Appl. Set-Valued Anal. Optim. 5 (2023), No. 1, pp. 31-47
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.5.2023.1.03

OPTIMALITY CONDITIONS AND DUALITY RELATIONS IN NONSMOOTH FRACTIONAL INTERVAL-VALUED MULTIOBJECTIVE OPTIMIZATION

NGUYEN HUY HUNG, NGUYEN VAN TUYEN*

Department of Mathematics, Hanoi Pedagogical University 2, Xuan Hoa, Phuc Yen, Vinh Phuc, Vietnam

Abstract. This paper deals with Pareto solutions of a nonsmooth fractional interval-valued multiobjective optimization. We first introduce four types of Pareto solutions of the considered problem by considering the lower-upper interval order relation and then apply some advanced tools of variational analysis and generalized differentiation to establish necessary optimality conditions for these solutions. Sufficient conditions for Pareto solutions of such a problem are also provided by means of introducing the concepts of (strictly) generalized convex functions defined in terms of the limiting/Mordukhovich subdifferential of locally Lipschitzian functions. Finally, a Mond–Weir type dual model is formulated, and weak, strong and converse-like duality relations are examined.

Keywords. Fractional interval-valued multiobjective optimization; Optimality conditions; Duality; Limiting/Mordukhovich subdifferential; Pareto solutions; Generalized convex-affine function.

2020 Mathematics Subject Classification. 90C29, 90C46.

1. INTRODUCTION

In this paper, we are interested in Pareto solutions of the following fractional multiobjective problem with interval-valued objective functions:

$$LU\text{-Min}\, F(x) := \left(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_m(x)}{g_m(x)}\right) \tag{FIMP}$$
$$\text{s.t. } x \in \Omega := \{x \in S : h_j(x) \le 0, j = 1, \dots, p\},$$

where $f_i, g_i \colon \mathbb{R}^n \to \mathscr{K}_c$, $i \in I := \{1, \dots, m\}$, are interval-valued functions defined respectively by $f_i(x) = [f_i^L(x), f_i^U(x)]$, $g_i(x) = [g_i^L(x), g_i^U(x)]$, f_i^L, f_i^U, g_i^L,$g_i^U \colon \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitzian functions satisfying $f_i^L(x) \le f_i^U(x)$ and

$$0 < g_i^L(x) \le g_i^U(x)$$

for all $x \in S$ and $i \in I$, $\mathscr{K}_c$ is the class of all closed and bounded intervals in $\mathbb{R}$, i.e.,

$$\mathscr{K}_c = \{[a^L, a^U] : a^L, a^U \in \mathbb{R},\ a^L \le a^U\},$$

$h_j \colon \mathbb{R}^n \to \mathbb{R}$, $j \in J := \{1, \dots, p\}$, are locally Lipschitzian functions, and S is a nonempty and closed subset of $\mathbb{R}^n$.

*Corresponding author.
E-mail addresses: nguyenhuyhung@hpu2.edu.vn (N.H. Hung), nguyenvantuyen83@hpu2.edu.vn (N.V. Tuyen).
Received September 3, 2022; Accepted January 1, 2023

An interval-valued optimization problem is one of the deterministic optimization models to deal with the uncertain/incomplete data. More precisely, in interval-valued optimization, the coefficients of objective and constraint functions are taken as closed intervals. The study of optimality conditions and duality relations for optimization problems with one or multiple interval-valued objective functions have recently received increasing interest in optimization community; see, e.g., [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and the references therein. However, there are quite few publications devoted to optimality conditions and duality relations for fractional interval-valued optimization problems; see [24, 25, 26].

Debnath and Gupta [24] presented some necessary and sufficient optimality conditions for nondifferentiable fractional interval-valued programming problems, where numerators of the objective function and constraint functions are convex, while denominators of the objective function are concave. Recently, these results have been extended to fractional interval-valued multiobjective problems [25, 26]. However, to the best of our knowledge, so far there have been no papers investigating optimality conditions and duality for fractional interval-valued multiobjective problems with locally Lipschitzian data.

Motivated by the above observations, in this paper, we introduce some kinds of Pareto solutions with respect to lower-upper (LU) interval order relation for problems of the form (FIMP). Then we employ the limiting/Mordukhovich subdifferential and the limiting/Mordukhovich normal cone to derive necessary and sufficient optimality conditions for these Pareto solutions of problem (FIMP). Along with optimality conditions, we state a dual problem in the sense of Mond–Weir to the primal one and examine weak, strong and converse duality relations under assumptions of (strictly) generalized convexity (cf. [2, 27, 28]). In addition, some examples are also given for analyzing the obtained results.

The paper is organized as follows. Section 2 contains some basic definitions from variational analysis, interval analysis and several auxiliary results. In Section 3, we first introduce four kinds of Pareto solutions of problem (FIMP) and then establish necessary conditions for these solutions. Sufficient optimality conditions for such solutions are provided by means of introducing (strictly) generalized convex functions defined in terms of the limiting subdifferential for locally Lipschitzian functions. Section 4 is devoted to describing duality relations.

2. Preliminaries

Throughout the paper, let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its nonnegative orthant. The topological closure of a set S is denoted by $\operatorname{cl} S$. As usual, the polar cone of a set $S \subset \mathbb{R}^n$ is defined by

$$S^\circ := \{x^* \in \mathbb{R}^n \ : \ \langle x^*, x\rangle \leq 0 \quad \forall x \in S\}.$$

Definition 2.1 (see [29, 30])**.** Given $\bar{x} \in \operatorname{cl} S$. The set

$$N(\bar{x}; S) := \{z^* \in \mathbb{R}^n : \exists x^k \xrightarrow{S} \bar{x}, \varepsilon_k \to 0^+, z_k^* \to z^*, z_k^* \in \widehat{N}_{\varepsilon_k}(x^k; S), \ \ \forall k \in \mathbb{N}\}$$

is called the *limiting/Mordukhovich normal cone* of S at $\bar{x}$, where

$$\widehat{N}_\varepsilon(x; S) := \left\{ z^* \in \mathbb{R}^n \ : \ \limsup_{u \xrightarrow{S} x} \frac{\langle z^*, u - x\rangle}{\| u - x \|} \leq \varepsilon \right\}$$

is the set of ε-*normals* of S at x and $u \xrightarrow{S} x$ means that $u \to x$ and $u \in S$.

Let $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty, \infty]$ be an *extended-real-valued function*. The *epigraph* and *domain* of φ are denoted, respectively, by

$$\text{epi } \varphi := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \, : \, \varphi(x) \leq \alpha\},$$
$$\text{dom } \varphi := \{x \in \mathbb{R}^n \, : \, \ |\varphi(x)| < +\infty\}.$$

Definition 2.2 (see [29, 30])**.** Let $\bar{x} \in \text{dom } \varphi$.

(i) The set

$$\partial\varphi(\bar{x}) := \{x^* \in \mathbb{R}^n \, : \, (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi)\},$$

is called the *limiting/Mordukhovich subdifferential* of φ at $\bar{x}$. If $\bar{x} \notin \text{dom } \varphi$, then we put $\partial\varphi(\bar{x}) = \emptyset$.

(ii) The set $\partial^+\varphi(\bar{x}) := -\partial(-\varphi)(\bar{x})$ is called the *upper subdifferential* of φ at $\bar{x}$.

We now summarize some properties of the limiting subdifferential that will be used in the sequel.

Proposition 2.1 (see [29, Theorem 3.36])**.** *Let $\varphi_l\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $l = 1, \ldots, p$, $p \geq 2$, be lower semicontinuous around $\bar{x}$ and let all but one of these functions be locally Lipschitzian around $\bar{x}$. Then we have the following inclusion*

$$\partial(\varphi_1 + \ldots + \varphi_p)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \ldots + \partial\varphi_p(\bar{x}).$$

Recall that, the function φ is called *lower semicontinuous* (*l.s.c.*) *at* a point $\bar{x} \in \text{dom}\varphi$ if

$$\varphi(\bar{x}) \leq \liminf_{x \to \bar{x}} \varphi(x).$$

We say that φ is l.s.c. around $\bar{x}$ when it is l.s.c. at any point of some neighborhood of $\bar{x}$. The function φ is called *locally Lipschitzian around* $\bar{x}$, or *Lipschitz continuous around* $\bar{x}$ if there is a neighborhood U of this point and a constant $l \geq 0$ such that

$$\|\varphi(v) - \varphi(u)\| \leq l\|v - u\| \ \text{ for all } \ u, v \in U.$$

Proposition 2.2 (see [29, Theorem 3.46])**.** *Let $\varphi_l\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $l = 1, \ldots, p$, be locally Lipschitzian around $\bar{x}$. Then the function $\phi(\cdot) := \max\{\varphi_l(\cdot) : l = 1, \ldots, p\}$ is also locally Lipschitzian around $\bar{x}$ and one has*

$$\partial\phi(\bar{x}) \subset \bigcup \left\{ \partial\left(\sum_{l=1}^{p} \lambda_l \varphi_l\right)(\bar{x}) \, : \, (\lambda_1, \ldots, \lambda_p) \in \Lambda(\bar{x}) \right\},$$

where

$$\Lambda(\bar{x}) := \big\{(\lambda_1, \ldots, \lambda_p) \, : \, \lambda_l \geq 0, \sum_{l=1}^{p} \lambda_l = 1, \lambda_l[\varphi_l(\bar{x}) - \phi(\bar{x})] = 0\big\}.$$

Proposition 2.3 (see [29, Corollary 1.111(ii)])**.** *Let $\varphi_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, 2$, be locally Lipschitzian around $\bar{x}$. If $\varphi_2(\bar{x}) \neq 0$, then we have*

$$\partial\left(\frac{\varphi_1}{\varphi_2}\right)(\bar{x}) \subset \frac{\partial(\varphi_2(\bar{x})\varphi_1)(\bar{x}) + \partial(-\varphi_1(\bar{x})\varphi_2)(\bar{x})}{[\varphi_2(\bar{x})]^2}.$$

Proposition 2.4 (see [29, Proposition 1.114])**.** *Let $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at $\bar{x}$. If $\bar{x}$ is a local minimizer of φ, then $0 \in \partial\varphi(\bar{x})$.*

Next we recall some definitions and facts in interval analysis. Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two intervals in $\mathscr{K}_c$. Then, we define

(i) $A+B := \{a+b : a \in A, b \in B\} = [a^L + b^L, a^U + b^U]$;

(ii) $A-B := \{a-b : a \in A, b \in B\} = [a^L - b^U, a^U - b^L]$;

(iii) For each $k \in \mathbb{R}$,
$$kA := \{ka : a \in A\} = \begin{cases} [ka^L, ka^U] & \text{if } k \geq 0, \\ [ka^U, ka^L] & \text{if } k < 0; \end{cases}$$

(iv) $\frac{A}{B} := \left[\min\left(\frac{a^L}{b^L}, \frac{a^L}{b^U}, \frac{a^U}{b^L}, \frac{a^U}{b^U}\right), \max\left(\frac{a^L}{b^L}, \frac{a^L}{b^U}, \frac{a^U}{b^L}, \frac{a^U}{b^U}\right)\right]$, if $0 \notin B$,

see, e.g., [31, 32, 33] for more details.

Definition 2.3 (see [5, 19])**.** Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two intervals in $\mathscr{K}_c$. We say that:

(i) $A \leq_{LU} B$ if $a^L \leq b^L$ and $a^U \leq b^U$.

(ii) $A <_{LU} B$ if $A \leq_{LU} B$ and $A \neq B$, or, equivalently, $A <_{LU} B$ if
$$\begin{cases} a^L < b^L \\ a^U \leq b^U, \end{cases} \quad \text{or} \quad \begin{cases} a^L \leq b^L \\ a^U < b^U, \end{cases} \quad \text{or} \quad \begin{cases} a^L < b^L \\ a^U < b^U. \end{cases}$$

(iii) $A <_{LU}^{s} B$ if $a^L < b^L$ and $a^U < b^U$.

3. Optimality conditions

We now introduce Pareto solutions of (FIMP) with respect to LU interval order relation. For the sake of convenience, *we always assume hereafter* that $f_i^L(x) \geq 0$, $\forall x \in S$ and $i \in I$. For each $i \in I$ and $x \in \mathbb{R}^n$, put $F_i(x) := \frac{f_i(x)}{g_i(x)}$. By definition, we have
$$F_i(x) := \frac{f_i(x)}{g_i(x)} = \left[\frac{f_i^L(x)}{g_i^U(x)}, \frac{f_i^U(x)}{g_i^L(x)}\right].$$

Definition 3.1. Let $\bar{x} \in \Omega$. We say that:

(i) $\bar{x}$ is a *type-1 Pareto solution* of (FIMP), denoted by $\bar{x} \in \mathscr{S}_1$(FIMP), if there is no $x \in \Omega$ such that
$$\begin{cases} F_i(x) \leq_{LU} F_i(\bar{x}), & \forall i \in I, \\ F_k(x) <_{LU} F_k(\bar{x}), & \text{for at least one } k \in I. \end{cases}$$

(ii) $\bar{x}$ is a *type-2 Pareto solution* of (FIMP), denoted by $\bar{x} \in \mathscr{S}_2$(FIMP), if there is no $x \in \Omega$ such that
$$\begin{cases} F_i(x) \leq_{LU} F_i(\bar{x}), & \forall i \in I, \\ F_k(x) <_{LU}^{s} F_k(\bar{x}), & \text{for at least one } k \in I. \end{cases}$$

(iii) $\bar{x}$ is a *type-1 weakly Pareto solution* of (FIMP), denoted by $\bar{x} \in \mathscr{S}_1^w$(FIMP), if there is no $x \in \Omega$ such that
$$F_i(x) <_{LU} F_i(\bar{x}), \ \forall i \in I.$$

(iv) $\bar{x}$ is a *type-2 weakly Pareto solution* of (FIMP), denoted by $\bar{x} \in \mathscr{S}_2^w$(FIMP), if there is no $x \in \Omega$ such that
$$F_i(x) <_{LU}^{s} F_i(\bar{x}), \ \forall i \in I.$$

Remark 3.1. The following relations are immediate from the definition of Pareto solutions.

(i) $\mathscr{S}_1$(FIMP) $\subseteq \mathscr{S}_2$(FIMP) $\subseteq \mathscr{S}_2^w$(FIMP).
(ii) $\mathscr{S}_1$(FIMP) $\subseteq \mathscr{S}_1^w$(FIMP) $\subseteq \mathscr{S}_2^w$(FIMP).

Furthermore, the above inclusions may be strict; see, e.g., [16, Examples 3.3–3.5].

The following result provides a Fritz-John type necessary condition for type-2 weakly Pareto solutions of problem (FIMP).

Theorem 3.1. *If $\bar{x} \in \mathscr{S}_2^w$(FIMP), then there exist $\lambda_i^L \geq 0$, $\lambda_i^U \geq 0$, $i \in I$, and $\mu_j \geq 0$, $j \in J$ with $\sum_{i\in I}(\lambda_i^L + \lambda_i^U) + \sum_{j\in J}\mu_j = 1$, such that*

$$0 \in \sum_{i\in I} \frac{\lambda_i^L}{g_i^U(\bar{x})}\left[\partial f_i^L(\bar{x}) - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}\partial^+ g_i^U(\bar{x})\right] + \sum_{i\in I} \frac{\lambda_i^U}{g_i^L(\bar{x})}\left[\partial f_i^U(\bar{x}) - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}\partial^+ g_i^L(\bar{x})\right]$$
$$+ \sum_{j\in J}\mu_j \partial h_j(\bar{x}) + N(\bar{x};S), \quad \mu_j h_j(\bar{x}) = 0, \quad j \in J. \tag{3.1}$$

Proof. Since $\bar{x} \in \mathscr{S}_2^w$(FIMP), there is no $x \in \Omega$ such that $F_i(x) <_{LU}^s F_i(\bar{x})$, $\forall i \in I$, i.e.,

$$\frac{f_i^L(x)}{g_i^U(x)} < \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} \text{ and } \frac{f_i^U(x)}{g_i^L(x)} < \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}, \quad \forall i \in I.$$

Hence for each $x \in \Omega$, there exists $i \in I$ such that

$$\frac{f_i^L(x)}{g_i^U(x)} \geq \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} \text{ or } \frac{f_i^U(x)}{g_i^L(x)} \geq \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}. \tag{3.2}$$

Let φ be a real-valued function defined by

$$\varphi(x) := \max_{i\in I, j\in J}\left\{\frac{f_i^L(x)}{g_i^U(x)} - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}, \frac{f_i^U(x)}{g_i^L(x)} - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}, h_j(x)\right\}, \quad \forall x \in \mathbb{R}^n.$$

By (3.2), we have

$$0 = \varphi(\bar{x}) \leq \varphi(x), \quad \forall x \in \Omega.$$

If $x \in S \setminus \Omega$, then there exists $j_0 \in J$ such that $h_{j_0}(x) > 0$ and so $\varphi(x) > 0$. This implies that

$$0 = \varphi(\bar{x}) \leq \varphi(x), \quad \forall x \in S,$$

or, equivalently, $\bar{x}$ is a minimizer to the following unconstrained optimization problem

$$\text{minimizer } \varphi(x) + \delta(x;S), \quad x \in \mathbb{R}^n,$$

where $\delta(\cdot,;S)$ is the indicator function of Ω and defined by

$$\delta(x;S) = \begin{cases} 0, & \text{if } x \in S, \\ +\infty, & \text{otherwise.} \end{cases}$$

By Proposition 2.4, we have

$$0 \in \partial(\varphi + \delta(\cdot;S))(\bar{x}).$$

Clearly, φ is locally Lipschitzian around $\bar{x}$ and $\delta(\cdot;S)$ is lower semicontinuous around this point. Hence by Proposition 2.1 and the fact that $\partial\delta(\cdot;S)(\bar{x}) = N(\bar{x};S)$ (see e.g., [29, Proposition 1.19]), we obtain

$$0 \in \partial\varphi(\bar{x}) + N(\bar{x};S). \tag{3.3}$$

By Proposition 2.2, we have

$$\partial\varphi(\bar{x}) \subset \Bigg\{ \sum_{i\in I} \lambda_i^L \partial\left(\frac{f_i^L}{g_i^U}\right)(\bar{x}) + \sum_{i\in I} \lambda_i^U \partial\left(\frac{f_i^U}{g_i^L}\right)(\bar{x}) + \sum_{j\in J} \mu_j \partial h_j(\bar{x}) \,:\, \lambda_i^L, \lambda_i^U \geq 0, i\in I,$$
$$\mu_j \geq 0, j\in J, \sum_{i\in I}(\lambda_i^L+\lambda_i^U) + \sum_{j\in J}\mu_j = 1, \mu_j h_j(\bar{x}) = 0, j\in J \Bigg\}. \tag{3.4}$$

Now, taking Proposition 2.3 into account, we arrive at

$$\begin{aligned}\partial\left(\frac{f_i^L}{g_i^U}\right)(\bar{x}) &\subset \frac{\partial(g_i^U(\bar{x})f_i^L)(\bar{x}) + \partial(-f_i^L(\bar{x})g_i^U)(\bar{x})}{[g_i^U(\bar{x})]^2}\\ &= \frac{g_i^U(\bar{x})\partial f_i^L(\bar{x}) + f_i^L(\bar{x})\partial(-g_i^U)(\bar{x})}{[g_i^U(\bar{x})]^2},\\ &= \frac{g_i^U(\bar{x})\partial f_i^L(\bar{x}) - f_i^L(\bar{x})\partial^+ g_i^U(\bar{x})}{[g_i^U(\bar{x})]^2}, \quad \forall i\in I,\end{aligned} \tag{3.5}$$

where the last equalities hold due to the fact that $f_i^L(\bar{x}) \geq 0$, $g_i^U(\bar{x}) > 0$, and

$$\partial(-g_i^U)(\bar{x}) = -\partial^+ g_i^U(\bar{x}), \quad \forall i\in I.$$

Similarly, we have

$$\partial\left(\frac{f_i^U}{g_i^L}\right)(\bar{x}) \subset \frac{g_i^L(\bar{x})\partial f_i^U(\bar{x}) - f_i^U(\bar{x})\partial^+ g_i^L(\bar{x})}{[g_i^L(\bar{x})]^2}, \quad \forall i\in I. \tag{3.6}$$

It follows from (3.3)–(3.6) that

$$0 \in \Bigg\{ \sum_{i\in I} \frac{\lambda_i^L}{g_i^U(\bar{x})}\left[\partial f_i^L(\bar{x}) - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}\partial^+ g_i^U(\bar{x})\right] + \sum_{i\in I} \frac{\lambda_i^U}{g_i^L(\bar{x})}\left[\partial f_i^U(\bar{x}) - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}\partial^+ g_i^L(\bar{x})\right]$$
$$+ \sum_{j\in J}\mu_j\partial h_j(\bar{x}) : \lambda_i^L, \lambda_i^U \geq 0, i\in I, \mu_j \geq 0, j\in J,$$
$$\sum_{i\in I}(\lambda_i^L+\lambda_i^U) + \sum_{j\in J}\mu_j = 1, \mu_j h_j(\bar{x}) = 0, j\in J \Bigg\} + N(\bar{x};S).$$

In other words, there exist $\lambda_i^L \geq 0$, $\lambda_i^U \geq 0$, $i\in I$, and $\mu_j \geq 0$, $j\in J$, with $\sum_{i\in I}(\lambda_i^L+\lambda_i^U) + \sum_{j\in J}\mu_j = 1$ satisfying (3.1). The proof is complete. □

The relation obtained in (3.1) suggests us to define a Karush–Kuhn–Tucker (KKT) type condition when dealing with Pareto solutions of problem (FIMP).

Definition 3.2. Let $\bar{x}\in\Omega$. We say that $\bar{x}$ satisfies the *KKT condition* if (3.1) holds with $\lambda_i^L \geq 0$, $\lambda_i^U \geq 0$, $i\in I$, and $\mu_j \geq 0$, $j\in J$ such that $\sum_{i\in I}(\lambda_i^L+\lambda_i^U) + \sum_{j\in J}\mu_j = 1$ and $(\lambda^L,\lambda^U) \neq (0,0)$, where $\lambda^L := (\lambda_1^L,\ldots,\lambda_m^L)$ and $\lambda^U := (\lambda_1^U,\ldots,\lambda_m^U)$.

In order to obtain optimality conditions of KKT-type for Pareto solutions of problem (FIMP), we use the following well known constraint qualification.

Definition 3.3 (see [34])**.** Let $\bar{x} \in \Omega$. We say that the *constraint qualification* (CQ) is satisfied at $\bar{x}$ if there do not exist $\mu_j \geq 0$, $j \in J(\bar{x})$ not all zero, such that

$$0 \in \sum_{j \in J(\bar{x})} \mu_j \partial h_j(\bar{x}) + N(\bar{x}; S), \tag{CQ}$$

where $J(\bar{x}) := \{j \in J \,:\, g_j(\bar{x}) = 0\}$.

It is worth to mentioning here that the above (CQ) reduces to the classical Mangasarian–Fromovitz constraint qualification when the functions $h_1, \ldots, h_p$ are strictly differentiable at such $\bar{x}$ and $S = \mathbb{R}^n$; see e.g., [35, 36].

Theorem 3.2. *If $\bar{x} \in \mathscr{S}_2^w$(FIMP) and the (CQ) holds at $\bar{x}$, then $\bar{x}$ satisfies the KKT condition.*

Proof. Assume that $\bar{x} \in \mathscr{S}_2^w$(FIMP) and the (CQ) holds at $\bar{x}$. Then by Theorem 3.1, there exist $\lambda_i^L \geq 0$, $\lambda_i^U \geq 0$, $i \in I$, and $\mu_j \geq 0$, $j \in J$ with $\sum_{i \in I}(\lambda_i^L + \lambda_i^U) + \sum_{j \in J} \mu_j = 1$ satisfying (3.1). If $(\lambda^L, \lambda^U) = (0,0)$, then $0 \in \sum_{j \in J} \mu_j \partial h_j(\bar{x}) + N(\bar{x}; S)$ and $\mu_j h_j(\bar{x}) = 0$ for all $j \in J$. Hence, by the condition (CQ), $\mu_j = 0$ for all $j \in J$. This contradicts to the fact that

$$\sum_{i \in I}(\lambda_i^L + \lambda_i^U) + \sum_{j \in J} \mu_j = 1.$$

Therefore, $(\lambda^L, \lambda^U) \neq (0,0)$. The proof is complete. □

The following example shows that the conclusion of Theorem 3.2 may fail if the (CQ) is not satisfied.

Example 3.1. We consider problem (FIMP) with $m = 2$, $n = p = 1$, $S = (-\infty, 0]$, $h(x) = x^2$, $f_1^L(x) = f_1^U(x) = -x + 2$, $f_2^L(x) = f_2^U(x) = -x + 3$, $g_1^L(x) = g_2^L(x) = -2x + 1$, and $g_1^U(x) = g_2^U(x) = -2x + 2$. Then

$$F_1(x) = \left[\frac{-x+2}{-2x+2}, \frac{-x+2}{-2x+1}\right], F_2(x) = \left[\frac{-x+3}{-2x+2}, \frac{-x+3}{-2x+1}\right],$$

and $\Omega = \{0\}$. Clearly, $\bar{x} = 0 \in \mathscr{S}_2^w$(FIMP), $\nabla h(\bar{x}) = 0$, and $N(\bar{x}, S) = [0, +\infty)$. By Theorem 3.1, there exist $\lambda_i^L \geq 0$, $\lambda_i^U \geq 0$, $i \in I = \{1, 2\}$, $\mu \geq 0$ with $\sum_{i \in I}(\lambda_i^L + \lambda_i^U) + \mu = 1$ satisfying (3.1), i.e.,

$$0 \in \frac{\lambda_1^L}{2} + \lambda_2^L + 3\lambda_1^U + 5\lambda_2^U + [0, +\infty).$$

This implies that $\lambda_1^L = \lambda_2^L = \lambda_1^U = \lambda_2^U = 0$ and so the KKT condition do not hold at $\bar{x}$. Actually, the (CQ) fails to hold at $\bar{x}$.

Next we present sufficient conditions for Pareto solutions of (FIMP). In order to obtain these sufficient conditions, we need to introduce concepts of (strictly) generalized convexity at a given point for a family of locally Lipschitzian functions. The following definition is motivated from [28].

Definition 3.4. (i) We say that (F, h) is *generalized convex on S* at $\bar{x} \in S$ if for any $x \in S$, $x_i^{*L} \in \partial f_i^L(\bar{x})$, $x_i^{*U} \in \partial f_i^U(\bar{x})$, $y_i^{*L} \in \partial^+ g_i^L(\bar{x})$, $y_i^{*U} \in \partial^+ g_i^U(\bar{x})$, $i \in I$, and $z_j^* \in \partial h_j(\bar{x})$, $j \in J$, there exists $\nu \in [N(\bar{x};S)]^\circ$ satisfying

$$\begin{aligned}
&f_i^L(x) - f_i^L(\bar{x}) \geq \langle x_i^{*L}, \nu \rangle, \ \ \forall i \in I,\\
&f_i^U(x) - f_i^U(\bar{x}) \geq \langle x_i^{*U}, \nu \rangle, \ \ \forall i \in I,\\
&g_i^L(x) - g_i^L(\bar{x}) \leq \langle y_i^{*L}, \nu \rangle, \ \ \forall i \in I,\\
&g_i^U(x) - g_i^U(\bar{x}) \leq \langle y_i^{*U}, \nu \rangle, \ \ \forall i \in I,\\
&h_j(x) - h_j(\bar{x}) \geq \langle z_j^{*}, \nu \rangle, \ \ \forall j \in J.
\end{aligned}$$

(ii) We say that (F,h) is *strictly generalized convex on S* at $\bar{x} \in S$ if for any $x \in S \setminus \{\bar{x}\}$, $x_i^{*L} \in \partial f_i^L(\bar{x})$, $x_i^{*U} \in \partial f_i^U(\bar{x})$, $y_i^{*L} \in \partial^+ g_i^L(\bar{x})$, $y_i^{*U} \in \partial^+ g_i^U(\bar{x})$, $i \in I$, and $z_j^* \in \partial h_j(\bar{x})$, $j \in J$, there exists $\nu \in [N(\bar{x};S)]^\circ$ satisfying

$$\begin{aligned}
&f_i^L(x) - f_i^L(\bar{x}) > \langle x_i^{*L}, \nu \rangle, \ \ \forall i \in I,\\
&f_i^U(x) - f_i^U(\bar{x}) > \langle x_i^{*U}, \nu \rangle, \ \ \forall i \in I,\\
&g_i^L(x) - g_i^L(\bar{x}) \leq \langle y_i^{*L}, \nu \rangle, \ \ \forall i \in I,\\
&g_i^U(x) - g_i^U(\bar{x}) \leq \langle y_i^{*U}, \nu \rangle, \ \ \forall i \in I,\\
&h_j(x) - h_j(\bar{x}) \geq \langle z_j^{*}, \nu \rangle, \ \ \forall j \in J.
\end{aligned}$$

Remark 3.2. We see that if S is convex and f_i^L, f_i^U, $-g_i^L$, $-g_i^U$, $i \in I$, and h_j, $j \in J$, are convex, then (F,h) is generalized convex on S at any $\bar{x} \in S$ with $\nu = x - \bar{x}$. Moreover, the class of generalized convex functions is properly larger than the one of convex functions; see, e.g., [37, Example 3.2] and [34, Example 3.12].

Theorem 3.3. *Let $\bar{x} \in \Omega$ satisfy the KKT condition.*

(i) *If (F,h) is generalized convex on S at $\bar{x}$, then $\bar{x} \in \mathscr{S}_2^w$(FIMP).*
(ii) *If (F,h) is strictly generalized convex on S at $\bar{x}$, then $\bar{x} \in \mathscr{S}_1$(FIMP) and so $\bar{x} \in \mathscr{S}_2$(FIMP) and $\bar{x} \in \mathscr{S}_1^w$(FIMP).*

Proof. Since $\bar{x}$ satisfies the KKT condition, there exist $(\lambda^L, \lambda^U) \in (\mathbb{R}_+^m \times \mathbb{R}_+^m) \setminus \{(0,0)\}$, $\mu_j \geq 0$, $j \in J$, and $x_i^{*L} \in \partial f_i^L(\bar{x})$, $x_i^{*U} \in \partial f_i^U(\bar{x})$, $y_i^{*L} \in \partial^+ g_i^L(\bar{x})$, $y_i^{*U} \in \partial^+ g_i^U(\bar{x})$, $i \in I$, $z_j^* \in \partial h_j(\bar{x})$, $j \in J$, and $\omega^* \in N(\bar{x};S)$ such that $\mu_j h_j(\bar{x}) = 0$, $j \in J$ and

$$\sum_{i \in I} \frac{\lambda_i^L}{g_i^U(\bar{x})} \left[x_i^{*L} - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} y_i^{*U} \right] + \sum_{i \in I} \frac{\lambda_i^U}{g_i^L(\bar{x})} \left[x_i^{*U} - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})} y_i^{*L} \right] + \sum_{j \in J} \mu_j z_j^* + \omega^* = 0,$$

or, equivalently,

$$\sum_{i \in I} \frac{\lambda_i^L}{g_i^U(\bar{x})} \left[x_i^{*L} - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} y_i^{*U} \right] + \sum_{i \in I} \frac{\lambda_i^U}{g_i^L(\bar{x})} \left[x_i^{*U} - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})} y_i^{*L} \right] + \sum_{j \in J} \mu_j z_j^* = -\omega^*. \tag{3.7}$$

First, we prove (i). Assume on the contrary that $\bar{x} \notin \mathscr{S}_2^w$. This means that there exists $\hat{x} \in \Omega$ such that $F_i(\hat{x}) <_{LU}^s F_i(\bar{x})$, $\forall i \in I$, or, equivalently,

$$\frac{f_i^L(\hat{x})}{g_i^U(\hat{x})} < \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} \text{ and } \frac{f_i^U(\hat{x})}{g_i^L(\hat{x})} < \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}, \ \ \forall i \in I. \tag{3.8}$$

By the generalized convexity of (F,h) at $\bar{x}$, for such $\hat{x}$, there is $\nu \in [N(\bar{x};S)]^\circ$ such that

$$\begin{aligned}
&\sum_{i\in I}\frac{\lambda_i^L}{g_i^U(\bar{x})}\left[\langle x_i^{*L},\nu\rangle - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}\langle y_i^{*U},\nu\rangle\right] + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(\bar{x})}\left[\langle x_i^{*U},\nu\rangle - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}\langle y_i^{*L},\nu\rangle\right] + \sum_{j\in J}\mu_j\langle z_j^*,\nu\rangle \\
&\le \sum_{i\in I}\frac{\lambda_i^L}{g_i^U(\bar{x})}\left[f_i^L(\hat{x}) - f_i^L(\bar{x}) - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}(g_i^U(\hat{x}) - g_i^U(\bar{x}))\right] \\
&\quad + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(\bar{x})}\left[f_i^U(\hat{x}) - f_i^U(\bar{x}) - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}(g_i^L(\hat{x}) - g_i^L(\bar{x}))\right] + \sum_{j\in J}\mu_j(h_j(\hat{x}) - h_j(\bar{x})) \\
&= \sum_{i\in I}\frac{\lambda_i^L}{g_i^U(\bar{x})}\left[f_i^L(\hat{x}) - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}g_i^U(\hat{x})\right] + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(\bar{x})}\left[f_i^U(\hat{x}) - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}g_i^L(\hat{x})\right] \\
&\qquad + \sum_{j\in J}\mu_j(h_j(\hat{x}) - h_j(\bar{x})).
\end{aligned}$$

It follows from (3.7) and relations $\omega^* \in N(\bar{x};S)$ and $\nu \in [N(\bar{x},S)]^\circ$ that

$$\begin{aligned}
0 \le \langle -\omega^*,\nu\rangle &= \sum_{i\in I}\frac{\lambda_i^L}{g_i^U(\bar{x})}\left[\langle x_i^{*L},\nu\rangle - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}\langle y_i^{*U},\nu\rangle\right] + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(\bar{x})}\left[\langle x_i^{*U},\nu\rangle - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}\langle y_i^{*L},\nu\rangle\right] \\
&\qquad + \sum_{j\in J}\mu_j\langle z_j^*,\nu\rangle \\
&\le \sum_{i\in I}\frac{\lambda_i^L}{g_i^U(\bar{x})}\left[f_i^L(\hat{x}) - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}g_i^U(\hat{x})\right] + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(\bar{x})}\left[f_i^U(\hat{x}) - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}g_i^L(\hat{x})\right] \\
&\qquad + \sum_{j\in J}\mu_j(h_j(\hat{x}) - h_j(\bar{x})). \qquad (3.9)
\end{aligned}$$

Furthermore, we see that $\mu_j h_j(\bar{x}) = 0$ and $\mu_j h_j(\hat{x}) \le 0$ for all $j \in J$. This and (3.9) imply that

$$\sum_{i\in I}\frac{\lambda_i^L}{g_i^U(\bar{x})}\left[f_i^L(\hat{x}) - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}g_i^U(\hat{x})\right] + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(\bar{x})}\left[f_i^U(\hat{x}) - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}g_i^L(\hat{x})\right] \ge 0.$$

Since $(\lambda^L,\lambda^U) \neq (0,0)$, there exists $i_0 \in I$ such that

$$f_{i_0}^L(\hat{x}) - \frac{f_{i_0}^L(\bar{x})}{g_{i_0}^U(\bar{x})}g_{i_0}^U(\hat{x}) \ge 0 \ \text{ or } \ f_{i_0}^U(\hat{x}) - \frac{f_{i_0}^U(\bar{x})}{g_{i_0}^L(\bar{x})}g_{i_0}^L(\hat{x}) \ge 0,$$

or, equivalently,

$$\frac{f_{i_0}^L(\hat{x})}{g_{i_0}^U(\hat{x})} \ge \frac{f_{i_0}^L(\bar{x})}{g_{i_0}^U(\bar{x})} \quad \text{or} \quad \frac{f_{i_0}^U(\hat{x})}{g_{i_0}^L(\hat{x})} \ge \frac{f_{i_0}^U(\bar{x})}{g_{i_0}^L(\bar{x})},$$

This together with (3.8) gives a contradiction, which completes the proof of (i).

We now prove (ii). Suppose on the contrary that $\bar{x} \notin \mathscr{S}_1$(FIMP). Then there exists $\hat{x} \in \Omega$ such that

$$\begin{cases} F_i(\hat{x}) \le_{LU} F_i(\bar{x}), \ \ \forall i \in I, \\ F_k(\hat{x}) <_{LU} F_k(\bar{x}), \ \ \text{for at least one } \ k \in I. \end{cases}$$

This implies that $\hat{x} \neq \bar{x}$ and

$$\frac{f_i^L(\hat{x})}{g_i^U(\hat{x})} \le \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})} \ \text{ and } \ \frac{f_i^U(\hat{x})}{g_i^L(\hat{x})} \le \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}, \ \ \forall i \in I, \qquad (3.10)$$

with at least one of the inequalities is strict. Hence, by the strictly generalized convexity of (F,h) at $\bar{x}$ and the assumption that $(\lambda^L,\lambda^U)\neq(0,0)$, for $\hat{x}$ above, there exists $\nu\in[N(\bar{x};S)]^\circ$ such that

$$\begin{aligned}0\leq\langle-\omega^*,\nu\rangle&=\sum_{i\in I}\frac{\lambda_i^L}{g_i^U(\bar{x})}\left[\langle x_i^{*L},\nu\rangle-\frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}\langle y_i^{*U},\nu\rangle\right]+\sum_{i\in I}\frac{\lambda_i^U}{g_i^L(\bar{x})}\left[\langle x_i^{*U},\nu\rangle-\frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}\langle y_i^{*L},\nu\rangle\right]\\&\qquad+\sum_{j\in J}\mu_j\langle z_j^*,\nu\rangle\\&<\sum_{i\in I}\frac{\lambda_i^L}{g_i^U(\bar{x})}\left[f_i^L(\hat{x})-\frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}g_i^U(\hat{x})\right]+\sum_{i\in I}\frac{\lambda_i^U}{g_i^L(\bar{x})}\left[f_i^U(\hat{x})-\frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}g_i^L(\hat{x})\right]\\&\qquad+\sum_{j\in J}\mu_j(h_j(\hat{x})-h_j(\bar{x}))\\&\leq\sum_{i\in I}\frac{\lambda_i^L}{g_i^U(\bar{x})}\left[f_i^L(\hat{x})-\frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}g_i^U(\hat{x})\right]+\sum_{i\in I}\frac{\lambda_i^U}{g_i^L(\bar{x})}\left[f_i^U(\hat{x})-\frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}g_i^L(\hat{x})\right].\end{aligned}$$

This implies that there exists $i_0\in I$ satisfying

$$f_{i_0}^L(\hat{x})-\frac{f_{i_0}^L(\bar{x})}{g_{i_0}^U(\bar{x})}g_{i_0}^U(\hat{x})>0\ \text{ or }\ f_{i_0}^U(\hat{x})-\frac{f_{i_0}^U(\bar{x})}{g_{i_0}^L(\bar{x})}g_{i_0}^L(\hat{x})>0,$$

or, equivalently,

$$\frac{f_{i_0}^L(\hat{x})}{g_{i_0}^U(\hat{x})}>\frac{f_{i_0}^L(\bar{x})}{g_{i_0}^U(\bar{x})}\ \text{ or }\ \frac{f_{i_0}^U(\hat{x})}{g_{i_0}^L(\hat{x})}>\frac{f_{i_0}^U(\bar{x})}{g_{i_0}^L(\bar{x})}.$$

It together with (3.10) gives a contradiction. The proof is complete. □

Remark 3.3. The condition (3.1) alone is not sufficient for Pareto solutions of (FIMP) if the (strict) generalized convexity of (F,h) at the point under consideration is violated. To see this, let us consider the following example.

Example 3.2. We consider problem (FIMP) with $m=2$, $n=p=1$, $S=(-\infty,1]$, $h(x)=-x^2$, $f_1^L(x)=f_1^U(x)=-x^3+1$, $f_2^L(x)=f_2^U(x)=-2x^3+3$, $g_1^L(x)=g_2^L(x)=x^2+1$, and $g_1^U(x)=g_2^U(x)=x^2+2$. Then

$$F_1(x)=\left[\frac{-x^3+1}{x^2+2},\frac{-x^3+1}{x^2+1}\right],F_2(x)=\left[\frac{-2x^3+3}{x^2+2},\frac{-2x^3+3}{x^2+1}\right],$$

and $\Omega=S$. Let $\bar{x}=0\in\Omega$. Then, we have $N(\bar{x};S)=\{0\}$ and

$$\partial f_i^L(\bar{x})=\partial f_i^U(\bar{x})=\partial^+g_i^L(\bar{x})=\partial^+g_i^U(\bar{x})=\partial h(\bar{x})=\{0\},i\in\{1,2\}.$$

Thus, $\bar{x}$ satisfies the KKT condition. However, since $\frac{1}{2}\in S$ and

$$F\left(\frac{1}{2}\right)=\left(\left[\frac{7}{18},\frac{7}{10}\right],\left[\frac{11}{9},\frac{11}{5}\right]\right)<_{LU}^{s}F(\bar{x})=\left(\left[\frac{1}{2},1\right],\left[\frac{3}{2},3\right]\right),$$

we arrive at $\bar{x}\notin\mathscr{S}_2^w(\text{FIMP})$. Meanwhile, it is easy to check that (F,h) is not generalized convex at $\bar{x}$.

4. Approximate duality theorems

Let $\mathscr{A} := (A_1, \ldots, A_m)$ and $\mathscr{B} := (B_1, \ldots, B_m)$, where A_i, B_i, $i \in I$, are intervals in $\mathscr{K}_c$. In what follows, we use the following notations for convenience.

$$\mathscr{A} \preceq_{LU} \mathscr{B} \Leftrightarrow \begin{cases} A_i \leq_{LU} B_i, & \forall i \in I, \\ A_k <_{LU} B_k, & \text{for at least one } k \in I. \end{cases}$$

$$\mathscr{A} \not\preceq_{LU} \mathscr{B} \text{ is the negation of } \mathscr{A} \preceq_{LU} \mathscr{B}.$$

$$\mathscr{A} \prec^s_{LU} \mathscr{B} \Leftrightarrow A_i <^s_{LU} B_i, \ \forall i \in I.$$

$$\mathscr{A} \not\prec^s_{LU} \mathscr{B} \text{ is the negation of } \mathscr{A} \prec^s_{LU} \mathscr{B}.$$

For $y \in \mathbb{R}^n$, $(\lambda^L, \lambda^U) \in (\mathbb{R}^m_+ \times \mathbb{R}^m)_+ \setminus \{(0,0)\}$, and $\mu \in \mathbb{R}^p_+$, put

$$\mathscr{L}(y, \lambda^L, \lambda^U, \mu) := F(y) = (F_1(y), \ldots, F_m(y)),$$

where

$$F_i(y) := \frac{f_i(y)}{g_i(y)} = \left[\frac{f_i^L(y)}{g_i^U(y)}, \frac{f_i^U(y)}{g_i^L(y)}\right], \ i \in I.$$

In connection with the primal problem (FIMP), we consider the following dual problem in the sense of Mond–Weir:

$$LU - \max \ \mathscr{L}(y, \lambda^L, \lambda^U, \mu) \qquad \text{s.t. } (y, \lambda^L, \lambda^U, \mu) \in \Omega_{MW}, \tag{FIMD$_{MW}$}$$

where the feasible set Ω_{MW} is defined by

$$\Omega_{MW} := \Big\{(y, \lambda^L, \lambda^U, \mu) \in S \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \mathbb{R}^p_+ : 0 \in \sum_{i \in I} \frac{\lambda_i^L}{g_i^U(y)} \left[\partial f_i^L(y) - \frac{f_i^L(y)}{g_i^U(y)} \partial^+ g_i^U(y)\right]$$
$$+ \sum_{i \in I} \frac{\lambda_i^U}{g_i^L(y)} \left[\partial f_i^U(y) - \frac{f_i^U(y)}{g_i^L(y)} \partial^+ g_i^L(y)\right] + \sum_{j \in J} \mu_j \partial h_j(y) + N(y; S),$$
$$\sum_{j \in J} \mu_j h_j(y) \geq 0, \ \sum_{i \in I} (\lambda_i^L + \lambda_i^U) + \sum_{j \in J} \mu_j = 1, (\lambda^L, \lambda^U) \neq (0,0)\Big\}.$$

Definition 4.1. Let $(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in \Omega_{MW}$. We say that

(i) $(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a *type-1 Pareto solution* of (FIMD$_{MW}$), denoted by

$$(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in \mathscr{S}_1(\text{FIMD}_{MW}),$$

if there is no $(y, \lambda^L, \lambda^U, \mu) \in \Omega_{MW}$ such that

$$\mathscr{L}(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \preceq_{LU} \mathscr{L}(y, \lambda^L, \lambda^U, \mu).$$

(ii) $(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a *type-2 weakly Pareto solution* of (FIMD$_{MW}$), denoted by

$$(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in \mathscr{S}^w_2(\text{FIMD}_{MW}),$$

if there is no $(y, \lambda^L, \lambda^U, \mu) \in \Omega_{MW}$ such that

$$\mathscr{L}(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \preceq^s_{LU} \mathscr{L}(y, \lambda^L, \lambda^U, \mu).$$

The following theorem describes weak duality relations between the primal problem (FIMP) and the dual problem (FIMD$_{MW}$).

Theorem 4.1 (Weak duality). *Let $x \in \Omega$ and $(y, \lambda^L, \lambda^U, \mu) \in \Omega_{MW}$.*

(i) *If (F, h) is generalized convex on S at y, then*

$$F(x) \not\prec^s_{LU} \mathscr{L}(y, \lambda^L, \lambda^U, \mu).$$

(ii) *If (F, h) is strictly generalized convex on S at y, then*

$$F(x) \not\preceq_{LU} \mathscr{L}(y, \lambda^L, \lambda^U, \mu).$$

Proof. Since $x \in \Omega$ and $(y, \lambda^L, \lambda^U, \mu) \in \Omega_{MW}$, we have $x, y \in S$,

$$h_j(x) \le 0, \ \sum_{j \in J} \mu_j h_j(y) \ge 0, \tag{4.1}$$

and

$$0 \in \sum_{i \in I} \frac{\lambda_i^L}{g_i^U(y)} \left[\partial f_i^L(y) - \frac{f_i^L(y)}{g_i^U(y)} \partial^+ g_i^U(y) \right] + \sum_{i \in I} \frac{\lambda_i^U}{g_i^L(y)} \left[\partial f_i^U(y) - \frac{f_i^U(y)}{g_i^L(y)} \partial^+ g_i^L(y) \right] + \sum_{j \in J} \mu_j \partial h_j(y) + N(y; S).$$

This implies that there exist $x_i^{*L} \in \partial f_i^L(y)$, $x_i^{*U} \in \partial f_i^U(y)$, $y_i^{*L} \in \partial^+ g_i^L(y)$, $y_i^{*U} \in \partial^+ g_i^U(y)$, $i \in I$, $z_j^* \in \partial h_j(y)$, $j \in J$, and $\omega^* \in N(y; S)$ such that

$$\sum_{i \in I} \frac{\lambda_i^L}{g_i^U(y)} \left[x_i^{*L} - \frac{f_i^L(y)}{g_i^U(y)} y_i^{*U} \right] + \sum_{i \in I} \frac{\lambda_i^U}{g_i^L(y)} \left[x_i^{*U} - \frac{f_i^U(y)}{g_i^L(y)} y_i^{*L} \right] + \sum_{j \in J} \mu_j z_j^* = -\omega^*. \tag{4.2}$$

We first prove (i). Suppose on the contrary that

$$F(x) \prec^s_{LU} \mathscr{L}(y, \lambda^L, \lambda^U, \mu),$$

or, equivalently,

$$F_i(x) \prec^s_{LU} \mathscr{L}_i(y, \lambda^L, \lambda^U, \mu), \ \ \forall i \in I.$$

Then,

$$\frac{f_i^L(x)}{g_i^U(x)} < \frac{f_i^L(y)}{g_i^U(y)} \ \text{ and } \ \frac{f_i^U(x)}{g_i^L(x)} < \frac{f_i^U(y)}{g_i^L(y)}, \ \ \forall i \in I. \tag{4.3}$$

By the generalized convex property of (F, h) on S at y, for such x, there exists $\nu \in [N(y, S)]^\circ$ such that

$$\begin{aligned}
&\sum_{i \in I} \frac{\lambda_i^L}{g_i^U(y)} \left[\langle x_i^{*L}, \nu \rangle - \frac{f_i^L(y)}{g_i^U(y)} \langle y_i^{*U}, \nu \rangle \right] + \sum_{i \in I} \frac{\lambda_i^U}{g_i^L(y)} \left[\langle x_i^{*U}, \nu \rangle - \frac{f_i^U(y)}{g_i^L(y)} \langle y_i^{*L}, \nu \rangle \right] + \sum_{j \in J} \mu_j \langle z_j^*, \nu \rangle \\
&\le \sum_{i \in I} \frac{\lambda_i^L}{g_i^U(y)} \left[f_i^L(x) - f_i^L(y) - \frac{f_i^L(y)}{g_i^U(y)} (g_i^U(x) - g_i^U(y)) \right] \\
&\quad + \sum_{i \in I} \frac{\lambda_i^U}{g_i^L(y)} \left[f_i^U(x) - f_i^U(y) - \frac{f_i^U(y)}{g_i^L(y)} (g_i^L(x) - g_i^L(y)) \right] + \sum_{j \in J} \mu_j (h_j(x) - h_j(y)) \\
&= \sum_{i \in I} \frac{\lambda_i^L}{g_i^U(y)} \left[f_i^L(x) - \frac{f_i^L(y)}{g_i^U(y)} g_i^U(x) \right] + \sum_{i \in I} \frac{\lambda_i^U}{g_i^L(y)} \left[f_i^U(x) - \frac{f_i^U(y)}{g_i^L(y)} g_i^L(x) \right] \\
&\qquad + \sum_{j \in J} \mu_j (h_j(x) - h_j(y)).
\end{aligned}$$

It follows from (4.2) and relations $\omega^* \in N(y;S)$ and $\nu \in [N(y,S)]^\circ$ that

$$0 \leq \sum_{i\in I} \frac{\lambda_i^L}{g_i^U(y)}\left[\langle x_i^{*L},\nu\rangle - \frac{f_i^L(y)}{g_i^U(y)}\langle y_i^{*U},\nu\rangle\right] + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(y)}\left[\langle x_i^{*U},\nu\rangle - \frac{f_i^U(y)}{g_i^L(y)}\langle y_i^{*L},\nu\rangle\right] + \sum_{j\in J}\mu_j\langle z_j^*,\nu\rangle.$$

Thus,

$$0 \leq \sum_{i\in I} \frac{\lambda_i^L}{g_i^U(y)}\left[f_i^L(x) - \frac{f_i^L(y)}{g_i^U(y)}g_i^U(x)\right] + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(y)}\left[f_i^U(x) - \frac{f_i^U(y)}{g_i^L(y)}g_i^L(x)\right] + \sum_{j\in J}\mu_j(h_j(x)-h_j(y)).$$

It together with (4.1) implies that

$$0 \leq \sum_{i\in I} \frac{\lambda_i^L}{g_i^U(y)}\left[f_i^L(x) - \frac{f_i^L(y)}{g_i^U(y)}g_i^U(x)\right] + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(y)}\left[f_i^U(x) - \frac{f_i^U(y)}{g_i^L(y)}g_i^L(x)\right]. \tag{4.4}$$

By (4.4) and the fact that $(\lambda^L,\lambda^U) \neq (0,0)$, it follows that there is $i_0 \in I$ such that

$$f_{i_0}^L(x) - \frac{f_{i_0}^L(y)}{g_{i_0}^U(y)}g_{i_0}^U(x) \geq 0 \ \text{ or } \ f_{i_0}^U(x) - \frac{f_{i_0}^U(y)}{g_{i_0}^L(y)}g_{i_0}^L(x) \geq 0,$$

or, equivalently,

$$\frac{f_{i_0}^L(x)}{g_{i_0}^U(x)} \geq \frac{f_{i_0}^L(y)}{g_{i_0}^U(y)} \ \text{ or } \ \frac{f_{i_0}^U(x)}{g_{i_0}^L(x)} \geq \frac{f_{i_0}^U(y)}{g_{i_0}^L(y)},$$

which contradicts (4.3) and therefore completes the proof of (i).

Next we prove (ii). Assume to the contrary that

$$F(x) \preceq_{LU} \mathscr{L}(y,\lambda^L,\lambda^U,\mu).$$

This means that

$$\begin{cases} F_i(x) \leq_{LU} F_i(y), & \forall i \in I, \\ F_k(x) <_{LU} F_k(y), & \text{for at least one } \ k \in I. \end{cases} \tag{4.5}$$

Hence, $x \neq y$. By the strictly of (F,h) on S at y and the assumption that $(\lambda^L,\lambda^U) \neq (0,0)$, for such x, there exists $\nu \in [N(y,S)]^\circ$ such that

$$\begin{aligned}
&\sum_{i\in I} \frac{\lambda_i^L}{g_i^U(y)}\left[\langle x_i^{*L},\nu\rangle - \frac{f_i^L(y)}{g_i^U(y)}\langle y_i^{*U},\nu\rangle\right] + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(y)}\left[\langle x_i^{*U},\nu\rangle - \frac{f_i^U(y)}{g_i^L(y)}\langle y_i^{*L},\nu\rangle\right] + \sum_{j\in J}\mu_j\langle z_j^*,\nu\rangle \\
&< \sum_{i\in I} \frac{\lambda_i^L}{g_i^U(y)}\left[f_i^L(x) - f_i^L(y) - \frac{f_i^L(y)}{g_i^U(y)}(g_i^U(x) - g_i^U(y))\right] \\
&\quad + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(y)}\left[f_i^U(x) - f_i^U(y) - \frac{f_i^U(y)}{g_i^L(y)}(g_i^L(x)-g_i^L(y))\right] + \sum_{j\in J}\mu_j(h_j(x)-h_j(y)) \\
&= \sum_{i\in I} \frac{\lambda_i^L}{g_i^U(y)}\left[f_i^L(x) - \frac{f_i^L(y)}{g_i^U(y)}g_i^U(x)\right] + \sum_{i\in I}\frac{\lambda_i^U}{g_i^L(y)}\left[f_i^U(x) - \frac{f_i^U(y)}{g_i^L(y)}g_i^L(x)\right] \\
&\quad + \sum_{j\in J}\mu_j(h_j(x)-h_j(y)).
\end{aligned}$$

Continuing a similar procedure as in the proof of (i), we arrive at

$$0 < \sum_{i\in I} \frac{\lambda_i^L}{g_i^U(y)} \left[f_i^L(x) - \frac{f_i^L(y)}{g_i^U(y)} g_i^U(x) \right] + \sum_{i\in I} \frac{\lambda_i^U}{g_i^L(y)} \left[f_i^U(x) - \frac{f_i^U(y)}{g_i^L(y)} g_i^L(x) \right].$$

Hence, there exists $i_0 \in I$ such that

$$\frac{f_{i_0}^L(x)}{g_{i_0}^U(x)} > \frac{f_{i_0}^L(y)}{g_{i_0}^U(y)} \text{ or } \frac{f_{i_0}^U(x)}{g_{i_0}^L(x)} > \frac{f_{i_0}^U(y)}{g_{i_0}^L(y)}.$$

It together with (4.5) gives a contradiction, which completes the proof. $\square$

The following example asserts the importance of the generalized convexity of (F,h) on S used in Theorem 4.1.This means that the conclusion of Theorem 4.1 may fail if this property has been violated.

Example 4.1. We consider problem (FIMP) with $m=2$, $n=p=1$, $S=(-\infty,1]$, $h(x)=-|x|$, $f_1^L(x)=f_1^U(x)=1-x^3$, $f_2^L(x)=f_2^U(x)=1-x^5$, $g_1^L(x)=g_2^L(x)=x^2+1$, and $g_1^U(x)=g_2^U(x)=x^2+2$. Then

$$F_1(x)=\left[\frac{1-x^3}{x^2+2},\frac{1-x^3}{x^2+1}\right], F_2(x)=\left[\frac{1-x^5}{x^2+2},\frac{1-x^5}{x^2+1}\right],$$

and $\Omega=S$. Let $\bar{x}=1\in\Omega$. We now consider the dual problem (FIMD$_{MW}$). By choosing $\bar{y}=0\in S$, $\bar{\lambda}_1^L=\bar{\lambda}_2^L=\bar{\lambda}_1^U=\bar{\lambda}_2^U=\frac{1}{4}$, $\bar{\mu}=0$, we have $(\bar{y},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu})\in\Omega_{MW}$ and that

$$F(\bar{x})=\big([0,0],[0,0]\big)\prec_{LU}^{s}\mathscr{L}(\bar{y},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu})=\left(\left[\frac{1}{2},1\right],\left[\frac{1}{2},1\right]\right).$$

The reason is that the generalized convexity of (F,h) on S has been violated at $\bar{y}$.

Next we present a theorem that formulates strong duality relations between the primal problem (FIMP) and the dual problem (FIMD$_{MW}$).

Theorem 4.2 (Strong duality). *Suppose that $\bar{x}\in\mathscr{S}_2^w$(FIMP) and the (CQ) is satisfied at this point. Then there exist $(\bar{\lambda}^L,\bar{\lambda}^U)\in(\mathbb{R}_+^m\times\mathbb{R}_+^m)\setminus\{(0,0)\}$, and $\bar{\mu}\in\mathbb{R}_+^p$ such that $(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu})\in\Omega_{MW}$ and $F(\bar{x})=\mathscr{L}(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu})$. Furthermore,*

(i) *If (F,h) is generalized convex on S at $\bar{x}$, then $(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu})$ is a type-2 weakly Pareto solution of (FIMD$_{MW}$).*

(ii) *If (F,h) is strictly generalized convex on S at $\bar{x}$, then $(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu})$ is a type-1 Pareto solution of (FIMD$_{MW}$).*

Proof. Since $\bar{x}\in\mathscr{S}_2^w$(FIMP) and the (CQ) is satisfied at this point, by Theorem 3.2, there exist $(\bar{\lambda}^L,\bar{\lambda}^U)\in(\mathbb{R}_+^m\times\mathbb{R}_+^m)\setminus\{(0,0)\}$ and $\bar{\mu}\in\mathbb{R}_+^p$ such that $(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu})\in\Omega_{MW}$. Clearly,

$$F(\bar{x})=\mathscr{L}(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu}).$$

(i) Since (F,h) is generalized convex on S at $\bar{x}$, by Theorem 4.1(i), we have

$$F(\bar{x})=\mathscr{L}(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu})\not\prec_{LU}^{s} L(y,\lambda^L,\lambda^U,\mu)$$

for all $(y,\lambda^L,\lambda^U,\mu)\in\Omega_{MW}$. This means that $(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu})$ is a type-2 weakly Pareto solution of (FIMD$_{MW}$).

(ii) If (F,h) is strictly generalized convex on S at $\bar{x}$, then by invoking Theorem 4.1(ii), we obtain

$$F(\bar{x}) = \mathscr{L}(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu}) \nleq_{LU} L(y,\lambda^L,\lambda^U,\mu)$$

for all $(y,\lambda^L,\lambda^U,\mu) \in \Omega_{MW}$. Thus, $(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu})$ is a type-1 Pareto solution of (FIMD$_{MW}$), which completes the proof. □

Remark 4.1. The (CQ) condition plays an important role in establishing the strong duality results in Theorem 4.2. This means that if the (CQ) is not satisfied at a type-2 weakly Pareto solution of (FIMP), then strong dual relations in Theorem 4.2 are no longer true at this point. Indeed, let us look back at Example 3.1. We see that $\bar{x} = 0 \in \mathscr{S}_2^w$(FIMP). Furthermore, the (CQ) is not satisfied at $\bar{x}$ and there do not exist a triple $(\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu})$ such that $(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu}) \in \Omega_{MW}$. Thus, in this case, we do not have strong dual relations.

We finish this section by establishing converse-like duality relations for Pareto solutions between the primal problem (FIMP) and the dual one (FIMD$_{MW}$).

Theorem 4.3 (Converse-like duality)**.** *Let* $(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu}) \in \Omega_{MW}$.

(i) *If $\bar{x} \in \Omega$ and (F,h) is generalized convex on S at $\bar{x}$, then $\bar{x}$ is a type-2 weakly Pareto solution of* (FIMP).

(ii) *If $\bar{x} \in \Omega$ and (F,h) is strictly generalized convex on S at $\bar{x}$, then $\bar{x}$ is a type-1 Pareto solution of* (FIMP).

Proof. (i) Since $(\bar{x},\bar{\lambda}^L,\bar{\lambda}^U,\bar{\mu}) \in \Omega_{MW}$, we have

$$\begin{aligned} 0 \in \sum_{i\in I} \frac{\lambda_i^L}{g_i^U(\bar{x})}\left[\partial f_i^L(\bar{x}) - \frac{f_i^L(\bar{x})}{g_i^U(\bar{x})}\partial^+ g_i^U(\bar{x})\right] + \sum_{i\in I} \frac{\lambda_i^U}{g_i^L(\bar{x})}\left[\partial f_i^U(\bar{x}) - \frac{f_i^U(\bar{x})}{g_i^L(\bar{x})}\partial^+ g_i^L(\bar{x})\right] \\ + \sum_{j\in J} \mu_j \partial h_j(\bar{x}) + N(\bar{x};S), \end{aligned}$$

$$\sum_{j\in J}\mu_j h_j(\bar{x}) \geq 0,\ \sum_{i\in I}(\lambda_i^L+\lambda_i^U) + \sum_{j\in J}\mu_j = 1, (\lambda^L,\lambda^U) \neq (0,0). \tag{4.6}$$

Since $\bar{x} \in \Omega$, one has $h_j(\bar{x}) \leq 0$ for all $j \in J$. Hence, $\mu_j h_j(\bar{x}) \leq 0$ for all $j \in J$. This together with (4.6) yields $\mu_j h_j(\bar{x}) = 0$ for all $j \in J$. Thus, by the generalized convexity of (F,h) on S at $\bar{x}$ and Theorem 3.3(i), $\bar{x}$ is a type-2 weakly Pareto solution of (FIMP).

The proof of (ii) is similar to that of (i) by using the strictly generalized convexity of (F,h) and Theorem 3.3(ii), so omitted. □

Acknowledgments

The authors would like to thank the two anonymous referees, whose suggestions and comments improved the paper. This research is funded by Hanoi Pedagogical University 2 under grant number HPU2.UT-2021.15.

References

[1] I. Ahmad, K. Kummari, S. Al-Homidan, Sufficiency and duality for nonsmooth interval-valued optimization problems via generalized invex-infine functions, J. Oper. Res. Soc. China (2022). doi: 10.1007/s40305-021-00381-6.

[2] K.D. Bae, T. Shitkovskaya, Z. Hong, D.S. Kim, Optimality conditions and duality in nonsmooth multiobjective fractional programming, J. Nonlinear Convex Anal. 23 (2022), 99–112.

[3] Y. Chalco-Cano, W.A. Lodwick, A. Rufian-Lizana, Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative, Fuzzy Optim. Decis. Mak. 12 (2013), 305–322.

[4] N.H. Hung, H.N. Tuan, N.V. Tuyen, On approximate quasi Pareto solutions in nonsmooth semi-infinite interval-valued vector optimization problems, Appl. Anal. (2022). doi: 10.1080/00036811.2022.2027385.

[5] H. Ishibuchi, H. Tanaka, Multiobjective programming in optimization of the interval objective function, European J. Oper. Res. 48 (1990), 219–225.

[6] M. Jennane, E.M. Kalmoun, L. Lafhim, Optimality conditions for nonsmooth interval-valued and multiobjective semi-infinite programming, RAIRO-Oper. Res. 55 (2021), 1–11.

[7] P. Kumar, B. Sharma, J. Dagar, Interval-valued programming problem with infinite constraints, J. Oper. Res. Soc. China 6 (2018), 611–626.

[8] K. Kummari, I. Ahmad, Sufficient optimality conditions and duality for nonsmooth interval-valued optimization problems via L-invex-infine functions, U.P.B. Sci. Bull. Ser. A 82 ((2020)), 45–54.

[9] D.V. Luu, T.T. Mai, Optimality and duality in constrained interval-valued optimization, 4OR-Q. J. Oper. Res. 16 (2018), 311–337.

[10] R. Osuna-Gómez, B. Hernádez-Jiménez, Y. Chalco-Cano, G. Ruiz-Gazón, New efficiency conditions for multiobjective interval-valued programming problems, Inf. Sci. 420 (2017), 235–248.

[11] X. Qian, K.R. Wang, X.B. Li, Solving vector interval-valued optimization problems with infinite interval constraints via integral-type penalty function, Optimization (2021). doi: 10.1080/02331934.2021.1906872.

[12] A.D. Singh, B.A. Dar, Optimality conditions in multiobjective programming problems with interval valued objective functions, Control Cybern. 44 (2015), 19–45.

[13] D. Singh, B.A. Dar, D.S. Kim, KKT optimality conditions in interval valued multiobjective programming with generalized differentiable functions, European J. Oper. Res. 254 (2016), 29–39.

[14] D. Singh, B.A. Dar, D.S. Kim, Sufficiency and duality in non-smooth interval valued programming problems, J. Ind. Manag. Optim. 15 (2019), 647–665.

[15] T.V. Su, D.H. Dinh, Duality results for interval-valued pseudoconvex optimization problem with equilibrium constraints with applications, Comput. Appl. Math. 39 (2020), 1–24.

[16] L.T. Tung, Karush–Kuhn–Tucker optimality conditions and duality for semi-infinite programming with multiple interval-valued objective functions, J. Nonlinear Funct. Anal. 2019 (2019), 1–21.

[17] L.T. Tung, Karush–Kuhn–Tucker optimality conditions and duality for convex semi-infinite programming with multiple interval-valued objective functions, J. Appl. Math. Comput. 62 (2020), 67–91.

[18] N.V. Tuyen, Approximate solutions of interval-valued optimization problems, Investigación Oper. 42 (2021), 223–237.

[19] H.C. Wu,, The Karush–Kuhn–Tuker optimality conditions in an optimization problem with interval valued objective functions, Eur. J. Oper. Res. 176 (2007), 46–59.

[20] H.C. Wu, The Karush–Kuhn–Tucker optimality conditions in multiobjective programming problems with interval-valued objective functions, Eur. J. Oper. Res. 196 (2009), 49–60.

[21] H.C. Wu, The Karush–Kuhn–Tucker optimality conditions for multi-objective programming problems with fuzzy-valued objective functions, Fuzzy Optim. Decis. Mak. 8 (2009), 1–28.

[22] H.C. Wu, Duality theory for optimization problems with interval-valued objective functions, J. Optim. Theory Appl. 144 (2010), 615–628.

[23] C.L. Zhang, N.J. Huang, On Ekeland's variational principle for interval-valued functions with applications, Fuzzy Sets and Systems, 436 (2022), 152–174.

[24] I.P. Debnath, S.K. Gupta, Necessary and sufficient optimality conditions for fractional interval-valued optimization problems. In: K. Deep, M. Jain, S. Salhi, (eds) Decision Science in Action. Asset Analytics, pp. 155–173, Springer, 2019.

[25] B.A. Dar, A. Jayswal, D. Singh, Optimality, duality and saddle point analysis for interval-valued nondifferentiable multiobjective fractional programming problems, Optimization 70 (2021), 1275–1305.

[26] I.P. Debnath, S.K. Gupta, The Karush–Kuhn–Tucker conditions for multiple objective fractional interval valued optimization problems, RAIRO Oper. Res. 54 (2020), 1161–1188.

[27] T.D. Chuong, D.S. Kim, A class of nonsmooth fractional multiobjective optimization problems, Ann. Oper. Res. 244 (2016), 367–383.
[28] T.D. Chuong, Nondifferentiable fractional semi-infinite multiobjective optimization problems, Oper. Res. Lett. 44 (2016), 260-–266.
[29] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, Vol. 1: Basic Theory, Springer, Berlin, 2006.
[30] B.S. Mordukhovich, Variational analysis and applications, Springer, Cham, 2018.
[31] G. Alefeld, J. Herzberger, Introduction to Interval Computations, Academic Press, New-York, 1983.
[32] R.E. Moore, Interval Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1966.
[33] R.E. Moore, Method and Applications of Interval Analysis, Studies in Applied Mathematics, SIAM, Philadelphia, 1979.
[34] T.D. Chuong, D.S. Kim, Approximate solutions of multiobjective optimization problems, Positivity 20 (2016), 187–207.
[35] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, Vol. 2: Applications, Springer, Berlin, 2006.
[36] T.D. Chuong, N.Q. Huy, J.-C. Yao, Subdifferentials of marginal functions in semi-infinite programming, SIAM J. Optim. 20 (2009), 1462–1477.
[37] T.D. Chuong, D.S. Kim, Nonsmooth semi-infinite multiobjective optimization problems, J. Optim. Theory Appl. 160 (2014), 748–762.