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# PARAMETERIZED DOUGLAS-RACHFORD DYNAMICAL SYSTEM FOR MONOTONE INCLUSION PROBLEMS

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**Abstract.** Douglas-Rachford splitting method with resolvent operator is a renowned algorithm to solve monotone inclusion problem involving sum of two monotone operators. In this paper, we investigate a Douglas-Rachford-based dynamical systems designed to approach the solution sets of inclusion problems involving the sum of two maximally monotone operators. Our main aim is to use parametrized resolvent instead of classical resolvent as the Douglas-Rachford operator in the framework of preconditioning. The convergence of the orbit is demonstrated. We also add a Tikhonov regularized term (both inner and outer regularization) to obtain strong convergence of the induced orbit.
**Keywords.** Douglas-Rachford; Monotone inclusion; Tikhonov regularization; Preconditioning.

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## 1. Introduction

The inclusion problem involving the sum of two operators has attracted the focus of a great number of researchers. The inclusion problem is to find

$$x \in H \text{ such that } 0 \in Ax + Bx, \tag{$\mathscr{P}$}$$

where $H$ is a Hilbert space with inner product $\langle , \rangle$ and $A, B : H \to 2^H$ are set-valued operators. This structure is quite common in various domains, such as partial differential equations, mechanics, signal and image processing, convex optimization, statics, and game theory [1, 2, 3, 4].

Dynamical systems approach towards monotone inclusion problems were recognized as a valuable tool for discovering and studying numerical algorithms (see [5] for a recent survey). In 2019, Csetnek et al. [6] proposed and studied the continuous Douglas-Rachford dynamical system for problem ($\mathscr{P}$), which is as follows:

$$\begin{cases} \dot{z}(t) + z(t) = \left(\frac{I + R_{\lambda A} R_{\lambda B}}{2}\right) z(t), \\ z(0) = z_0 \in H, \end{cases}$$

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where $R_{\lambda A}, R_{\lambda B}$ are reflected resolvents of $A$ and $B$, respectively, $A, B : H \to 2^H$ are maximally monotone operators, and $\lambda > 0$. In 2020, Zhu et al. [7] analyzed the Douglas-Rachford dynamical system for minimizing the sum of strongly convex and weakly convex functions:

$$\begin{cases} \dot{z}(t) = \alpha(R_{\lambda f} R_{\lambda g} z(t) - z(t)), \\ z(0) = z_0 \in H, \end{cases}$$

where $\alpha > 0$, $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ are proper and closed, $f$ is strongly convex, and $g$ is weakly convex. Here, $R_{\lambda f} = 2\text{prox}_{\lambda f} - I$ and $\text{prox}_{\lambda f} x = \arg\min_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}$. The authors also studied the global exponential convergence of the generated orbit under some regularity conditions.

In 2021, Zhu et al. [8] investigated the convergence of the orbit of following dynamical system for problem ($\mathscr{P}$):

$$\begin{cases} \dot{z}(t) = \theta(t)(R^{\gamma}_{\lambda A} R^{\mu}_{\delta B} z(t) - z(t)) + f(t), \\ z(0) = z_0 \in H, \end{cases}$$

where $R^{\gamma}_{\lambda A} = (1 - \gamma I) + \gamma J_{\lambda A}, R^{\mu}_{\delta B} = (1 - \mu)I + \mu J_{\delta B}$, and $\gamma, \lambda, \delta, \mu > 0$, $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, and $f : \mathbb{R}_+ \to H$ are locally integrable function. Here, $A, B : H \to 2^H$ are maximally $\alpha, \beta$-monotone operators, respectively.

All these dynamical system have been formed in terms of the resolvents of the associated maps $A$ and $B$. Recently, in the papers [9, 10, 11], the authors introduced preconditioned resolvents for an set-valued operator involving an associated operator. The authors demonstrated some properties of preconditioned resolvents that generalize those of resolvents for an operator and also studied forward-backward splitting methods in the framework of preconditioning. Motivated by this fact, in this paper, we study the following dynamical system corresponding to problem ($\mathscr{P}$):

$$\begin{cases} \dot{z}(t) = \kappa(t)(R^{M}_{\beta A} R^{M}_{\beta B} z(t) - z(t)), \\ z(0) = z_0 \in H, \end{cases} \tag{1.1}$$

where $R^{M}_{\beta A}$ and $R^{M}_{\beta B}$ are preconditioned parameterized reflected resolvent corresponding to a preconditioning map $M : H \to H$.

Our main aim in the paper is to analyze dynamical system (1.1) and its asymptotic behaviour. More precisely,

(i) we study the weak convergence of the orbit under mild conditions on the associated preconditioning $M$ and the operators $A$ and $B$.

(ii) we introduce and study the Tikhonov regularized (both inner and outer) system associated with (1.1) and derive the strong convergence of the orbit.

The structure of the paper is as follows. In Section 2, we recollect some standard basics of monotone operators and resolvents. In Section 3, we study in detail the Douglas-Rachford dynamical system corresponding to the monotone inclusion problem involving preconditioning. In Section 4, the last section, we study Tikhonov regularization methods for the dynamical system.

## 2. Preliminaries

Throughout the paper, $H$ is assumed to be a Hilbert space. By $I$, we denote the identity operator. Let $T : H \to 2^H$. By $\mathscr{G}(T) := \{(x, x^*) \in H \times H \mid x^* \in T(x)\}$, we denote the graph of $T$. By $dom(T) := \{x \in H \mid T(x) \neq \emptyset\}$ and $ran(T) := \{x^* \in H \mid \exists\, x \in H$ such that $x^* \in T(x)\}$, we denote the domain and range of $T$, respectively. We also denote the fixed points of $T$ by $Fix(T) := \{x \in H \mid x \in T(x)\}$.

**Definition 2.1.** An operator $T : H \to 2^H$ is said to be

(i) monotone if $\langle x - y, u - v\rangle \geq \sigma \|x - y\|^2$, $\forall (x,u); (y,v) \in \mathscr{G}(T)$;
(ii) maximally monotone if there exists no monotone operator $S : H \to 2^H$ such that $\mathscr{G}(S)$ properly contains $\mathscr{G}(T)$;
(iii) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in H$;
(iv) firmly nonexpansive if $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$, $\forall x, y \in H$.

**Definition 2.2.** A nonexpansive operator $T : dom(T) \to H$ is said to be $\alpha$-averaged for $\alpha \in (0,1)$ if there exists a nonexpansive operator $R : dom(T) \to H$ such that $T = \alpha R + (1 - \alpha)I$.

The following proposition is a well-known characterization of averaged operators:

**Proposition 2.1.** [1] *Let $T : dom(T) \to H$ be a nonexpansive map and $\alpha \in (0,1)$. Then the following are equivalent:*

(i) *$T$ is $\alpha$-averaged.*
(ii) *For $x, y \in dom(T)$, $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1-\alpha}{\alpha}\|(I - T)x - (I - T)y\|^2$, $\forall x, y \in dom(T)$.*

The resolvent of a set-valued monotone operator $T : H \to 2^H$ is a single-valued Lipschitz continuous operator that is associated with $T$.

**Definition 2.3.** Let $T : H \to 2^H$ be a set-valued operator. Then resolvent of $T$ of index $\gamma > 0$ is defined by: $J_{\gamma T} := (I + \gamma T)^{-1}$.

The resolvents of operators play an important role in designing algorithms for monotone inclusion problem ($\mathscr{P}$). One refers to forward backward algorithms, Douglas-Rachford algorithms, forward-backward-forward algorithms, and so on; see [1] for an extensive overview.

Recently, in [9], the authors introduced and studied the preconditioned resolvent with the help of an auxiliary bounded linear operator, which generalize the concepts of the resolvent.

Let $M$ be a bounded linear operator on $H$. A self-adjoint operator $M$ on $H$ is said to be strongly positive of order $m > 0$ if $M - mI$ is positive definite and we denote all such operators by $\mathscr{S}_m(H) := \{M : H \to H \mid M$ is positive definite of order $m\}$. We define the $M$-inner product $\langle \cdot, \cdot \rangle_M$ on $H$ by $\langle x, y\rangle_M = \langle x, M(y)\rangle$ for all $x, y \in H$, where $M$ is a positive definite operator. The corresponding $M$-norm is defined by $\|x\|_M = \langle x, x\rangle_M$ for all $x \in H$.

**Definition 2.4** ([9])**.** Let $M \in \mathscr{S}_m(H)$, and $T : H \to 2^H$ be a set-valued operator. The preconditioned resolvent of $T$ of index $\gamma > 0$ is defined by $J^M_{\gamma T} = (I + \gamma MT)^{-1}$.

The preconditioned resolvent is sometime easier to compute than the actual resolvent (see [12]). The following property of preconditioned resolvent is taken from [9]

**Proposition 2.2** ([9])**.** *Let $M \in \mathscr{S}_m(H)$, and $T : H \to 2^H$ be monotone. Then $J^M_T$ is firmly nonexpansive.*

## 3. The Douglas-Rachford Dynamical System

In this section, we study the convergence of the orbit of a parametrized Douglas-Rachford dynamical system (1.1) for problem ($\mathscr{P}$). In an explicit form, it takes the following structure:

For $A, B : H \to 2^H$, consider the dynamical system

$$\begin{cases} x(t) = J_B^M z(t) \\ y(t) = J_A^M(\beta x(t) - z(t)) \\ \dot{z}(t) = \beta\kappa(t)(y(t) - x(t)) \\ z(0) = z_0 \in H, \end{cases} \tag{3.1}$$

where $k : [0,\infty) \to [0,\infty)$ is a Lebesgue measurable function and $\beta \in [1,2)$. Here $R_{\beta A}^M = \beta J_A^M - I$ and $R_{\beta B}^M = \beta J_B^M - I$ are called parametrized preconditioned reflected resolvents. In case of $\beta = 2$, we call them as preconditioned reflected resolvents.

For $M \in \mathscr{S}_m(H)$, the parametrized preconditioned reflected resolvent satisfies the property of averaged operators as we show below.

**Proposition 3.1.** *Let $T : H \to 2^H$ be a maximal monotone operator and $M \in \mathscr{S}_m(H)$. Then $-R_{\beta T}^M$ is $\frac{\beta}{2}$-averaged.*

*Proof.* For all $x, y \in dom(R_{\beta T}^M)$, using the fact that $J_T^M$ is firmly nonexpansive, we have

$$\begin{aligned} \left\| -R_{\beta T}^M x + R_{\beta T}^M y \right\|^2 &= \beta^2 \|J_T^M x - J_T^M y\|^2 - 2\beta\langle x - y, J_T^M x - J_T^M y\rangle + \|x - y\|^2 \\ &= \|x - y\|^2 + \left[\beta^2 - 2\beta\right] \left\|J_T^M x - J_T^M y\right\|^2 \\ &= \|x - y\|^2 + \frac{\beta^2 - 2\beta}{\beta^2} \|(I + R_{\beta T}^M)x - (I + R_{\beta T}^M)y\|^2 \\ &= \|x - y\|^2 - \left(\frac{2}{\beta} - 1\right) \|(I + R_{\beta T}^M)x - (I + R_{\beta T}^M)y\|^2. \end{aligned}$$

From Proposition 2.1, we see that $-R_{\beta T}^M$ is $\frac{\beta}{2}$-averaged. □

**Lemma 3.1.** *Let $A, B : H \to 2^H$ be maximal monotone operators and $0 \in int(dom(A) - dom(B))$. Let $\beta \in [1,2)$. Then*

(i) *$zer(A + B + \lambda I) \neq \emptyset$ for $\lambda \in \mathbb{R}_+$.*
(ii) *$J_B^M(Fix(R_{\beta A}^M R_{\beta B}^M)) = zer(A + B + (2-\beta)M^{-1})$ and $zer(A + B + (2-\beta)M^{-1})$ is a singleton.*
(iii) *$Fix(R_{\beta A}^M R_{\beta B}^M) \neq \emptyset$.*

*Proof.* (i) Since $A$ and $B$ are maximal monotone operators and $0 \in int(dom(A) - dom(B))$, $A + B$ is maximal monotone operator. Consider $S = \frac{1}{\lambda}(A + B)$, which is also a maximal monotone operator. From Minty's theorem, $ran(S + I) = H \Rightarrow 0 \in zer(S + I)$, which implies that $zer(S + I) = zer(A + B + \lambda I) \neq \emptyset$.

(ii) Since $MA$ and $MB$ are maximal monotone operators [9, Lemma 3.7], from (i), $zer(A + B + (2-\beta)M^{-1}) = zer(MA + MB + (2-\beta)I) \neq \emptyset$. Let $u \in H$ be an arbitrary. Then $0 \in MAu + MBu + (2-\beta)u$, and then there exixts $v \in H$ such that

$$u - v \in MAu + (2-\beta)u \text{ and } v - u \in MBu,$$

i.e.,

$$(\beta-1)u-v\in MAu \text{ and } u=J_B^M v.$$

Hence,

$$\begin{aligned}\beta J_B^M v-v\in MA\circ J_B^M v-J_B^M v &\iff J_B^M v=J_A^M(\beta J_B^M v-v)\\ &\iff 0\in \beta J_A^M(\beta J_B^M v-v)-\beta J_B^M v\\ &\iff v=\beta J_A^M(\beta J_B^M v-v)-(\beta J_B^M v-v)\\ &\iff v=R_{\beta A}^M R_{\beta B}^M v.\end{aligned}$$

Note that $u=J_B^M v$, $v\in Fix(R_{\beta A}^M R_{\beta B}^M)$, and $u\in zer(A+B+(2-\beta)M^{-1})$. Indeed, we have

$$J_B^M(Fix(R_{\beta A}^M R_{\beta B}^M))=zer(A+B+(2-\beta)M^{-1})=zer(MA+MB+(2-\beta)I).$$

As $2-\beta>0$, $MA+MB+(2-\beta)I$ is strongly monotone, and hence

$$zer(A+B+(2-\beta)M^{-1})=zer(MA+MB+(2-\beta)I)$$

is a singleton.

(iii) Since $zer(A+B+(2-\beta)M^{-1})\neq\emptyset$, we have $Fix(R_{\beta A}^M R_{\beta B}^M)\neq\emptyset$.

□

**Proposition 3.2.** [13] *Let $T:H\to H$ be an $\alpha$-averaged operator with $\alpha\in(0,1)$ and $Fix(T)\neq\emptyset$. Assume that $z:[0,\infty)\to H$ is an unique strong global solution of the following dynamical system*

$$\begin{cases}\dot z(t)=\kappa(t)[Tz(t)-z(t)],\\ z(0)=z_0\in H,\end{cases}$$

*where $\kappa:[0,\infty)\to[0,\infty)$ is a Lebesgue measurable function and*

$$0<\underline{\kappa}\le\inf_{t\ge0}\kappa(t)\le\sup_{t\ge0}\kappa(t)\le\overline{\kappa}, \tag{3.2}$$

*where $\underline{\kappa},\overline{\kappa}\in\mathbb{R}$. Then*

(i) *The orbit $z$ is bounded and $\dot z$, $(I-T)z\in L^2([0,\infty);H)$.*
(ii) $\lim_{t\to\infty}\dot z(t)=\lim_{t\to\infty}(I-T)(z(t))=0$.
(iii) *$z(t)\rightharpoonup z\in Fix(T)$ as $t\to\infty$.*

We now show the weak convergence of the orbit of the dynamical system (1.1) and strong convergence of the shadow orbits as given explicitly in (3.1).

**Theorem 3.1.** *Let $A,B:H\to 2^H$ be maximal monotone operators such that $0\in int(dom(A)-dom(B))$ and $M\in\mathscr{S}_m(H)$. Let $\beta\in(1,2)$. Let $z:[0,\infty)\to H$ be a unique strong global solution to dynamical system (3.1). Then*

(i) *The orbit $z$ is bounded and $\dot z$, $(I-R_{\beta A}^M R_{\beta B}^M)z\in L^2([0,\infty);H)$.*
(ii) $\lim_{t\to\infty}\dot z(t)=\lim_{t\to\infty}(I-R_{\beta A}^M R_{\beta B}^M)(z(t))=0$.
(iii) *$z(t)\rightharpoonup z\in Fix(R_{\beta A}^M R_{\beta B}^M)$ as $t\to\infty$.*

*Let $x=J_B^M z$. Moreover, we have the following:*

(iv) *$x(t)\to x$ and $y(t)\to x$.*

*Proof.* Since the composition of two averaged operators is averaged, we have that $R^M_{\beta A} R^M_{\beta B}$ is $\alpha$-averaged with full domain. Using Lemma 3.1 and Proposition 3.2, we obtain that $Fix(R^M_{\beta A} R^M_{\beta B}) \neq \emptyset$ and (i), (ii), and (iii) holds for some $z \in Fix(R^M_{\beta A} R^M_{\beta B})$. Since $R^M_{\beta B}$ is single-valued, we have $x = J^M_B z \in \mathscr{P}$.

(iv) From Proposition 3.1, $\left(-R^M_{\beta A}\right)$ is $\frac{\beta}{2}(=\alpha_1)$-averaged and $\left(-R^M_{\beta B}\right)$ is $\frac{\beta}{2}(=\alpha_2)$-averaged. Setting $s_1 = \frac{1-\alpha_1}{\alpha_1}$ and $s_2 = \frac{1-\alpha_2}{\alpha_2}$, we have $s_1 + s_2 > 0$ and $\frac{1-\alpha}{\alpha} = \frac{s_1 s_2}{s_1+s_2}$. By Proposition 2.1, for $t \in [0,\infty)$, we have

$$\begin{aligned}
&\left\|R^M_{\beta A} R^M_{\beta B} z(t) - z\right\|^2 \\
&= \|z(t)-z\|^2 - s_2 \left\|\left(I + R^M_{\beta B}\right) z(t) - \left(I + R^M_{\beta B}\right) z\right\|^2 \\
&- s_1 \left\|\left(-R^M_{\beta B} - R^M_{\beta A} R^M_{\beta B}\right) z(t) - \left(-R^M_{\beta B} - R^M_{\beta A} R^M_{\beta B}\right) z\right\|^2 \\
&= \|z(t)-z\|^2 - \frac{s_1 s_2}{s_1+s_2} \left\|(I - R^M_{\beta A} R^M_{\beta B}) z(t)\right\|^2 - \frac{1}{s_1+s_2} \\
&\left\|s_1 \left[\left(-R^M_{\beta B} - R^M_{\beta A} R^M_{\beta B}\right) z(t) - \left(-R^M_{\beta B} - R^M_{\beta A} R^M_{\beta B}\right) z\right] - s_2 \left[\left(I + R^M_{\beta B}\right) z(t) - \left(I + R^M_{\beta B}\right) z\right]\right\|^2.
\end{aligned} \tag{3.3}$$

Note that $I + R^M_{\beta B} = \beta J^M_B$ and $-R^M_{\beta B} - R^M_{\beta A} R^M_{\beta B} = -\beta J^M_A R^M_{\beta B}$. Also for given $z \in Fix(R^M_{\beta A} R^M_{\beta B})$, $x = J_B z = J^M_A R^M_{\beta B} z$. So, from (3.3), we have

$$\begin{aligned}
&\left\|R^M_{\beta A} R^M_{\beta B} z(t) - z\right\|^2 \\
&= \|z(t)-z\|^2 - \frac{1-\alpha}{\alpha} \|(I - R^M_{\beta A} R^M_{\beta B}) z(t)\|^2 - \frac{\beta^2}{s_1+s_2} \|s_1(y(t)-x) + s_2(x(t)-x)\|^2.
\end{aligned} \tag{3.4}$$

Now,

$$\begin{aligned}
\frac{d}{dt}\|z(t)-z\|^2 &= 2\langle z(t)-z, \dot{z}(t)\rangle \\
&= 2\langle z(t)-z, \kappa(t)(R^M_{\beta A} R^M_{\beta B} z(t) - z(t))\rangle \\
&= \kappa(t)\left(\|R^M_{\beta A} R^M_{\beta B} z(t) - z\|^2 - \|R^M_{\beta A} R^M_{\beta B} z(t) - z(t)\|^2 - \|z(t)-z\|^2\right).
\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we have

$$\frac{d}{dt}\|z(t)-z\|^2 = -\kappa(t)\left(\frac{1}{\alpha}\|z(t) - R^M_{\beta A} R^M_{\beta B} z(t)\|^2 - \frac{\beta^2}{s_1+s_2}\|s_1(y(t)-x) + s_2(x(t)-x)\|^2\right). \tag{3.6}$$

Integrating (3.6) from 0 to $s$, we obtain

$$\begin{aligned}
&\frac{1}{\alpha}\int_0^s \kappa(t)\|z(t) - R^M_{\beta A} R^M_{\beta B} z(t)\|^2 dt + \int_0^s \beta^2 \frac{\kappa(t)}{s_1+s_2}\|s_1(y(t)-x) + s_2(x(t)-x)\|^2 dt \\
&\leq \|z(0)-z\|^2 - \|z(s)-z\|^2.
\end{aligned}$$

Taking the limit $s \to \infty$, we obtain from results (ii) and (iii) that

$$\lim_{s\to\infty}\int_0^s \beta^2 \frac{\kappa(t)}{s_1+s_2}\|s_1(y(t)-x)+s_2(x(t)-x)\|^2 dt < \infty.$$

Using the fact that $y(t)$ and $x(t)$ are continuous and (3.2), we have

$$\|s_1(y(t)-x)+s_2(x(t)-x)\|^2 \to 0$$

as $t \to \infty$. Also, from result (ii), $(I-R^M_{\beta A}R^M_{\beta B})(z(t)=\beta(x(t)-y(t)) \to 0$ as $t\to\infty$, and $s_1+s_2>0$, we deduce that $x(t)\to x$ and $y(t)\to x$. □

From Lemma 3.1 and Theorem 3.1, we can see that the orbit of dynamical system (1.1) converges to the solution of the inclusion problem involving the operator $A+B+(2-\beta)M^{-1}$, which, in the limiting case of $\beta \to 2$, boils down to original inclusion problem ($\mathscr{P}$).

## 4. Tikhonov Regularized Douglas-Rachford Dynamical Systems

In this section, we studied dynamical system (3.1). We introduce a perturbed operator and study the Tikhonov regularization. Let $\varepsilon > 0$ and consider two maximal monotone operators $A,B : H \to 2^H$. Since $B$ is maximal monotone, the perturbed operator $B_\varepsilon := B+\varepsilon I$ is $\varepsilon$-strongly monotone. Hence, operator $A+B_\varepsilon$ is $\varepsilon$-strongly monotone. Therefor, for $\varepsilon>0$, $zer(A+B_\varepsilon)$ is a singleton and we denote its unique element by $z_\varepsilon$.

The following lemmas are taken from [14].

**Lemma 4.1.** *Assume $\varepsilon > 0$ and consider two maximal monotone operators $A,B : H\to 2^H$ such that $zer(A+B)\neq\emptyset$. Then $z_\varepsilon \to z^* := \inf\{\|x\| : x\in zer(A+B)\}$ as $\varepsilon\to 0$.*

**Lemma 4.2.** *Assume $\varepsilon_1,\varepsilon_2>0$ and consider two maximal monotone operators $A,B : H\to 2^H$ such that $zer(A+B)\neq\emptyset$. Then $\|z_{\varepsilon_1}-z_{\varepsilon_2}\| \le \frac{\|z_{\varepsilon_1}\|}{\varepsilon_2}|\varepsilon_1-\varepsilon_2|$, i.e., $\varepsilon\mapsto z_\varepsilon$ is locally Lipschitz continuous on $(0,\infty)$, and then differentiable almost everywhere. Moreover,*

$$\left\|\frac{d}{d\varepsilon}x_\varepsilon\right\| \le \frac{\|z_\varepsilon\|}{\varepsilon},\ \forall\varepsilon\in(0,\infty).$$

### 4.1. Outer Tikhonov Regularization.

Consider the following dynamical system:

$$\begin{cases} \dot z(t) = \kappa(t)[R^M_{\beta A}R^M_{\beta B}z(t)-z(t)]-\varepsilon(t)z(t), \\ z(0)=z_0\in H, \end{cases} \tag{4.1}$$

where $\kappa : [0,\infty)\to[0,\infty)$ and $\varepsilon : [0,\infty)\to[0,\infty)$ are Lebesgue measurable functions.

**Theorem 4.1.** *Let $A,B : H\to 2^H$ be maximal monotone operators such that $zer(A+B)\neq\emptyset$ and $M\in\mathscr{S}_m(H)$ for some $m>0$. Let $z : [0,\infty)\to H$ be a unique strong global solution to dynamical system (4.1). Suppose that*

(i) $\int_0^\infty \varepsilon(t)dt=\infty$,
(ii) $\int_0^\infty \kappa(t)dt=\infty$,
(iii) *$\varepsilon$ and $\kappa$ are absolutely continuous and* $\frac{\varepsilon(t)}{\kappa(t)}\to 0$ *as* $t\to\infty$,
(iv) $\int_0^\infty \left|\frac{d}{dt}\left(\frac{\varepsilon(t)}{\kappa(t)}\right)\right| dt<\infty$.

*Then $z(t)\to z^*$ as $t\to\infty$.*

*Proof.* Since the composition of two averaged operators is still averaged, $R^M_{\beta A}R^M_{\beta B}$ is also averaged and hence nonexpansive. From [14, Theorem 3.4], we conclude the result immediately. □

4.2. **Inner Tikhonov Regularization.** Consider the following dynamical system:

$$\begin{cases} \dot{z}(t) = \kappa(t)[R_{\beta A}R_{\beta(B+\varepsilon(t)I)}z(t) - z(t)], \\ z(0) = z_0 \in H, \end{cases} \tag{4.2}$$

where $\kappa : [0,\infty) \to [0,\infty)$ and $\varepsilon : [0,\infty) \to [0,\infty)$ are Lebesgue measurable functions. Here we have taken $M = I$.

The following lemma is useful for deriving the main result, which we quote from [1] without proof.

**Lemma 4.3.** [1] *Let $T : H \to 2^H$ be a maximal monotone operator and $\alpha \in \mathbb{R}_+$. Then, for $S = T + \alpha I$, $J_S = J_{(1+\alpha)^{-1}T}((I+\alpha)^{-1}I)$.*

**Lemma 4.4.** *Let $t \to z(t)$ be a strong global solution to (4.2). Then, for almost all $t \in [0,\infty)$,*

$$\langle \dot{z}(t), z(t) - \bar{z}(\varepsilon(t))\rangle \leq \frac{\kappa(t)}{2}\left(\frac{\beta^2 - 2\beta}{(1+\varepsilon(t))^2}\right)\|z(t) - \bar{z}(\varepsilon(t))\|^2.$$

*Proof.* For almost all $t \in [0,\infty)$, we have

$$\begin{aligned}
&2\langle \dot{z}(t), z(t) - \bar{z}(\varepsilon(t))\rangle \\
&= \|\dot{z}(t) + z(t) - \bar{z}(\varepsilon(t))\|^2 - \|\dot{z}(t)\|^2 - \|z(t) - \bar{z}(\varepsilon)(t)\|^2 \\
&= \|\kappa(t)(T_t(z(t) - \bar{z}(\varepsilon(t))) + (1-\kappa(t))(z(t) - \bar{z}(\varepsilon(t)))\|^2 - \|\dot{z}(t)\|^2 - \|z(t) - \bar{z}(\varepsilon(t))\|^2 \\
&= \kappa(t)\|T_t(z(t) - \bar{z}(\varepsilon(t))\|^2 + (1-\kappa(t))\|z(t) - \bar{z}(\varepsilon(t))\|^2 - \kappa(t)(1-\kappa(t))\|T_t(z(t)) - z(t)\|^2 \\
&\quad - \|\dot{z}(t)\|^2 - \|z(t) - \bar{z}(\varepsilon(t))\|^2 \\
&= \kappa(t)\|T_t(z(t) - \bar{z}(\varepsilon(t))\|^2 - \kappa(t)\|z(t) - \bar{z}(\varepsilon(t))\|^2 - \kappa(t)(1-\kappa(t))\|T_t(z(t)) - z(t)\|^2 - \|\dot{z}(t)\|^2.
\end{aligned} \tag{4.3}$$

Also, for $x, y \in H$ and $\forall t \in [0,\infty)$, we have

$$\begin{aligned}
&\|R_{\beta(B+\varepsilon(t)I)}x - R_{\beta(B+\varepsilon(t)I)}y\|^2 \\
&= \|\beta J_{\gamma(B+\varepsilon(t)I)}x - J_{\gamma(B+\varepsilon(t)I)}y - x + y\|^2 \\
&= \beta^2\|J_{\gamma(B+\varepsilon(t)I)}x - J_{\gamma(B+\varepsilon(t)I)}y\|^2 + \|x-y\|^2 - 2\beta\langle J_{\gamma(B+\varepsilon(t)I)}x - J_{\gamma(B+\varepsilon(t)I)}y, x-y\rangle \\
&= \beta^2\|J_{\gamma(1+\varepsilon(t))^{-1}B}(1+\varepsilon(t))^{-1}x - J_{\gamma(1+\varepsilon(t))^{-1}B}(1+\varepsilon(t))^{-1}y\|^2 + \|x-y\|^2 \\
&\quad - 2\beta\langle J_{\gamma(1+\varepsilon(t))^{-1}B}(1+\varepsilon(t))^{-1}x - J_{\gamma(1+\varepsilon(t))^{-1}B}(1+\varepsilon(t))^{-1}y, x-y\rangle.
\end{aligned}$$

Using the nonexpansiveness and firmly-nonexpansiveness of the resolvent, we obtain

$$\begin{aligned}
&\|R_{\beta(B+\varepsilon(t)I)}x - R_{\beta(B+\varepsilon(t)I)}y\|^2 \\
&\leq \|x-y\|^2 + (\beta^2 - 2\beta)\|(1+\varepsilon(t))^{-1}x - (1+\varepsilon(t))^{-1}y\|^2 \\
&= \left(1 + \frac{\beta^2 - 2\beta}{(1+\varepsilon(t))^2}\right)\|x-y\|^2.
\end{aligned}$$

Again by the nonexpansiveness of the resolvent, we have

$$\|T_t x - T_t y\|^2 \le \left(1 + \frac{\beta^2 - 2\beta}{(1+\varepsilon(t))^2}\right)\|x-y\|^2. \tag{4.4}$$

From (4.3) and (4.4), we have

$$\begin{aligned}
&2\langle \dot{z}(t), z(t) - \bar{z}(\varepsilon(t))\rangle \\
&\le \kappa(t)\left(1 + \frac{\beta^2 - 2\beta}{(1+\varepsilon(t))^2}\right)\|z(t) - \bar{z}(\varepsilon(t))\|^2 - \kappa(t)\|z(t) - \bar{z}(\varepsilon(t))\|^2 \\
&= \kappa(t)\left(\frac{\beta^2 - 2\beta}{(1+\varepsilon(t))^2}\right)\|z(t) - \bar{z}(\varepsilon(t))\|^2.
\end{aligned}$$

□

**Theorem 4.2.** *Let $t \mapsto z(t)$ be the strong solution of (4.2). Assume that the following assertions hold true:*

(i) *$\varepsilon$ is absolutely continuous and $\varepsilon(t)$ decreases to $0$ as $t \to \infty$,*

(ii) *$\frac{\dot{\varepsilon}(t)}{\varepsilon(t)\kappa(t)} \to 0$ as $t \to \infty$,*

(iii) *$\int_0^\infty \frac{\kappa(t)}{(1+\varepsilon(t))^2} = \infty$.*

*Then $z(t) \to P_{zer(A+B)(0)}$ as $t \to \infty$.*

*Proof.* Let $h(t) = \frac{1}{2}\|z(t) - \bar{z}(\varepsilon(t))\|^2$. From Lemma 4.4, we deduce that

$$\begin{aligned}
\dot{h}(t) &= \left\langle z(t) - \bar{z}(\varepsilon(t)), \dot{z}(t) - \dot{\varepsilon}(t)\frac{d}{d\varepsilon}\bar{z}(\varepsilon(t))\right\rangle \\
&= \langle z(t) - \bar{z}(\varepsilon(t)), \dot{z}(t)\rangle - \left\langle z(t) - \bar{z}(\varepsilon(t)), \dot{\varepsilon}(t)\frac{d}{d\varepsilon}\bar{z}(\varepsilon(t))\right\rangle \\
&\le \frac{\kappa(t)}{2}\left(\frac{\beta^2 - 2\beta}{(1+\varepsilon(t))^2}\right)\|z(t) - \bar{z}(\varepsilon(t))\|^2 - \left\langle z(t) - \bar{z}(\varepsilon(t)), \dot{\varepsilon}(t)\frac{d}{d\varepsilon}\bar{z}(\varepsilon(t))\right\rangle.
\end{aligned}$$

By denoting $\Gamma(t) = \frac{\kappa(t)}{2}\left(\frac{2\beta - \beta^2}{(1+\varepsilon(t))^2}\right)$, we obtain that

$$\dot{h}(t) \le -2\Gamma(t)h(t) - \dot{\varepsilon}(t)\left\|\frac{d}{d\varepsilon}\bar{z}(\varepsilon(t))\right\|\sqrt{h(t)}. \tag{4.5}$$

Putting $\theta := \sqrt{2h}$ gives $h = \frac{\theta^2}{2}$ and $\dot{h} = \theta\dot{\theta}$. From (4.5), we have

$$\dot{\theta}(t) + \Gamma(t)\theta(t) \le -\dot{\varepsilon}(t)\left\|\frac{d}{d\varepsilon}\bar{z}(\varepsilon(t))\right\|.$$

By Lemma 4.2, we see that

$$\dot{\theta}(t) + \Gamma(t)\theta(t) \le -\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}\|\bar{z}(\varepsilon(t))\|.$$

Define the integrating factor $I(t) = \int_0^t \Gamma(s)ds$, we have

$$\frac{d}{dt}\left(\theta(t)e^{I(t)}\right) \le -\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}\|\bar{z}(\varepsilon(t))\|e^{I(t)}.$$

Thus

$$0 \leq \theta(t) \leq e^{-I(t)} \left[ \theta(0) - \int_0^t \frac{\dot{\varepsilon}(s)}{\varepsilon(s)} \|\bar{z}(\varepsilon(s))\| e^{I(s)} ds \right]. \tag{4.6}$$

If $\int_0^t \frac{\dot{\varepsilon}(s)}{\varepsilon(s)} \|\bar{z}(\varepsilon(s))\| e^{I(s)} ds$ is bounded, then $\lim_{t\to\infty} \theta(t) = 0$; otherwise by (i), (ii), (iii), the L'Hospital rule, and Lemma 4.1, we obtain

$$\begin{aligned} &\lim_{t\to\infty} e^{-I(t)} \int_0^t \frac{\dot{\varepsilon}(s)}{\varepsilon(s)} \|\bar{z}(\varepsilon(s))\| e^{I(s)} ds \\ &= \lim_{t\to\infty} \frac{\dot{\varepsilon}(t)(1+\varepsilon(t))^2 \|\dot{z}(\varepsilon(t))\|}{\varepsilon(t)\kappa(t)(2\beta - \beta^2)} = 0. \end{aligned}$$

From (iii) and (4.6), we see that $\theta(t) \to 0$ as $t \to \infty$. Hence

$$\|z(t) - \bar{z}(\varepsilon(t))\| \to \text{ as } t \to \infty. \tag{4.7}$$

In view of

$$\|z(t) - P_{zer(A+B)}(0)\| \leq \|z(t) - \bar{z}(\varepsilon(t))\| + \|\bar{z}(\varepsilon()t) - P_{zer(A+B)}(0)\|,$$

Lemma 4.1, and (4.7), we have that $z(t) \to P_{zer(A+B)}(0)$ as $t \to \infty$. □

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