting since it give more information to verify the solution of semi-infinite programming on Riemannian manifolds.

Example 3.2. Let $\mathscr{E} = \mathbb{R}^3$ and $M^2 = \mathbb{S}^2 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. Then, M^2 is a Riemannian manifold (see, e.g., [13, 21, 27]) with the usual metric $\langle u, v\rangle_{\bar{x}} = \langle u, v\rangle$, $\forall u, v \in T_{\bar{x}}M^2$, where $=$ $\langle .,.\rangle$ is standard inner product on $\mathbb{R}^3$. The Riemannian distance function is $d(x,y) = \arccos\langle x, y\rangle$,

$\forall x, y \in M^2$, where $\arccos\langle x,y\rangle := \theta \in [0,\pi]$ satisfies $\cos\theta = \frac{\langle x,y\rangle}{\|x\|.\|y\|} = \langle x,y\rangle$. For $x \in M^2$, the exponent map $\exp_x : T_xM^2 \to M^2$ is defined by

$$\exp_x(v) = \begin{cases} \cos(\|v\|)x + \sin(\|v\|)\frac{v}{\|v\|}, & v \in T_xM^2 \setminus \{0\} \\ x, & v = 0. \end{cases}$$

Moreover, $\exp_x^{-1} : M^2 \to T_xM^2$ is

$$\exp_x^{-1}(y) = \begin{cases} \frac{\arccos\langle x,y\rangle}{\sin(\arccos\langle x,y\rangle)}(y - \langle x,y\rangle x), & y \notin \{x,-x\} \\ 0, & y = x. \end{cases}$$

Consider the problem

$$(P) : \min f(x) = -x_3^2$$
$$\text{s.t.}\quad g_\ell(x) = \ell x_3 + \ell - 1 \leq 0,\ \ \ell \in L = \left[0, \frac{2}{2-\sqrt{3}}\right],$$

where $f, g_\ell : \mathbb{S}^2 \to \mathbb{R}$. Then, $g_\ell(x) \leq 0, \forall \ell \in L$ if and only if

$$x_3 \leq \min_{\ell \in \left(0, \frac{2}{2-\sqrt{3}}\right]} \left\{-1 + \frac{1}{\ell}\right\} \Leftrightarrow x_3 \leq -\frac{\sqrt{3}}{2}.$$

Hence, $\Omega = \{x \in \mathbb{S}^2 \mid x_3 \leq -\frac{\sqrt{3}}{2}\}$ and we can check directly that

$$S = \{x \in \mathbb{S}^2 \mid x_3 = -1\} = \{(0,0,-1)\}.$$

By using some formulas in [13, 21, 27] to calculate, we have

$$\text{grad} f(x) = P_x((0,0,-2x_3)) = (2x_1x_3^2, 2x_2x_3^2, -2x_3 + 2x_3^3) = 2x_3(x_1x_3, x_2x_3, -1 + x_3^2),$$

$$\text{grad} g_\ell(x) = P_x((0,0,\ell)) = (-\ell x_1x_3, -\ell x_2x_3, \ell - \ell x_3^2), \ell \in L,$$

where $P_x = I_3 - xx^T$. Taking $\bar{x} = (0,0,-1) \in S$, one has

$$L(\bar{x}) = L, T_{\bar{x}}M^2 = \{v \in \mathbb{R}^3 \mid \langle v, \bar{x}\rangle = -v_3 = 0\} \text{ is a subspace of } \mathbb{R}^3.$$

Let us take $v \in \mathscr{T}(\Omega,\bar{x}) \setminus \{0\}$. Then, there exist $\tau_k \downarrow 0$ and $v_k \in T_{\bar{x}}M^n$ with $v^k = (v_1^k, v_2^k, v_3^k) \to v = (v_1, v_2, v_3)$ such that

$$\exp_{\bar{x}}(\tau_k v^k) = \cos(\|\tau_k v^k\|)\bar{x} + \sin(\|\tau_k v^k\|)\frac{\tau_k v^k}{\|\tau_k v^k\|} \in \Omega, \forall k,$$

which implies

$$\cos(\|\tau_k v^k\|)(-1) + \frac{\sin(\|\tau_k v^k\|)}{\|\tau_k v^k\|}\tau_k v_3^k \leq -\frac{\sqrt{3}}{2}, \forall k.$$

Letting k to infinity, one has that $-1 \leq -\frac{\sqrt{3}}{2}$, which satisfies for all $v \in \mathscr{T}(\Omega,\bar{x}) \setminus \{0\}$. It should be noted that $0 \in \mathscr{T}(\Omega,\bar{x})$. Hence,

$$\mathscr{T}(\Omega,\bar{x}) = \{v \in T_{\bar{x}}M^2 \mid v_3 \leq 0\} = \text{clcone}\mathscr{T}(\Omega,\bar{x}).$$

Moreover,

$$\text{grad} f(\bar{x}) = (0,0,0), \text{grad} g_\ell(\bar{x}) = (0,0,0), \bigcup_{\ell \in L(\bar{x})} \text{grad} g_\ell(\bar{x}) = \{(0,0,0)\},$$

$$\left(\bigcup_{\ell\in L(\bar{x})}\operatorname{grad}g_\ell(\bar{x})\right)^- = \left\{x^*\in T_{\bar{x}}M^2 \mid \langle x^*,(0,0,0)\rangle\le 0\right\} = T_{\bar{x}}M^2 \subseteq \mathscr{T}(\Omega,\bar{x}),$$

$$\operatorname{pos}\bigcup_{\ell\in L(\bar{x})}\operatorname{grad}g_\ell(\bar{x}) = \operatorname{pos}\{(0,0,0)\} = \{(0,0,0)\}$$

is closed, i.e., (GCQ) holds at $\bar{x}$. Now, let $\bar{\lambda}: L\to\mathbb{R}$ be defined by

$$\bar{\lambda}(\ell) = \begin{cases} 1, & \text{if } \ell = 0,\\ 0, & \text{otherwise.}\end{cases}$$

Then, $\bar{\lambda}\in\Lambda(\bar{x})$ and

$$\operatorname{grad}f(\bar{x}) + \sum_{\ell\in L}\bar{\lambda}_\ell\operatorname{grad}g_\ell(\bar{x}) = 0.$$

Furthermore, $\widetilde{L}(\bar{x}) = \{0\}$ and $\Omega_1 = \Omega$. Then,

$$\langle \operatorname{grad}f(\bar{x}), \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} = 0, \forall x\in\Omega_1\setminus\{\bar{x}\},$$

$$\begin{aligned}\frac{\sin(\arccos\langle x,\bar{x}\rangle)}{\arccos\langle x,\bar{x}\rangle}\langle \operatorname{grad}f(x), \exp_x^{-1}(\bar{x})\rangle_x &= \langle 2x_3(x_1x_3, x_2x_3, -1+x_3^2), \bar{x}-\langle\bar{x},x\rangle x)\rangle_x\\
&= 2x_3\left\langle (x_1x_3,x_2x_3,-1+x_3^2),(x_1x_3,x_2x_3,-1+x_3^2)\right\rangle\\
&= 2x_3\left(x_1^2x_3^2+x_2^2x_3^2+(-1+x_3^2)^2\right)\\
&= 2x_3\left(x_3^2(x_1^2+x_2^2)+1-2x_3^2+x_3^4\right)\\
&= 2x_3\left(x_3^2(1-x_3^2)+1-2x_3^2+x_3^4\right)\\
&= 2x_3(1-x_3)(1+x_3), \forall x\in\Omega\setminus\{\bar{x}\}.\end{aligned}$$

Since $-1\le x_3\le -\frac{\sqrt{3}}{2}\ \forall x\in\Omega\setminus\{\bar{x}\}$, one has

$$0<\arccos\langle x,\bar{x}\rangle\le\frac{\pi}{6},\ 0<\sin(\arccos\langle x,\bar{x}\rangle)\le\frac{1}{2}.$$

Hence,

$$\langle \operatorname{grad}f(x),\exp_x^{-1}(\bar{x})\rangle_x = 0 \Leftrightarrow 2x_3(1-x_3)(1+x_3)=0 \Leftrightarrow x_3=-1,$$
$$\langle \operatorname{grad}f(x),\exp_x^{-1}(\bar{x})\rangle_x \ge 0 \Leftrightarrow 1+x_3\le 0 \Leftrightarrow x_3=-1.$$

Moreover, it follows from Example 2.1 that f is geodesic convex and g_ℓ are geodesic convex for all $\ell\in L$. Hence,

$$S_1 = S_2 = \{x\in\Omega \mid x_3=-1\} = \{(0,0,-1)\}$$

and $S_1=S_2=S_3=S_4=S_5=S$, which yield that the conclusions of Proposition 3.2 hold.

Remark 3.1. (i) The semi-infinite programming could be served in formulating the robotics trajectory planning problems in Euclidean spaces in [35]. However, in some circumstances in real life, the robotics trajectory is planned on manifolds including surfaces, such as spheres, tori, and cylinders. The viewpoint of the authors is that the theory and algorithms of semi-infinite programming on manifolds in the future could be applied in these problems.
(ii) In the case that L is a finite set, the main results in the paper are also new since the characterization of solution sets of convex constrained optimizations have not been yet examined in [13, 18, 19]. Considering the characterization of solution sets of nonsmooth optimizations

on Riemannian manifolds by using the generalized subdifferentials on Riemannian manifolds could be an interesting subject for the future research.

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