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A SELF-ADAPTIVE INERTIAL ALGORITHM FOR SOLVING SPLIT NULL POINT PROBLEMS AND COMMON FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce a new self-adaptive inertial algorithm for finding a common solution of a split null point problem and a common fixed point problem of two infinite families of strict pseudo-contractive mappings and multivalued demicontractive mappings. We demonstrate a strong convergence result without a priori estimate of the norm of the linear operator in Hilbert spaces. As applications, we apply our main result to the split feasibility problem and the split minimization problem. Finally, we present a numerical example to show the efficient of the proposed algorithm.

Keywords. Null point problems; Self-adaptive inertial algorithm; Split null point problem; Strict pseudo-contractive mapping.

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1. Introduction

Throughout this paper, let $\mathbb{N}$ be the set of all positive integers, and let H be the real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$, and let I be the identity operator on H. One denotes the strong and weak convergence of a sequence $\{x_n\} \subset H$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let K be a closed, convex, and nonempty subset of H. The (metric) projection from H onto K, denoted by P_K, is defined, for each $x \in H$, by

$$\|x - P_K x\| = d(x,K) := \inf\{\|x - z\| : z \in K\}.$$

where $P_K x$ is the unique element in K.

Let C be a nonempty subset of H, and let $CB(C)$ be the family of nonempty, bounded and closed subsets of C. A mapping $f : H \to H$ is called a τ-contractive mapping if $\|f(x) - f(z)\| \leq \tau\|x - z\|$ with $\tau \in [0,1)$ for all $x,z \in H$. The Pompeiu-Hausdorff metric on $CB(C)$ is defined by

$$H(A,B) := \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\}$$

for all $A,B \in CB(C)$. Let $S : C \to 2^C$ be a multivalued mapping. An element $p \in C$ is called a fixed point of S if $p \in Sp$. The set of all fixed points of S is denoted by $F(S)$. We say that S satisfies the endpoint condition if $Sp = \{p\}$ for all $p \in F(S)$.

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In this paper, we study the split null point problem (SNPP), proposed by Byrne et al. [1] in 2012. That is, given two multivalued mappings $B_1: H_1 \to 2^{H_1}$ and $B_2: H_2 \to 2^{H_2}$, a bounded linear operator $A: H_1 \to H_2$, the SNPP is formulated as finding a point $x^* \in H_1$ such that

$$x^* \in B_1^{-1}0 \quad and \quad Ax^* \in B_2^{-1}0, \tag{1.1}$$

where $B_1^{-1}0 := \{x \in H_1 : 0 \in B_1 x\}$ and $B_2^{-1}0$ are the null point sets of B_1 and B_2, respectively. We set the solutions of the SNPP to $\Gamma = \{x^* \in B_1^{-1}0 \ \ and \ \ Ax^* \in B_2^{-1}0\}$.

Byrne et al. [1] proposed two iterative algorithms for solving the SNPP (1.1) with two maximal monotone mappings B_1 and B_2 as follows:

$$x_{n+1} = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \quad n \in \mathbb{N}, \tag{1.2}$$

and

$$\begin{cases} u \in H_1, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are the resolvents of B_1 and B_2, respectively, A^* is the adjoint operator of A. With the conditions that $\gamma \in (0, 2/\|A\|^2)$, $\alpha_n \in (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$, they proved the weak and strong convergence results of algorithms (1.2) and (1.3), respectively.

Notice that the parameter γ in the above algorithms depends on the norm of the operator A, which is not easy to calculate. In order to solve this problem, Suantai et al. [2] proposed the following self-adaptive algorithm (1.4) for two maximal monotone mappings B_1 and B_2 and a multivalued demicontractive mapping U:

$$\begin{cases} y_n = J_{\lambda_n}^{B_1}(x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n), \\ u_n = (1-\vartheta)y_n + \vartheta z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)u_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $z_n \in Uy_n$, and the stepsize γ_n is selected by:

$$\gamma_n = \begin{cases} \frac{\xi_n \|(I - J_{\lambda_n}^{B_2})Ax_n\|^2}{\|A^*(I - J_{\lambda_n}^{B_2})Ax_n\|^2}, & if\ Ax_n \notin B_2^{-1}0, \\ 1, & otherwise. \end{cases}$$

They obtained a strong convergence result without a priori estimate of the norm of the linear operator.

Recently, Timilehin et al. [3] proposed the following iterative algorithm with self-adaptive stepsizes for split equilibrium and fixed point problems:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = T_{r_n}^{F_1}(w_n + \gamma_n A^*(T_{s_n}^{F_2} - I)Aw_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n D)[(1-\beta_n)z_n + \beta_n W_n z_n], \end{cases} \tag{1.5}$$

where $\{W_n\}$ is defined by the sequence of strictly pseudo-contractive mappings. They also obtained the strong convergence result. The term $\theta_n(x_n - x_{n-1})$ in (1.5), known as the inertial step, is a crucial factor that makes the algorithm perform effectively. It is first proposed to speed up the convergence properties of iterative algorithms by Polyak [4] in 1964.

Inspired and motivated by the above works, we first use an inertial iterative method and a self-adaptive stepsize to construct an iterative algorithm for finding a common solution of the split null point problem and common fixed point problems. Next, we demonstrate a strong convergence result under some mild conditions. Then, we apply our main results to the split feasibility problems and the split minimization problems. Finally, we give a numerical example to support our algorithm.

2. Preliminaries

Now we recall some definitions and lemmas that we need in our study.

Definition 2.1. Let C be a nonempty, closed, and convex subset of H. A mapping $S : C \to C$ is said to be
(i) firmly nonexpansive if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 - \|(I - S)x - (I - S)y\|^2, \ \forall x, y \in C;$$

(ii) directed

$$\|x - Sx\|^2 \leq \langle x - Sx, x - p\rangle, \ \forall x \in C, p \in F(S).$$

(iii) nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \ \forall x, y \in C;$$

(iv) k-strict pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \ \forall x, y \in C;$$

We notice that a firmly nonexpansive mapping with nonempty fixed point sets is a directed mapping.

Definition 2.2. Let C be a nonempty, closed, and convex subset of H. A mapping $S : C \to CB(C)$ is said to be
(i) multivalued nonexpansive if

$$H(Sx, Sz) \leq \|x - z\|, \ \ \forall x, z \in C;$$

(ii) multivalued quasi-nonexpansive if $F(S) \neq \emptyset$ and

$$H(Sx, Sp) \leq \|x - p\|, \ \ \forall x \in C, p \in F(S);$$

(iii) multivalued demicontractive [5] if $F(S) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$H(Sx, Sp)^2 \leq \|x - p\|^2 + kd(x, Sx)^2, \ \ \forall x \in C, p \in F(S);$$

We notice that the classes of multivalued quasi-nonexpansive mappings and multivalued nonexpansive mappings with nonempty fixed point sets are exactly the class of demicontractive mappings. Then, is the demicontractive mapping quasi-nonexpansive? The answer is negative. We can see this via the following example.

Example 2.1. Let $H = R$. For each $i \in \mathbb{N}$, define

$$S_i x = \begin{cases} \left[-\frac{3i}{2i+1}x, -\frac{4i}{2i+1}x\right], & if\ x \leq 0, \\ \left[-\frac{4i}{2i+1}x, -\frac{3i}{2i+1}x\right], & if\ x > 0. \end{cases}$$

Then $S_i : R \to CB(R)$ is a multivalued demicontractive mapping, which is not quasi-nonexpansive.

Indeed, for each $i \in \mathbb{N}$, it is easy to check that $F(S_i) = \{0\}$. For each $0 \neq x \in R$,

$$H(S_i x, S_i 0)^2 = |-\frac{4i}{2i+1}x - 0|^2 = |x-0|^2 + (\frac{16i^2}{(2i+1)^2} - 1)|x|^2 = |x-0|^2 + \frac{12i^2-4i-1}{4i^2+4i+1}|x|^2.$$

Clearly, S_i is not quasi-nonexpansive. We also have

$$d(x, S_i x)^2 = |x - (-\frac{3i}{2i+1}x)|^2 = (\frac{5i+1}{2i+1})^2|x|^2 = \frac{25i^2+10i+1}{4i^2+4i+1}|x|^2.$$

Therefore,

$$H(S_i x - S_i 0)^2 = |x-0|^2 + \frac{12i^2-4i-1}{25i^2+10i+1}d(x, S_i x)^2.$$

Hence S_i is demicontractive with a constant $\widetilde{k}_i = \frac{12i^2-4i-1}{25i^2+10i+1} \in (0,1)$.

Definition 2.3. A bounded linear operator D on H is called strongly positive if there exists a constant $\overline{\gamma} > 0$ such that $\langle Dx, x\rangle \geq \overline{\gamma}\|x\|^2$ for all $x \in H$.

Definition 2.4. [2] Let $S : C \to CB(C)$ be a multivalued mapping. The $I - S$ is demiclosed at zero if for any sequence $\{x_n\}$ in C which converges weakly to $p \in C$ and the sequence $\{\|x_n - z_n\|\}$ converges strongly to 0, where $z_n \in Sx_n$, $p \in F(S)$.

Let C be a closed, convex, and nonempty subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $x, y \in C$. P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive and firmly nonexpansive. Moreover, P_C is characterized by the following property:

$$\langle u - P_C u, v - P_C u\rangle \leq 0, \quad \forall v \in C. \tag{2.1}$$

For a maximal monotone operator $B : H \to 2^H$ and $\lambda > 0$, we define the resolvent of B with parameter λ by $J_\lambda^B = (I + \lambda B)^{-1}$. From [6], $J_\lambda^B : H \to dom(B)$ is single-valued, firmly nonexpansive, and

$$\|J_\lambda^B x - J_\lambda^B y\|^2 \leq \langle J_\lambda^B x - J_\lambda^B y, x - y\rangle, \ \forall x, y \in H,$$

which is equivalent to

$$\langle (J_\lambda^B x - x) - (J_\lambda^B y - y), J_\lambda^B x - J_\lambda^B y\rangle \leq 0$$

and $F(J_\lambda^B) = B^{-1}0 = \{x \in H, 0 \in Bx\}$. Moreover, $I - J_\lambda^B$ is demiclosed at 0, and $B^{-1}0$ is convex and closed.

Lemma 2.1. [2] *Let C be a closed, convex, and nonempty subset of H. Let $S : C \to CB(C)$ be a multivalued $\widetilde{k}$-demicontractive mapping. Then*
(i) $F(S)$ is closed;
(ii) If S satisfies the endpoint condition, then $F(S)$ is convex.

Lemma 2.2. [2] *Let $x, z \in H$, and $t \in R$. Then the following inequalities hold on H:*
(i) $\|x + z\|^2 \leq \|x\|^2 + 2\langle z, x + z\rangle$;
(ii) $\|tx + (1-t)z\|^2 = t\|x\|^2 + (1-t)\|z\|^2 - t(1-t)\|x - z\|^2$.

Lemma 2.3. [7] *Let H be a real Hilbert space, $x_i \in H$ and $\{\alpha_i\}_{i=1}^m \subset (0,1)$ with $\sum_{i=1}^m \alpha_i = 1 (1 \leq i \leq m)$. Then the following identity holds:*

$$\|\sum_{i=1}^m \alpha_i x_i\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{i,j=1, i\neq j}^m \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.4. [2] *Let $A : H_1 \to H_2$ be a bounded linear operator, and let $S : H_2 \to H_2$ be a directed mapping with $A^{-1}(F(S)) \neq \emptyset$. If $x \in H_1$ with $Ax \neq S(Ax)$ and $p \in A^{-1}(F(S))$, then*

$$\|x - \gamma A^*(I-S)Ax - p\|^2 \leq \|x-p\|^2 - (2-\xi)\xi \frac{\|(I-S)Ax\|^4}{\|A^*(I-S)Ax\|^2},$$

where

$$\gamma := \xi \frac{\|(I-S)Ax\|^2}{\|A^*(I-S)Ax\|^2}$$

and $\xi \in (0,2)$.

Lemma 2.5. [8] *Let C be a closed, convex, and nonempty subset of a real Hilbert space H and $T : C \to C$ be a k-strict pseudo-contractive mapping. Define a mapping $T' : C \to C$ by $T'x = \alpha x + (1-\alpha)Tx$ for all $x \in C$ and $\alpha \in [k,1)$. Then T' is a nonexpansive mapping such that $F(T') = F(T)$.*

Definition 2.5. [9] Let $\{T_n\}$ be a sequence of k_n-strict pesudo-contractive mappings. Define $T_n' = t_n I + (1-t_n)T_n$, $t_n \in [k_n, 1)$. Then, by Lemma 2.5, T_n' is nonexpansive. Define the mapping W_n by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \zeta_n T_n' U_{n,n+1} + (1-\zeta_n)I, \\ U_{n,n-1} = \zeta_{n-1} T_{n-1}' U_{n,n} + (1-\zeta_{n-1})I, \\ \cdots, \\ U_{n,k} = \zeta_k T_k' U_{n,k+1} + (1-\zeta_k)I, \\ U_{n,k-1} = \zeta_{k-1} T_{k-1}' U_{n,k} + (1-\zeta_{k-1})I, \\ \cdots, \\ U_{n,2} = \zeta_2 T_2' U_{n,3} + (1-\zeta_2)I, \\ W_n = U_{n,1} = \zeta_1 T_1' U_{n,2} + (1-\zeta_1)I, \end{cases}$$

where $\{\zeta_n\}$ is a sequence of real numbers such that $0 \leq \zeta_n \leq 1$. For each $n \geq 1$, such a mapping W_n is nonexpansive.

The following lemmas relating to the mapping W_n are needed in proving our main result.

Lemma 2.6. [10] *Let C be a nonempty, closed, and convex subset of a strictly convex Banach space E. Let $\{T_i'\}$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(T_i') \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq \widetilde{b} < 1$ for all $i \geq 1$. Then*
(i) W_n is nonexpansive and $F(W_n) = \cap_{i=1}^{n} F(T_i') \neq \emptyset$ for each $n \geq 1$;
(ii) for each $x \in C$ and for each positive integer k, the $\lim_{n\to\infty} U_{n,k}x$ exists;
(iii) the mapping W of C into itself, defined by $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$ for all $x \in C$, is a nonexpansive mapping satisfying $F(W) = \cap_{i=1}^{\infty} F(T_i')$, which is called the modified W-mapping generated by $T_1, T_2, \ldots, \zeta_1, \zeta_2, \ldots$ and $t_1, t_2, \ldots$.

By Lemma 2.5 and Lemma 2.6, it follows that $F(W) = \cap_{i=1}^{\infty} F(T_i') = \cap_{i=1}^{\infty} F(T_i)$.

Lemma 2.7. [11] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{T_i'\}$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(T_i') \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq \widetilde{b} < 1$ for all $i \geq 1$, where $\widetilde{b}$ is a positive real number. If K is any bounded subset of C, then $\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0$.*

Lemma 2.8. [12] *Each Hilbert space H satisfies the Opial condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$ holds for every $y \in H$ with $y \neq x$.*

Lemma 2.9. [13] *Suppose that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\mu_n)a_n + \mu_n\sigma_n + \tau_n,\ n \in \mathbb{N},$$

where $\{\mu_n\}, \{\sigma_n\}$ and $\{\tau_n\}$ satisfy the following conditions:
(i) $\{\mu_n\} \subset [0,1]$, $\sum_{n=1}^{\infty}\mu_n = \infty$;
(ii) $\limsup_n \sigma_n \leq 0$ or $\sum_{n=1}^{\infty}|\mu_n\sigma_n| < \infty$;
(iii) $\tau_n \geq 0$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty}\tau_n < \infty$.
Then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.10. [14] *Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ which satisfies $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Define a sequence of positive integers $\{\rho(n)\}$ by $\rho(n) := \max\{m < n : t_m < t_{m+1}\}$ for all $n \geq n_0$ (for some n_0 large enough). Then $\{\rho(n)\}$ is a nondecreasing sequence such that $\rho(n) \to \infty$ as $n \to \infty$, and it holds that $t_{\rho(n)} \leq t_{\rho(n)+1}$, $t_n \leq t_{\rho(n)+1}$.*

3. Main Results

In this section, we present our proposed algorithm and prove a strong convergence theorem.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces, and let C be a nonempty, convex, and closed subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator. For each $i \in \mathbb{N}$, let $S_i : C \to CB(C)$ be a multivalued $\widetilde{k_i}$-demicontractive mapping such that $I - S_i$ is demiclosed at zero and S_i satisfies the endpoint condition. Assume that $T_i : C \to C$ is a k_i-strictly pseudo-contractive mapping and $\{\zeta_i\}$ is a real sequence such that $0 < \zeta_i \leq \widetilde{b} < 1$ for all $i \in \mathbb{N}$. Let $f : H_1 \to H_2$ be a τ-contrative mapping with $\tau \in (0,1)$. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings such that $dom(B_1)$ is included in C. Suppose that $\Omega := (\cap_{i=1}^{\infty}F(S_i)) \cap \Gamma \cap (\cap_{i=1}^{\infty}F(T_i)) \neq \emptyset$, where $\Gamma = \{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\}$. Assume that $\{x_n\}$ is a sequence iteratively generated by $x_1,\ x_2 \in H_1$ and*

$$\begin{cases} v_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_{\lambda_n}^{B_1}(v_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Av_n), \\ u_n = (1-\delta_n)y_n + \delta_n \sum\limits_{i=1}^{n}\alpha_{n,i}z_{n,i}, \\ x_{n+1} = \vartheta_n f(x_n) + (1-\vartheta_n)[(1-\beta_n)u_n + \beta_n W_n u_n], \end{cases} \tag{3.1}$$

where $n \geq 2$ and $z_{n,i} \in S_i y_n$, and θ_n is defined by

$$\theta_n = \begin{cases} \min\left\{\frac{\tau_n}{\|x_n - x_{n-1}\|},\ \frac{n-1}{n+\theta-1}\right\}, & if\ x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1}, & otherwise, \end{cases}$$

where $\theta \geq 3$ and the stepsize γ_n is selected in a way:

$$\gamma_n = \begin{cases} \frac{\xi_n\|(I-J_{\lambda_n}^{B_2})Av_n\|^2}{\|A^*(I-J_{\lambda_n}^{B_2})Av_n\|^2}, & if\ Av_n \notin B_2^{-1}0, \\ 1, & otherwise, \end{cases}$$

and the following conditions hold:

(C_1) $\{\lambda_n\} \subset (0,\infty)$ and $\liminf\limits_{n\to\infty} \lambda_n > 0$;
(C_2) $\{\alpha_{n,i}\} \subset [0,1]$, $\sum_{i=1}^{n} \alpha_{n,i} = 1$, $1-\delta_n \in (k,1)$ and $\liminf_{n\to\infty} \alpha_{n,i}(1-\delta_n-k) > 0$, $\forall i \in \mathbb{N}$, where $k = \sup\{\widetilde{k_i} : i \in \mathbb{N}\} < 1$;
(C_3) $\liminf\limits_{n\to\infty} \delta_n > 0$, $\liminf\limits_{n\to\infty} \delta_n(1-\delta_n-k) > 0$;
(C_4) $0 < \liminf\limits_{n\to\infty} \beta_n \le \limsup\limits_{n\to\infty} \beta_n < 1$;
(C_5) $\{\vartheta_n\} \subset (0,1)$, $\lim\limits_{n\to\infty} \vartheta_n = 0$, $\sum_{n=2}^{\infty} \vartheta_n = \infty$ and $\lim\limits_{n\to\infty} \frac{\tau_n}{\vartheta_n} = 0$;
(C_6) $\{\xi_n\} \subset (a,b) \subset (0,2)$.
Then, the sequence $\{x_n\}$ generated by algorithm (3.1) converges strongly to a point $x^* \in \Omega$, which is the unique solution of the following variational inequality problem:

$$\langle f(x^*) - x^*, p - x^* \rangle \le 0, \ \ \forall\, p \in \Omega. \tag{3.2}$$

Remark 3.1. From the definition of θ_n and condition (C_5), we have

$$\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0 \ \ \textit{and} \ \ \lim_{n\to\infty} \frac{\theta_n}{\vartheta_n}\|x_n - x_{n-1}\| = 0.$$

Proof. Step1. Prove that problem (3.2) has a unique solution $x^* \in \Omega$.

By Lemma 2.1, for each $i \in \mathbb{N}$, we have $F(S_i)$ is closed and convex. Then, $\cap_{i=1}^{\infty} F(S_i)$ is also closed and convex. Since $B_1^{-1}0$ and $B_2^{-1}0$ are closed and convex and A is a linear operator, then Γ is closed and convex. From Lemma 2.6 (iii), W is a nonexpansive mapping, which indicates that $F(W) = \cap_{i=1}^{\infty} F(T_i)$ is also closed and convex. Hence Ω is closed and convex. It is easy to know that $P_\Omega f$ is a contractive mapping. So by Banach fixed point theorom, there exists a unique element $x^* \in \Omega$ such that $x^* = P_\Omega f(x^*)$. It follows from (2.1) that variational inequality problem (3.2) has a unique solution $x^* \in \Omega$.

Step 2. Prove that $\{x_n\}$ is bounded.

Since $x^* \in \Omega$, we have $S_i x^* = \{x^*\}$ for all $i \in \mathbb{N}$, $J_{\lambda_n}^{B_1} x^* = x^*$, and $J_{\lambda_n}^{B_2}(Ax^*) = Ax^*$. It follows from the conditions (C_2)-(C_4) that there exists a positive integer n_0 such that $n > n_0^*$, $\lambda_n > c$, $\alpha_{n,i}(1-\delta_n-k) > c$, $\delta_n(1-\delta_n-k) > c$, and $c < \beta_n < d < 1$, where $0 < c < 1$ and $0 < d < 1$ for all $i \in \mathbb{N}$. From (3.1) we have the following inequalities:

$$\|v_n - x^*\| = \|x_n + \theta_n(x_n - x_{n-1}) - x^*\| \le \|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\| \tag{3.3}$$

and

$$\begin{aligned} \|v_n - x^*\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\|^2 \\ &\le \|x_n - x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - x^*\| + \theta_n^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.4}$$

From the definition of W_n and Lemma 2.6 (i), we have

$$\|W_n u_n - x^*\| = \|W_n u_n - W_n x^*\| \le \|u_n - x^*\|. \tag{3.5}$$

Put $\bar{u}_n = (1-\beta_n)u_n + \beta_n W_n u_n$. By (3.5), we obtain

$$\begin{aligned} \|\bar{u}_n - x^*\| &\le (1-\beta_n)\|u_n - x^*\| + \beta_n\|W_n u_n - x^*\| \\ &\le (1-\beta_n)\|u_n - x^*\| + \beta_n\|u_n - x^*\| \\ &= \|u_n - x^*\|. \end{aligned} \tag{3.6}$$

Since $J_{\lambda_n}^{B_1}$ is firmly nonexpansive, by Lemma 2.4, we get

$$\begin{aligned}\|y_n-x^*\|^2 &= \|J_{\lambda_n}^{B_1}(v_n-\gamma_n A^*(I-J_{\lambda_n}^{B_2}))Av_n-J_{\lambda_n}^{B_1}x^*\|^2\\ &\leq \|v_n-\gamma_n A^*(I-J_{\lambda_n}^{B_2})Av_n-x^*\|^2\\ &\leq \|v_n-x^*\|^2-(2-\xi_n)\xi_n\frac{\|(I-J_{\lambda_n}^{B_2})Av_n\|^4}{\|A^*(I-J_{\lambda_n}^{B_2})Av_n\|^2} \qquad (3.7)\\ &\leq \|v_n-x^*\|^2-\frac{(2-\xi_n)\xi_n}{\|A\|^2}\|(I-J_{\lambda_n}^{B_2})Av_n\|^2. \qquad (3.8)\end{aligned}$$

Using Lemma 2.2 (ii), Lemma 2.3, and (3.8), we have

$$\begin{aligned}\|u_n-x^*\|^2 &= \|\sum_{i=1}^n \alpha_{n,i}[(1-\delta_n)(y_n-x^*)+\delta_n(z_{n,i}-x^*)]\|^2\\ &\leq \sum_{i=1}^n \alpha_{n,i}\|(1-\delta_n)(y_n-x^*)+\delta_n(z_{n,i}-x^*)\|^2\\ &= \sum_{i=1}^n \alpha_{n,i}[(1-\delta_n)\|y_n-x^*\|^2+\delta_n\|z_{n,i}-x^*\|^2-\delta_n(1-\delta_n)\|y_n-z_{n,i}\|^2]\\ &= \sum_{i=1}^n \alpha_{n,i}[(1-\delta_n)\|y_n-x^*\|^2+\delta_n d(z_{n,i},S_i x^*)^2-\delta_n(1-\delta_n)\|y_n-z_{n,i}\|^2]\\ &\leq \sum_{i=1}^n \alpha_{n,i}[(1-\delta_n)\|y_n-x^*\|^2+\delta_n H(S_i y_n,S_i x^*)^2-\delta_n(1-\delta_n)\|y_n-z_{n,i}\|^2]\\ &\leq \sum_{i=1}^n \alpha_{n,i}[(1-\delta_n)\|y_n-x^*\|^2+\delta_n[\|y_n-x^*\|^2+k_i d(y_n,S_i y_n)^2]\\ &\quad -\delta_n(1-\delta_n)\|y_n-z_{n,i}\|^2]\\ &\leq \|y_n-x^*\|^2-\delta_n(1-\delta_n-k)\sum_{i=1}^n \alpha_{n,i}\|y_n-z_{n,i}\|^2 \qquad (3.9)\\ &\leq \|v_n-x^*\|^2-\frac{(2-\xi_n)\xi_n}{\|A\|^2}\|(I-J_{\lambda_n}^{B_2})Av_n\|^2\\ &\quad -\delta_n(1-\delta_n-k)\sum_{i=1}^n \alpha_{n,i}\|y_n-z_{n,i}\|^2. \qquad (3.10)\end{aligned}$$

Hence, when $n>n_0^*$, we arrive at

$$\|u_n-x^*\|\leq\|v_n-x^*\|. \qquad (3.11)$$

Using (3.3), (3.6), and (3.11), we see that

$$\|\overline{u}_n-x^*\|\leq\|x_n-x^*\|+\theta_n\|x_n-x_{n-1}\|. \qquad (3.12)$$

Therefore, by (3.12), we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|\vartheta_n f(x_n)+(1-\vartheta_n)\overline{u}_n - x^*\| \\
&\le \vartheta_n\|f(x_n)-x^*\|+(1-\vartheta_n)\|\overline{u}_n-x^*\| \\
&\le \vartheta_n(\|f(x_n)-f(x^*)\|+\|f(x^*)-x^*\|)+(1-\vartheta_n)\|\overline{u}_n-x^*\| \\
&\le \vartheta_n(\tau\|x_n-x^*\|+\|f(x^*)-x^*\|)+(1-\vartheta_n)\|x_n-x^*\|+\theta_n(1-\vartheta_n)\|x_n-x_{n-1}\| \\
&\le [1-\vartheta_n(1-\tau)]\|x_n-x^*\|+\vartheta_n\|f(x^*)-x^*\|+\theta_n(1-\vartheta_n)\|x_n-x_{n-1}\| \\
&\le [1-\vartheta_n(1-\tau)]\|x_n-x^*\|+\vartheta_n(1-\tau)M,
\end{aligned}$$

where

$$M = 2\max\left\{\frac{\|f(x^*)-x^*\|}{1-\tau}, \sup_{n\ge 2}\frac{\theta_n(1-\vartheta_n)\|x_n-x_{n-1}\|}{\vartheta_n(1-\tau)}\right\}.$$

It follows by induction that

$$\begin{aligned}
\|x_{n+1}-x^*\| &\le [1-\vartheta_n(1-\tau)]\|x_n-x^*\|+\vartheta_n(1-\tau)M, \\
&\le \max\{\|x_n-x^*\|, M\} \\
&\vdots \\
&\le \max\{\|x_{n_0}-x^*\|, M\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{y_n\}$, $\{v_n\}$, $\{\overline{u}_n\}$, and $\{f(x_n)\}$.

Step 3. Prove that sequence $\{x_n\}$ converges strongly to x^*.

From Remark 3.1, we obtain $\lim_{n\to\infty}\theta_n\|x_n-x_{n-1}\|=0$. Using (3.4), (3.6) and (3.10), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\vartheta_n f(x_n)+(1-\vartheta_n)\overline{u}_n-x^*\|^2 \\
&\le \vartheta_n\|f(x_n)-x^*\|^2+(1-\vartheta_n)\|\overline{u}_n-x^*\|^2 \\
&\le \vartheta_n\|f(x_n)-x^*\|^2+\|u_n-x^*\|^2 \\
&\le \vartheta_n\|f(x_n)-x^*\|^2+\|v_n-x^*\|^2-\frac{(2-\xi_n)\xi_n}{\|A\|^2}\|(I-J_{\lambda_n}^{B_2})Av_n\|^2 \\
&\quad -\delta_n(1-\delta_n-k)\sum_{i=1}^{n}\alpha_{n,i}\|y_n-z_{n,i}\|^2 \\
&\le \vartheta_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2+2\theta_n\|x_n-x_{n-1}\|\|x_n-x^*\| \\
&\quad +\theta_n^2\|x_n-x_{n-1}\|^2-\frac{(2-\xi_n)\xi_n}{\|A\|^2}\|(I-J_{\lambda_n}^{B_2})Av_n\|^2 \\
&\quad -\delta_n(1-\delta_n-k)\sum_{i=1}^{n}\alpha_{n,i}\|y_n-z_{n,i}\|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\delta_n(1-\delta_n-k)\alpha_{n,i}\|y_n-z_{n,i}\|^2 \\
\le\ &\frac{(2-\xi_n)\xi_n}{\|A\|^2}\|(I-J_{\lambda_n}^{B_2})Av_n\|^2+\delta_n(1-\delta_n-k)\sum_{i=1}^{n}\alpha_{n,i}\|y_n-z_{n,i}\|^2 \\
\le\ &\vartheta_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&+2\theta_n\|x_n-x_{n-1}\|\|x_n-x^*\|+\theta_n^2\|x_n-x_{n-1}\|^2. \qquad (3.13)
\end{aligned}$$

Moreover, from Lemma 2.2 (ii), (3.5), and (3.11), we obtain

$$\begin{aligned}\|\bar{u}_n-x^*\|^2 &= (1-\beta_n)\|u_n-x^*\|^2+\beta_n\|W_nu_n-x^*\|^2-\beta_n(1-\beta_n)\|W_nu_n-u_n\|^2\\ &\leq (1-\beta_n)\|u_n-x^*\|^2+\beta_n\|u_n-x^*\|^2-\beta_n(1-\beta_n)\|W_nu_n-u_n\|^2\\ &= \|u_n-x^*\|^2-\beta_n(1-\beta_n)\|W_nu_n-u_n\|^2\\ &\leq \|v_n-x^*\|^2-\beta_n(1-\beta_n)\|W_nu_n-u_n\|^2,\end{aligned}$$

which together with (3.4) implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \vartheta_n\|f(x_n)-x^*\|^2+(1-\vartheta_n)\|\bar{u}_n-x^*\|^2\\ &\leq \vartheta_n\|f(x_n)-x^*\|^2+\|v_n-x^*\|^2-\beta_n(1-\beta_n)\|W_nu_n-u_n\|^2\\ &\leq \vartheta_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2+2\theta_n\|x_n-x_{n-1}\|\|x_n-x^*\|\\ &\quad+\theta_n^2\|x_n-x_{n-1}\|^2-\beta_n(1-\beta_n)\|W_nu_n-u_n\|^2.\end{aligned}$$

So

$$\begin{aligned}\beta_n(1-\beta_n)\|W_nu_n-u_n\|^2 &\leq \vartheta_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\quad+2\theta_n\|x_n-x_{n-1}\|\|x_n-x^*\|+\theta_n^2\|x_n-x_{n-1}\|^2. \end{aligned}\tag{3.14}$$

From condition (C_5) and (3.12), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \langle\vartheta_n(f(x_n)-x^*)+(1-\vartheta_n)(\bar{u}_n-x^*),x_{n+1}-x^*\rangle\\ &= \langle\vartheta_n(f(x_n)-f(x^*)),x_{n+1}-x^*\rangle+\langle\vartheta_n(f(x^*)-x^*),x_{n+1}-x^*\rangle\\ &\quad+(1-\vartheta_n)\langle\bar{u}_n-x^*,x_{n+1}-x^*\rangle\\ &\leq \vartheta_n\tau\|x_n-x^*\|\|x_{n+1}-x^*\|+\vartheta_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\\ &\quad+(1-\vartheta_n)(\|x_n-x^*\|+\theta_n\|x_n-x_{n-1}\|)\|x_{n+1}-x^*\|\\ &= [1-\vartheta_n(1-\tau)]\|x_n-x^*\|\|x_{n+1}-x^*\|+\vartheta_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\\ &\quad+\theta_n(1-\vartheta_n)\|x_n-x_{n-1}\|\|x_{n+1}-x^*\|\\ &\leq \frac{1-\vartheta_n(1-\tau)}{2}(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)+\vartheta_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\\ &\quad+\theta_n(1-\vartheta_n)\|x_n-x_{n-1}\|\|x_{n+1}-x^*\|\\ &\leq \frac{1-\vartheta_n(1-\tau)}{2}\|x_n-x^*\|^2+\frac{1}{2}\|x_{n+1}-x^*\|^2+\vartheta_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\\ &\quad+\theta_n(1-\vartheta_n)\|x_n-x_{n-1}\|\|x_{n+1}-x^*\|,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq [1-\vartheta_n(1-\tau)]\|x_n-x^*\|^2+2\vartheta_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\\ &\quad+2\theta_n(1-\vartheta_n)\|x_n-x_{n-1}\|\|x_{n+1}-x^*\|\\ &= [1-\vartheta_n(1-\tau)]\|x_n-x^*\|^2+\vartheta_n(1-\tau)b_n,\end{aligned}\tag{3.15}$$

where

$$b_n=\frac{2}{1-\tau}\langle f(x^*)-x^*,x_{n+1}-x^*\rangle+\frac{2\theta_n(1-\vartheta_n)}{\vartheta_n(1-\tau)}\|x_n-x_{n-1}\|\|x_{n+1}-x^*\|.$$

Case 1. Assume that $\{\|x_n - x^*\|\}$ is a monotonicaly decreasing sequence. Hence $\{\|x_n - x^*\|\}$ is convergent and

$$\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0, \tag{3.16}$$

as $n \to \infty$. Combining conditions (C_2)-$(C6)$, (3.13), and (3.14), we have

$$\lim_{n\to\infty} \|(I - J_{\lambda_n}^{B_2})Av_n\| = 0, \tag{3.17}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \alpha_{n,i}\|y_n - z_{n,i}\|^2 = 0, \tag{3.18}$$

$$\lim_{n\to\infty} \|y_n - z_{n,i}\|^2 = 0, \tag{3.19}$$

$$\lim_{n\to\infty} \|W_n u_n - u_n\| = 0, \tag{3.20}$$

for all $i \in \mathbb{N}$. Using (3.4), (3.6), (3.7), and (3.9), we have

$$\begin{aligned}
\|\overline{u}_n - x^*\|^2 &\leq \|u_n - x^*\|^2 \leq \|y_n - x^*\|^2 \\
&\leq \|v_n - x^*\|^2 - (2 - \xi_n)\xi_n \frac{\|(I - J_{\lambda_n}^{B_2})Av_n\|^4}{\|A^*(I - J_{\lambda_n}^{B_2})Av_n\|^2} \\
&= \|v_n - x^*\|^2 - (2 - \xi_n)\gamma_n\|(I - J_{\lambda_n}^{B_2})Av_n\|^2 \\
&\leq \|x_n - x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - x^*\| \\
&\quad +\theta_n^2\|x_n - x_{n-1}\|^2 - (2 - \xi_n)\gamma_n\|(I - J_{\lambda_n}^{B_2})Av_n\|^2.
\end{aligned}$$

So

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \vartheta_n\|f(x_n) - x^*\|^2 + (1 - \vartheta_n)\|\overline{u}_n - x^*\|^2 \\
&\leq \vartheta_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - x^*\| \\
&\quad +\theta_n^2\|x_n - x_{n-1}\|^2 - (2 - \xi_n)\gamma_n\|(I - J_{\lambda_n}^{B_2})Av_n\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
(2 - \xi_n)\gamma_n\|(I - J_{\lambda_n}^{B_2})Av_n\|^2 &\leq \vartheta_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad +2\theta_n\|x_n - x_{n-1}\|\|x_n - x^*\| + \theta_n^2\|x_n - x_{n-1}\|^2.
\end{aligned}$$

Therefore, it follows from conditions (C_5) and (C_6) and (3.16) that

$$\lim_{n\to\infty} \gamma_n\|(I - J_{\lambda_n}^{B_2})Av_n\| = 0. \tag{3.21}$$

By using Lemma 2.4, we have

$$\begin{aligned}
& \|y_n - x^*\|^2 \\
= & \|J_{\lambda_n}^{B_1}(v_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Av_n) - J_{\lambda_n}^{B_1}x^*\|^2 \\
\leq & \langle J_{\lambda_n}^{B_1}(v_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2}))Av_n - J_{\lambda_n}^{B_1}x^*, v_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Av_n - x^*\rangle \\
= & \langle y_n - x^*, v_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Av_n - x^*\rangle \\
= & \frac{1}{2}[\|y_n - x^*\|^2 + \|v_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Av_n - x^*\|^2 - \|y_n - v_n + \gamma_n A^*(I - J_{\lambda_n}^{B_2})Av_n\|^2] \\
\leq & \frac{1}{2}[\|y_n - x^*\|^2 + \|v_n - x^*\|^2 - \|y_n - v_n + \gamma_n A^*(I - J_{\lambda_n}^{B_2})Av_n\|^2] \\
\leq & \frac{1}{2}[\|y_n - x^*\|^2 + \|v_n - x^*\|^2 - \|y_n - v_n\|^2 - \gamma_n^2\|A^*(I - J_{\lambda_n}^{B_2})Av_n\|^2 \\
& +2\gamma_n\|y_n - v_n\|\|A^*(I - J_{\lambda_n}^{B_2})Av_n\|],
\end{aligned}$$

which implies that

$$\|y_n - x^*\|^2 \leq \|v_n - x^*\|^2 - \|y_n - v_n\|^2 + 2\gamma_n\|y_n - v_n\|\|A^*(I - J_{\lambda_n}^{B_2})Av_n\|. \qquad (3.22)$$

Combining (C_5), (3.6), and (3.22) we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq & \vartheta_n\|f(x_n) - x^*\|^2 + (1 - \vartheta_n)\|\overline{u}_n - x^*\|^2 \\
\leq & \vartheta_n\|f(x_n) - x^*\|^2 + (1 - \vartheta_n)\|u_n - x^*\|^2 \\
\leq & \vartheta_n\|f(x_n) - x^*\|^2 + (1 - \vartheta_n)[(1 - \delta_n)\|y_n - x^*\|^2 + \delta_n\|\sum_{i=1}^{n}\alpha_{n,i}z_{n,i} - x^*\|^2] \\
\leq & \vartheta_n\|f(x_n) - x^*\|^2 + (1 - \delta_n)\|y_n - x^*\|^2 + \delta_n\sum_{i=1}^{n}\alpha_{n,i}d(z_{n,i}, S_i x^*)^2 \\
\leq & \vartheta_n\|f(x_n) - x^*\|^2 + (1 - \delta_n)\|y_n - x^*\|^2 + \delta_n\sum_{i=1}^{n}\alpha_{n,i}H(S_i y_n, S_i x^*)^2 \\
\leq & \vartheta_n\|f(x_n) - x^*\|^2 + (1 - \delta_n)\|y_n - x^*\|^2 \\
& +\delta_n\sum_{i=1}^{n}\alpha_{n,i}[\|y_n - x^*\|^2 + k_i d(y_n, S_i y_n)^2] \\
\leq & \vartheta_n\|f(x_n) - x^*\|^2 + \|y_n - x^*\|^2 + \delta_n k\sum_{i=1}^{n}\alpha_{n,i}\|y_n - z_{n,i}\|^2 \\
\leq & \vartheta_n\|f(x_n) - x^*\|^2 + \|v_n - x^*\|^2 - \|y_n - v_n\|^2 \\
& +2\gamma_n\|y_n - v_n\|\|A^*(I - J_{\lambda_n}^{B_2})Av_n\| + \delta_n k\sum_{i=1}^{n}\alpha_{n,i}\|y_n - z_{n,i}\|^2.
\end{aligned}$$

Hence, by (3.4), we have

$$\begin{aligned}
\|y_n - v_n\|^2 &\leq \vartheta_n\|f(x_n)-x^*\|^2 + \|v_n - x^*\|^2 - \|x_{n+1}-x^*\|^2 \\
&\quad +2\gamma_n\|y_n - v_n\|\|A^*(I-J_{\lambda_n}^{B_2})Av_n\| + \delta_n k\sum_{i=1}^{n}\alpha_{n,i}\|y_n, z_{n,i}\|^2 \\
&\leq \vartheta_n\|f(x_n)-x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1}-x^*\|^2 \\
&\quad +2\theta_n\|x_n - x_{n-1}\|\|x_n - x^*\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad +2\gamma_n\|y_n - v_n\|\|A^*(I-J_{\lambda_n}^{B_2})Av_n\| + \delta_n k\sum_{i=1}^{n}\alpha_{n,i}\|y_n - z_{n,i}\|^2 \\
&\leq \vartheta_n\|f(x_n)-x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1}-x^*\|^2 \\
&\quad +2\theta_n\|x_n - x_{n-1}\|\|x_n - x^*\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad +2\gamma_n\|y_n - v_n\|\|A\|\|(I-J_{\lambda_n}^{B_2})Av_n\| + (1-k)k\sum_{i=1}^{n}\alpha_{n,i}\|y_n - z_{n,i}\|^2.
\end{aligned}$$

So, using condition (C_5), (3.16), (3.18) and (3.21), we obtain

$$\lim_{n\to\infty}\|y_n - v_n\| = 0. \tag{3.23}$$

From condition (C_2), Remark 3.1, (3.18), and (3.20), we can obtain the following formulae:

$$\|u_n - y_n\|^2 \leq \delta_n \sum_{i=1}^{n}\alpha_{n,i}\|y_n - z_{n,i}\|^2 \leq (1-k)\sum_{i=1}^{n}\alpha_{n,i}\|y_n - z_{n,i}\|^2 \to 0, \tag{3.24}$$

$$\|\overline{u}_n - u_n\|^2 \leq \beta_n\|W_n u_n - u_n\|^2 \to 0, \tag{3.25}$$

$$\|v_n - x_n\| = \theta_n\|x_n - x_{n-1}\| \to 0. \tag{3.26}$$

Hence, it follows from condition (C_5) and (3.23)-(3.26) that

$$\begin{aligned}
\|x_{n+1}-x_n\| &\leq \|x_{n+1}-\overline{u}_n\| + \|\overline{u}_n - u_n\| + \|u_n - y_n\| + \|y_n - v_n\| + \|v_n - x_n\| \\
&\leq \vartheta_n\|f(x_n)-\overline{u}_n\| + \|\overline{u}_n - u_n\| + \|u_n - y_n\| + \|y_n - v_n\| + \|v_n - x_n\| \\
&\to 0.
\end{aligned}$$

Next we show that $\limsup_{n\to\infty}\langle f(x^*)-x^*, x_n - x^*\rangle \leq 0$. To show this, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle f(x^*)-x^*, x_{n_j} - x^*\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ and $p \in H_1$ such that $x_{n_{j_k}} \rightharpoonup p$. Without loss of generality, we may assume that $x_{n_j} \rightharpoonup p$. By (3.23), (3.24), and (3.26), we have $v_{n_j} \rightharpoonup p$, $u_{n_j} \rightharpoonup p$ and $y_{n_j} \rightharpoonup p$. So, for any $i \in \mathbb{N}$, by (3.19) and the demiclosedness of $I - S_i$ at zero, we obtain $p \in \cap_{i=1}^{\infty}F(S_i)$. Since A is a bounded linear, we have $\langle z, Ax_{n_j} - Ap\rangle = \langle A^*z, x_{n_j} - p\rangle \to 0$ as $j \to \infty$ for all $z \in H_2$. This implies that $Ax_{n_j} \rightharpoonup Ap$. From (3.17) and the demiclosedness of $I - J_{\lambda_n}^{B_2}$ at zero, we have that $Ap \in F(J_{\lambda_n}^{B_2}) = B_2^{-1}0$.

Now we show that $p \in B_1^{-1}0$. Indeed, due to $y_n = J_{\lambda_n}^{B_1}(v_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Av_n)$, we have

$$\frac{1}{\lambda_n}(v_n - y_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Av_n) \in B_1 y_n.$$

By the monotonicity of B_1, we have $\langle y_n - v, \frac{1}{\lambda_n}(v_n - y_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Av_n) - w\rangle \geq 0$ for all $(v,w) \in G(B_1)$. Taking $n = n_j$, condition (C_1), $y_{n_j} \rightharpoonup p$, and (3.23) imply that $\langle p - v, -w\rangle \geq 0$. Thus $0 \in B_1 p$, i.e., $p \in B_1^{-1}0$. Hence, $p \in \Gamma$. Suppose that $p \notin F(W)$, i.e., $Wp \neq p$. Since $\{u_n\}$ is bounded, there exists a bounded set $K \subset C$ such that $\{u_n\} \subset K$. It follows from Lemma 2.7 and (3.20) that

$$\begin{aligned} \|Wu_n - u_n\| &\leq \|Wu_n - W_n u_n\| + \|W_n u_n - u_n\| \\ &\leq \sup_{x \in K} \|Wx - W_n x\| + \|Wu_n - W_n u_n\| \to 0. \end{aligned} \tag{3.27}$$

So it follows from Lemma 2.8 and (3.27) that

$$\begin{aligned} \liminf_{j\to\infty} \|u_{n_j} - p\| &< \liminf_{j\to\infty} \|u_{n_j} - Wp\| \\ &\leq \liminf_{j\to\infty} \{\|u_{n_j} - Wu_{n_j}\| + \|Wu_{n_j} - Wp\|\} \\ &\leq \liminf_{j\to\infty} \{\|u_{n_j} - Wu_{n_j}\| + \|u_{n_j} - p\|\} \\ &\leq \liminf_{j\to\infty} \|u_{n_j} - p\|, \end{aligned}$$

which is a contradiction. Therefore, Lemma 2.6 yields $p \in F(W) = \cap_{i=1}^{\infty} F(T_i)$. Hence, $p \in \Omega$. Since x^* satisfies inequality (3.2), we obtain

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{j\to\infty} \langle f(x^*) - x^*, x_{n_j} - x^*\rangle = \langle f(x^*) - x^*, p - x^*\rangle \leq 0.$$

It follows from (C_5) and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ that

$$\begin{aligned} \limsup_{n\to\infty} b_n &\leq \frac{2}{1-\tau} \limsup_{n\to\infty} \langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad + \frac{2}{1-\tau} \limsup_{n\to\infty} \frac{2\theta_n(1-\vartheta_n)}{\vartheta_n} \|x_n - x_{n-1}\| \|x_{n+1} - x^*\| \\ &\leq \frac{2}{1-\tau} \limsup_{n\to\infty} \langle f(x^*) - x^*, x_{n+1} - x_n\rangle \\ &\quad + \frac{2}{1-\tau} \limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle \\ &\leq \frac{2}{1-\tau} \lim_{j\to\infty} \langle f(x^*) - x^*, x_{n_j} - x^*\rangle \\ &\leq \frac{2}{1-\tau} \langle f(x^*) - x^*, p - x^*\rangle \\ &\leq 0. \end{aligned}$$

Hence, by applying Lemma 2.9 to (3.15), we can deduce that $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that $\{\|x_n - x^*\|\}$ is not a monotonicaly decreasing sequence. Then there exsits a subsequence $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i} - x^*\| < \|x_{n_{i+1}} - x^*\|$ for all $i \in \mathbb{N}$. Define a positive integer sequence $\{\rho(n)\}$ by

$$\rho(n) := \max\{m \leq n : \|x_m - x^*\| < \|x_{m+1} - x^*\|\},$$

for all $n \geq n_0^*$. Combining this with Lemma 2.10, we obtain that

$$\|x_{\rho(n)} - x^*\|^2 \leq \|x_{\rho(n)+1} - x^*\|^2, \tag{3.28}$$

and

$$\|x_n - x^*\|^2 \leq \|x_{\rho(n)+1} - x^*\|^2. \tag{3.29}$$

Take $n = \rho(n)$ in (3.13) and (3.14). From the conditions (C_2)-(C_6), we have

$$\lim_{n\to\infty} \|(I - J_{\lambda_{\rho(n)}}^{B_2})Av_{\rho(n)}\| = 0, \tag{3.30}$$

$$\lim_{n\to\infty} \textstyle\sum_{i=1}^{n} \alpha_{\rho(n),i}\|y_{\rho(n)} - z_{\rho(n),i}\|^2 = 0, \tag{3.31}$$

$$\lim_{n\to\infty} \|y_{\rho(n)} - z_{\rho(n),i}\|^2 = 0, \tag{3.32}$$

$$\lim_{n\to\infty} \|W_{\rho(n)}u_{\rho(n)} - u_{\rho(n)}\| = 0. \tag{3.33}$$

From (3.2), (3.30)-(3.33) and by the similar proof of Case 1, we obtain

$$\limsup_{n\to\infty} b_{\rho(n)} \leq \frac{2\langle f(x^*) - x^*, p - x^*\rangle}{1-\tau} \leq 0. \tag{3.34}$$

From (3.15), we have

$$\|x_{\rho(n)+1} - x^*\|^2 \leq [1 - \vartheta_{\rho(n)}(1-\tau)]\|x_{\rho(n)} - x^*\|^2 + \vartheta_{\rho(n)}(1-\tau)b_{\rho(n)}, \tag{3.35}$$

which together with (3.28) implies

$$\begin{aligned}\vartheta_{\rho(n)}(1-\tau)\|x_{\rho(n)} - x^*\|^2 &\leq \|x_{\rho(n)} - x^*\|^2 - \|x_{\rho(n)+1} - x^*\|^2 + \vartheta_{\rho(n)}(1-\tau)b_{\rho(n)} \\ &\leq \vartheta_{\rho(n)}(1-\tau)b_{\rho(n)}.\end{aligned}$$

Hence, it follows that $\|x_{\rho(n)} - x^*\|^2 \leq b_{\rho(n)}$. Then combining this with (3.34), we have

$$\lim_{n\to\infty} \|x_{\rho(n)} - x^*\|^2 = 0.$$

Meanwhile, we also have $\lim_{n\to\infty} \|x_{\rho(n)+1} - x^*\|^2 = 0$ due to (3.35) and condition (C_5). Hence, it follows from (3.29) that $\lim_{n\to\infty} \|x_n - x^*\|^2 = 0$, i.e., the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

4. Applications

Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) is to find a point

$$x^* \in C \textit{ such that } Ax^* \in Q. \tag{4.1}$$

Applying Theorem 3.1, we obtain a strongly convergent result without a priori estimate of the operator norm for finding a common solution of the SFP (4.1) and the common fixed point problem for multivalued demicontractive mappings and strict pseudo-contractive mappings as follows.

Theorem 4.1. *Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. Let $S_i : C \to CB(C)$ be a multivalued $\widetilde{k}_i$-demicontractive mapping. Let $T_i : C \to C$ be a k_i-strictly pseudo-contractive mapping, and let $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq \widetilde{b} < 1$ for all $i \in \mathbb{N}$. Suppose that $\Omega := (\cap_{i=1}^{\infty} F(S_i)) \cap A^{-1}(Q) \cap (\cap_{i=1}^{\infty} F(T_i)) \neq \emptyset$. Let $f : H_1 \to H_1$ be a τ-contractive mapping with $\tau \in (0,1)$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1, x_2 \in C$ and*

$$\begin{cases} v_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(v_n - \gamma_n A^*(I - P_Q)Av_n), \\ u_n = (1-\delta_n)y_n + \delta_n \sum\limits_{i=1}^{n} \alpha_{n,i} z_{n,i}, \\ x_{n+1} = \vartheta_n f(x_n) + (1-\vartheta_n)[(1-\beta_n)u_n + \beta_n W_n u_n], \end{cases} \tag{4.2}$$

where $n \geq 2$, $z_{n,i} \in S_i y_n$, and θ_n is defined by

$$\theta_n = \begin{cases} \min\left\{ \frac{\tau_n}{\|x_n - x_{n-1}\|}, \ \frac{n-1}{n+\theta-1} \right\}, & if\ x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1}, & otherwise, \end{cases}$$

where $\theta \geq 3$ and the stepsize γ_n is selected in a way:

$$\gamma_n = \begin{cases} \frac{\xi_n \|(I-P_Q)Av_n\|^2}{\|A^*(I-P_Q)Av_n\|^2}, & if\ Av_n \notin Q, \\ 1, & otherwise, \end{cases}$$

and the sequences $\{\alpha_{n,i}\}, \{\vartheta_n\}, \{\delta_n\}, \{\xi_n\}, \{\tau_n\}$ and $\{\beta_n\}$ satisfy the conditions (C_2)-(C_6) in Theorem 3.1. If S_i satisfies the endpoint condition and $I - S_i$ is demiclosed at zero for all $i \in \mathbb{N}$, then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, where $x^* = P_\Omega f(x^*)$.

Proof. Setting $B_1 := N_C = \partial i_C$ and $B_2 := N_Q = \partial i_Q$, we have that B_1 and B_2 are maximal monotone. We also have $J_\lambda^{B_1} = P_C$ and $J_\lambda^{B_2} = P_Q$ for $\lambda > 0$, and $B_1^{-1}0 = C$ and $B_2^{-1}0 = Q$. Hence, the result is obtained by Theorem 3.1 immediately. □

Let $g_1 : H_1 \to (-\infty, \infty]$ and $g_2 : H_2 \to (-\infty, \infty]$ be two proper, lower semicontinuous, and convex functions, and let $A : H_1 \to H_2$ be a bounded linear operator. The split minimization problem (SMP) is reduced to finding a point $x^* \in H_1$ such that

$$x^* \in Argmin\, g_1 \ \ and \ \ Ax^* \in Argmin\, g_2. \tag{4.3}$$

Applying Theorem 3.1, we can obtain a strongly convergent result without a priori estimate of the operator norm for finding a common solution of the SMP (4.3) and the common fixed point problem of the multivalued demicontractive mappings and strict pseudo-contractive mappings as follows.

Theorem 4.2. *Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S_i : H_1 \to CB(H_1)$ be a multivalued $\widetilde{k}_i$-demicontractive mapping. Let $T_i : H_1 \to H_1$ be k_i-strictly pseudo-contractive mapping, and let $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq \widetilde{b} < 1$ for all $i \in \mathbb{N}$. Let $g_1 : H_1 \to (-\infty, \infty]$ and $g_2 : H_2 \to (-\infty, \infty]$ be two proper, lower semicontinuous, and convex functions. Suppose that $\Omega := (\cap_{i=1}^{\infty} F(S_i)) \cap \Theta \cap (\cap_{i=1}^{\infty} F(T_i)) \neq \emptyset$, where $\Theta = \{x \in Argmin\, g_1 : Ax \in Argmin\, g_2\}$. Let $f : H_1 \to H_1$ be a τ-contractive mapping with $\tau \in (0,1)$. Let $\{x_n\}$ be a sequence generated* *iteratively by;* $x_1, x_2 \in H_1$, *and*

$$\begin{cases} v_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = Prox_{\lambda_n g_1}(v_n - \gamma_n A^*(I - Prox_{\lambda_n g_2})Av_n), \\ u_n = (1-\delta_n)y_n + \delta_n \sum_{i=1}^{n} \alpha_{n,i} z_{n,i}, \\ x_{n+1} = \vartheta_n f(x_n) + (1-\vartheta_n)[(1-\beta_n)u_n + \beta_n W_n u_n], \end{cases}$$

where $n \geq 2$, $z_{n,i} \in S_i y_n$, *and* θ_n *is defined by*

$$\theta_n = \begin{cases} \min\left\{\frac{\tau_n}{\|x_n - x_{n-1}\|}, \ \frac{n-1}{n+\theta-1}\right\}, & if\ x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1}, & otherwise, \end{cases}$$

where $\theta \geq 3$ *and the stepsize* γ_n *is selected in a way:*

$$\gamma_n = \begin{cases} \frac{\xi_n \|(I - Prox_{\lambda_n g_2})Av_n\|^2}{\|A^*(I - Prox_{\lambda_n g_2})Av_n\|^2}, & if\ Av_n \notin Argmin\ g_2, \\ 1, & otherwise, \end{cases}$$

and the sequences $\{\lambda_n\}, \{\alpha_{n,i}\}, \{\vartheta_n\}, \{\delta_n\}, \{\xi_n\}, \{\tau_n\}$ *and* $\{\beta_n\}$ *satisfy the conditions* (C_1)-(C_6) *in Theorem 3.1. If* S_i *satisfies the endpoint condition and* $I - S_i$ *is demiclosed at zero for all* $i \in \mathbb{N}$, *then the sequence* $\{x_n\}$ *converges strongly to a point* $x^* \in \Omega$, *where* $x^* = P_\Omega f(x^*)$.

Proof. Taking $C = H_1$, $B_1 = \partial g_1$ *and* $B_2 = \partial g_2$, we have B_1 and B_2 are maximal monotone. One can show that $Argmin\ g_1 = (\partial g_1)^{-1}0 = B_1^{-1}0$ and $Argmin\ g_2 = (\partial g_2)^{-1}0 = B_2^{-1}0$. Obviously, the result is obtained by Theorem 3.1. □

5. NUMERICAL EXAMPLE

In this section, we present a numerical example to demonstrate the efficiency of our algorithm.

Example 5.1. Let $H_1 = R$ and $H_2 = R^3$. For each $i \in \mathbb{N}$, define a multivalued mapping $S_i : R \to CB(R)$ as follows:

$$S_i x = \begin{cases} \left[-\frac{3i}{2i+1}x, -\frac{4i}{2i+1}x\right], & if\ x \leq 0, \\ \left[-\frac{4i}{2i+1}x, -\frac{3i}{2i+1}x\right], & if\ x > 0. \end{cases}$$

Define a bounded linear operator $A : R \to R^3$ by $Ax := (15x, 6x, -27x)$. For each $n \in \mathbb{N}, i \geq 1$, let

$$\alpha_{n,i} = \begin{cases} \frac{1}{2^i}\left(\frac{n}{n+1}\right), & if\ n > i, \\ 1 - \frac{n}{n+1}\left(1 - \left(\frac{1}{2}\right)^{n-1}\right), & if\ n = i, \\ 0, & otherwise. \end{cases}$$

Let $B_1 : R \to 2^R$ be defined by

$$B_1(x) := \begin{cases} \{u \in R : z^2 + xz - 2x^2 \geq (z-x)u,\ \forall\, z \in [-9,3]\}, & if\ x \in [-9,3], \\ \emptyset, & otherwise. \end{cases}$$

Define a maximal monotone mapping $B_2 : R^3 \to 2^{R^3}$ by $B_2 := \partial g$, where $g : R^3 \to R$ is a function defined by $g(x,y,z) = \frac{|2x-5y+3z|^2}{2}$. Take $\lambda_n = 1, \tau_n = 1/(n+3)^2, \theta = 3, \lambda_n = 1, \delta_n = n/(6n+1), \vartheta_n = 1/(n+3), \beta_n = (n+1)/(2n+3)$, and $\xi_n = \sqrt{\frac{3n+1}{n+2}}$. Define a contraction f by $f(x) = (1/2)x$ with $\tau = 1/2$ and an infinite family of mappings $T_n : R \to R$ by $T_n x = -\frac{2}{n}x$ for all $x \in R$, $T_n' = t_n I + (1-t_n)T_n$, $t_n \in [k_n, 1)$. Let $\{\zeta_n\}$ be a sequence of nonnegative real numbers defined by $\zeta_n = n/(3n-1)$ for all $n \in \mathbb{N}$ and $\{W_n\}$ be generated by $\{T_n\}, \{\zeta_n\} and \{t_n\}$. Choose $k_n = t_n = 1/(n+2)$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $0 = P_\Omega f(0)$.

Indeed, from Example 2.1, we know that S_i is multivalued $\widetilde{k}_i$-demicontractive mapping with $\widetilde{k}_i = \frac{12i^2-4i-1}{25i^2+10i+1} \in (0,1)$, $I - S_i$ is demiclosed at zero for all $i \in \mathbb{N}$, and $k = \sup_{i\in\mathbb{N}} \widetilde{k}_i = \frac{12}{25} < 1$. By [15, Theorem 4.2], B_1 is maximal monotone. The resolvents of B_1 and B_2 can be written by $J_1^{B_1} x = x/4$ and $J_1^{B_2} = Prox_g = P^{-1}$, where

$$P = \begin{pmatrix} 5 & -10 & 6 \\ -10 & 26 & -15 \\ 6 & -15 & 10 \end{pmatrix}.$$

It is no difficult to verify that T_n is k_n-strict pseudo-contractive mapping for each $n \in \mathbb{N}$. It can also easily be checked that all the conditions on the control sequences in Theorem 3.1 are satisfied. Then algorithm (3.1) reduces to

$$\begin{cases} v_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \frac{1}{4}(v_n - \gamma_n A^*(I - P^{-1})Av_n), \\ u_n = \frac{5n+1}{6n+1} y_n + \frac{n}{6n+1} \sum_{i=1}^n \alpha_{n,i} z_{n,i}, \\ x_{n+1} = \frac{1}{2(n+3)} x_n + \frac{n+2}{n+3}[\frac{n+2}{2n+3} u_n + \frac{n+1}{2n+3} W_n u_n], \end{cases} \tag{5.1}$$

for all $n \geq 2$, where

$$\gamma_n := \begin{cases} \sqrt{\frac{3n+1}{n+2} \frac{\|(I-P^{-1})Av_n\|^2}{\|A^\top(I-P^{-1})Av_n}} \|^2, & if\ Av_n \neq P^{-1}(Av_n), \\ 1, & otherwise, \end{cases}$$

and

$$z_{n,i} := \begin{cases} -\frac{4i}{2i+1} y_n, & if\ y_n \leq 0, \\ -\frac{3i}{2i+1} y_n, & if\ y_n > 0. \end{cases}$$

Hence, the sequence $\{x_n\}$ generated by (5.1) converges strongly to $0 \in \Omega$ by Theorem 3.1.

We choose a pair of initial values to demonstrate the efficiency of our algorithm.

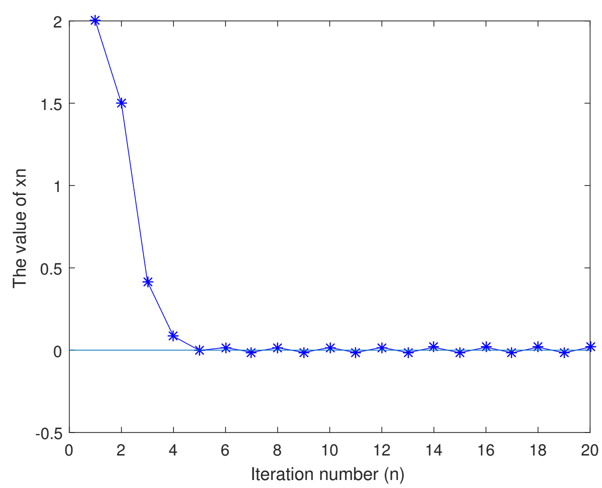

FIGURE 1. Numerical result for Example 5.1 with $x_1 = 2$ and $x_2 = 1.5$.

TABLE 1. The value of x_n

n	x_n	n	x_n
1	2	11	-0.0147
2	1.5	12	0.0171
3	0.4161	13	-0.0152
4	0.0842	14	0.0176
5	-0.0016	15	-0.0156
6	0.0162	16	0.0180
7	-0.0127	17	-0.0160
8	0.0154	18	0.0183
9	-0.0139	19	-0.0162
10	0.0164	20	0.0186

Thus we can obtain that the sequence $\{x_n\}$ generated by (5.1) converges to $0 \in \Omega = \{0\}$. And we can see from the figure and the table that $\{x_n\}$ converges to 0. Therefore, the iterative algorithm of Theorem 3.1 is well defined and efficient.

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