Appl. Set-Valued Anal. Optim. 5 (2023), No. 1, pp. 69-84
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.5.2023.1.05

NONSMOOTH INTERVAL-VALUED MULTIOBJECTIVE OPTIMIZATION PROBLEMS AND GENERALIZED VARIATIONAL INEQUALITIES ON HADAMARD MANIFOLDS

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Abstract. In this paper, we study the classes of approximate Minty and Stampacchia type vector variational inequalities along with their local and weak versions on a Hadamard manifold, and a class of nonsmooth interval-valued multiobjective optimization problems, respectively. We obtain the equivalence between the solutions of considered approximate vector variational inequalities and LU-efficient solutions of nonsmooth interval-valued multiobjective optimization problems in which the cost function is assumed to be geodesic approximately LU-convex or geodesic LU-α-convex.
Keywords. Clarke subgradient; Geodesic approximate LU-convexity; Hadamard manifold; Multiobjective optimization problem; Vector variational inequality.

2020 Mathematics Subject Classification. 49J52, 58E17, 58E35.

1. INTRODUCTION

In recent years, several concepts and techniques related to optimization problems have been extended from Euclidean spaces to Hadamard manifolds. In this direction, linear structure is replaced by a geodesic. This extension has a plenty of significant advantages. For example, the optimization problems involving nonconvex and non-monotone functions can be reduced to generalized convex optimization problems via applying the relevant properties of Riemannian metrics. Rapcsák [1] and Udriste [2] generalized the concept of the line segments between two points by geodesic segments, and introduced the notion of geodesic convex functions. A Hadamard manifold is a simply connected, complete Riemannian manifold with non-positive sectional curvature. Recently, numerous authors studied the generalizations of convex sets and functions on Riemannian and Hadamard manifolds; see, e.g., [3, 4, 5, 6, 7] and the references cited therein.

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Received January 2, 2023; Accepted March 8, 2023

Since there are numerous optimization problems which arise from real world applications always occurring with uncertainty. The theory of interval-valued has become a powerful mathematical tool to investigate the optimization problems with uncertainty because it can formulate the deterministic optimization models to deal with the inexact, imprecise, or uncertain data. In the research of interval-valued optimizations, it leads to an enormous difficulty that the objective and constraint functions are usually considered by interval-valued functions. To deal with the functions with interval coefficients, Moore [8, 9] introduced the concept of interval analysis. Wu [10] established the Karush-Kuhn-Tucker (KKT) optimality conditions for interval-valued optimization problems. After that, Ghosh et al. [11] extended the KKT conditions for constrained interval-valued optimization problems. Recently, Antczak [12] established the Fritz John, KKT necessary and sufficient optimality conditions for a new kind of nonsmooth interval-valued multiobjective optimization problems.

Convexity plays a central role in optimization theory and related areas. However, there are a lot of mathematical modeling from the real world applications which have not convex construct. Based on this motivation, several generalizations of the convexity notion were introduced and studied. For recent survey and more exposition about generalized convex functions, we refer to [13, 14].

In an effort to generalize the convexity notion, Luc et al. [15] introduced a new class of generalized convex functions, namely, ε-convex functions which have applications in approximate calculus. Ngai et al. [16] introduced an interesting class of generalized convex functions, namely, approximately convex functions by employing ε-convex functions as a tool. The class of approximately convex functions includes the classes of convex functions and weakly and strongly convex functions, and is stable under the finite sum and finite supremum. Moreover, most of the known subdifferentials, such as Clarke [17], Ioffe [18], and Mordukhovich [19, 20] have been coincided for approximately convex functions. Ngai and Penot [16] derived several characterizations for approximately convex functions in terms of generalized subdifferential. Amini-Harandi and Farajzadeh [21] extended and refined the results of Daniilidis and Georgiev [22] from Banach spaces to locally convex spaces. Upadhyay et al. [23] introduced a class of generalized approximate LU-convex functions and derived the relations between a nonsmooth interval-valued multiobjective programming problem and a generalized Stampacchia vector variational inequality.

Giannessi [24] introduced the vector valued version of variational inequalities studied by Minty [25] and Stampacchia [26] in finite dimensional Euclidean spaces. Since then, vector variational inequalities and their generalizations have been widely used as an efficient tool to study multiobjective optimization problems; see, e.g., [27, 28, 29, 30] and the references cited therein. Németh [31] introduced variational inequalities on Hadamard manifolds and established some results concerning the existence of solutions to a variational inequality. Li et al. [32] investigated some existence results for the solutions and convexity of the solution set for the variational inequalities with multivalued mappings on Riemannian manifolds. Chen and Huang [33] studied the equivalence among Stampacchia and Minty vector variational inequalities and nonsmooth multiobjective optimization problems by using Clarke subdifferential, and proved certain existence theorems under relaxed compactness assumption. Chen and Fang [34] investigated the equivalence among Stampacchia and Minty vector variational inequalities and nonsmooth multiobjective optimization problems under pseudoconvexity assumptions. Recently, Upadhyay et al. [35] studied the equivalence between the solutions of an interval-valued multiobjective optimization problem and a generalized vector variational inequality involving strongly geodesic LU-convex functions on a Hadamard manifold.

Motivated by the works of [33, 34, 35, 36, 37, 38], we consider the classes of approximate Minty and Stampacchia vector variational inequalities and nonsmooth interval-valued multiobjective optimization problems. Under approximate geodesic LU-convexity and geodesic LU-α-convexity assumptions, we establish the equivalence between the solutions of considered approximate vector variational inequalities and LU-efficient solution of nonsmooth interval-valued multiobjective optimization problem.

The organization of this paper is given as follows. In Section 2, we recall some basic definitions and preliminaries. In Section 3, we consider the classes of nonsmooth interval-valued multiobjective optimization problems and approximate Stampacchia and Minty vector variational inequalities along with their weak and local versions. Certain equivalence relations between the solutions of considered approximate vector variational inequalities and nonsmooth interval-valued multiobjective optimization problems are established under geodesic approximate LU-convexity and geodesic LU-α-convexity assumptions on Hadamard manifold. Section 4 ends this paper by providing conclusions and further research.

2. Preliminaries

In what follows, we denote by $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\text{int}(\mathbb{R}^n_+)$ the n-dimensional Euclidean space, the non-negative orthant of $\mathbb{R}^n$, and the positive orthant of $\mathbb{R}^n$, respectively.

For the reader's convenience, we use the following symbols to describe the relation between the vectors $y, z \in \mathbb{R}^n$

(i) $y = z \iff y_j = z_j, \forall j = 1, \dots, n;$
(ii) $y < z \iff y_j < z_j, \forall j = 1, \dots, n;$
(iii) $y \leqq z \iff y_j \leq z_j, \forall j = 1, \dots, n;$
(iv) $y \leq z \iff y_j \leq z_j, \forall j = 1, \dots, n,\ j \neq k$ and $y_k < z_k$ for some k.

Also, we adopt the notion $\mathscr{I}$ to denote the class of all compact intervals in $\mathbb{R}$, i.e., $C = [c^L, c^U] \in \mathscr{I}$ is a compact interval, where c^L and c^U denote the lower and upper bounds of C, respectively. Let us recall the following algebra operations to intervals $C = [c^L, c^U], D = [d^L, d^U] \in \mathscr{I}$

(i) $C + D = \{c + d : c \in C \text{ and } d \in D\} = [c^L + d^L, c^U + d^U];$
(ii) $-C = \{-c : c \in C\} = [-c^U, -c^L];$
(iii) $C \times D = \{cd : c \in C \text{ and } d \in D\} = [\min_{cd}, \max_{cd}]$, where
$\min_{cd} = \min\{c^L d^L, c^L d^U, c^U d^L, c^U d^U\}$ and $\max_{cd} = \max\{c^L d^L, c^L d^U, c^U d^L, c^U d^U\}$.

Then, the minus operator and scalar multiplication to intervals $C = [c^L, c^U], D = [d^L, d^U] \in \mathscr{I}$ can be defined by

$$C - D = C + (-D) = [c^L - d^U, c^U - d^L],$$

$$\alpha C = \{\alpha c : c \in C\} = \begin{cases} [\alpha c^L, \alpha c^U], & \alpha \geq 0, \\ |\alpha|[-c^U, -c^L], & \alpha < 0, \end{cases}$$

with $\alpha \in \mathbb{R}$. The real number c can be considered as a closed interval $C_c = [c, c]$.

For any $C = [c^L, c^U], D = [d^L, d^U] \in \mathscr{I}$, we define

(i) $C \preceq_{LU} D \iff c^L \leq d^L$ and $c^U \leq d^U$,
(ii) $C \prec_{LU} D \iff C \preceq_{LU} D$ and $C \neq D$, that is, one of the following is satisfied:
(a) $c^L < d^L$ and $c^U < d^U$; or
(b) $c^L \leq d^L$ and $c^U < d^U$; or
(c) $c^L < d^L$ and $c^U \leq d^U$.

Remark 2.1. In what follows, we say that intervals $C = [c^L, c^U]$, $D = [d^L, d^U] \in \mathscr{I}$ are comparable if and only if $C \preceq_{LU} D$ or $C \succeq_{LU} D$. So, it is easy to see that C and D are not comparable if one of the following holds:

$$c^L \leq d^L \text{ and } c^U > d^U;\; c^L < d^L \text{ and } c^U \geq d^U;\; c^L < d^L \text{ and } c^U > d^U;$$
$$c^L \geq d^L \text{ and } c^U < d^U;\; c^L > d^L \text{ and } c^U \leq d^U;\; c^L > d^L \text{ and } c^U < d^U.$$

Let $\mathbf{C} = (C_1, \ldots, C_p)$ be an interval-valued vector, where each component $C_k = [c_k^L, c_k^U]$, $k = 1, 2, \ldots, p$ is a compact interval. Let $\mathbf{C}$ and $\mathbf{D}$ be two interval-valued vectors. If C_k and D_k are comparable for each $k = 1, 2, \ldots, p$, then

(i) $\mathbf{C} \preceq_{LU} \mathbf{D}$ if and only if $C_k \preceq_{LU} D_k$ for all $k = 1, 2, \ldots, p$;
(ii) $\mathbf{C} \prec_{LU} \mathbf{D}$ if and only if $C_k \preceq_{LU} D_k$ for all $k = 1, 2, \ldots, p$, $k \neq r$ and $C_r \prec_{LU} D_r$ for some r.

The function $\Psi : \mathbb{R}^n \to \mathscr{I}$ is called an interval-valued function, where $\Psi(y) = [\Psi^L(y), \Psi^U(y)]$ and Ψ^L, $\Psi^U : \mathbb{R}^n \to \mathbb{R}$ are real valued functions satisfying $\Psi^L(y) \leq \Psi^U(y)$, for all $y \in \mathbb{R}^n$.

We recall the following definitions and preliminaries about Riemannian manifold (see [1, 2]).

Let H be a connected manifold with finite dimension n. For $z \in H$, $T_z H$ and $TH = \cup_{z \in H} T_z H$ denote by the tangent space of H at z and the tangent bundle of H, respectively. Assume that H is a Riemannian manifold endowed with a Riemannian metric $\langle ., . \rangle_z$ on the tangent space $T_z H$ with associated norm denoted by $\|.\|_z$, and $\Omega : [a, b] \to H$ is a piecewise differentiable curve joining $\Omega(a) = p$ to $\Omega(b) = q$. Then the length of Ω is defined by

$$L(\Omega) := \int_a^b \left\|\Omega'(\lambda)\right\|_{\Omega(\lambda)} d\lambda.$$

For any $p, q \in H$, the Riemannian distance between p and q is defined by $d(p, q) := \inf_\Omega L(\Omega)$, namely, the Riemannian distance between p and q is the infimum over all piecewise differentiable curve joining p and q. It is not difficult to see that this distance function d induces the original topology on H. Let $\chi(H)$ denote the space of all vector fields on H. Then, the Riemannian metric induces a map $\Psi \mapsto$ *grad* $\Psi \in \chi(H)$, which associates to each Ψ its gradient via the rule $\langle d\Psi, X \rangle = d\Psi(X)$, for each $X \in \chi(H)$. It is well-known that on every Riemannian manifold there exists one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields $X, Y \in H$. We also recall that a geodesic is a C^∞ smooth path Ω whose tangent is parallel along the path Ω, that is, Ω satisfies the equation

$$\nabla_{\frac{d\Omega(\lambda)}{d\lambda}} \frac{d\Omega(\lambda)}{d\lambda} = 0.$$

It is known that a Levi-Civita connection ∇ can induce an isometry $P_{\lambda_1,\Omega}^{\lambda_2} : T_{\Omega(\lambda_1)} H \to T_{\Omega(\lambda_2)} H$, which is referred to as parallel translation along Ω from $\Omega(\lambda_1)$ to $\Omega(\lambda_2)$. We say that Ω is a minimal geodesic, if Ω is a path joining p and q in H such that $L(\Omega) = d(p, q)$ is a geodesic. A Riemannian manifold is said to be complete, if for any $z \in H$, all geodesics emananting from z

are defined for all $-\infty < \lambda < \infty$. If H is complete, then any points in H can be joined by minimal geodesic. Suppose that H is complete. The exponential map $exp_z : T_zH \to H$ at z is defined by $exp_z v = \Omega_v(1,z)$, for every $v \in T_zH$, where $\Omega(\cdot) = \Omega_v(\cdot,z)$ is the geodesic starting at z with velocity v, that is $\Omega(0) = z$ and $\Omega'(0) = v$. It is easy to see that $exp_z(\lambda v) = \Omega_v(\lambda,z)$, for each real number λ. We note that the map exp_z is differentiable on T_zH, for every $z \in H$. We say that a Hadamard manifold is a simply connected complete Riemannian manifold with nonpositive sectional curvature, if H is a Hadamard manifold such that $exp_z : T_pH \to H$ is a diffeomorphism for every $p \in H$ and for any $z, y \in H$, there exists unique minimal geodesic joining z and y.

Let $A : H \to 2^H$ be a multivalued vector field such that $Az \subseteq T_zH$ for each $z \in H$. Then the domain $D(A)$ of A is defined by $D(A) := \{z \in H : A(z) \neq \emptyset\}$.

Definition 2.1. [37] Let H be a Hadamard manifold. A is said to be

(i) geodesic approximately monotone or geodesic submonotone at $\bar{z} \in H$ if, for every $\alpha > 0$, there exists $\delta > 0$, such that, for every $z, y \in B(\bar{z},\delta) \cap D(A)$, and for every $\xi \in A(z), \eta \in A(y)$,
$$\left\langle P^{0}_{1,\Omega}\xi - \eta, \exp_y^{-1} z \right\rangle \geq -\alpha \left\| \exp_y^{-1} z \right\|;$$

(ii) geodesic α-monotone on H if, for given $\alpha > 0$, for every $z, y \in D(A)$, and for every $\xi \in A(z), \eta \in A(y)$,
$$\left\langle P^{0}_{1,\Omega}\xi - \eta, \exp_y^{-1} z \right\rangle \geq -\alpha \left\| \exp_y^{-1} z \right\|,$$
where $\Omega(\lambda) := \exp_y(\lambda \exp_y^{-1} z), \lambda \in [0,1]$.

In addition, we also recall the following definitions concerning nonsmooth analysis from [39, 40].

Definition 2.2. Let $\Psi : H \to]-\infty,\infty]$ be a proper function. Ψ is said to be Lipschitz near $\bar{z} \in H$ if there exists a positive constant $L_{\bar{z}}$, and $\delta_z > 0$ such that $|\Psi(z) - \Psi(y)| \leq L_{\bar{z}} d(z,y)$ for all $z, y \in B(\bar{z},\delta_z)$, where $L_{\bar{z}}$ is called Lipschitz constant of Ψ in the neighbourhood of $\bar{z}$ and $B(\bar{z},\delta_z) := \{z \in H : \; d(\bar{z},z) < \delta_z\}$. Moreover, Ψ is locally Lipschitz on H, if it is Lipschitz near $\bar{z}$ for every $\bar{z} \in H$.

Definition 2.3. A nonempty subset Γ of H is said to be geodesic convex set if, for any points $z, y \in \Gamma$, the geodesic joining z to y is contained in Γ.

From now onwards, H is assumed to be a Hadamard manifold of dimension n, $\emptyset \neq \Gamma$ is assumed to be a geodesic convex subset of H, and $\Psi : \Gamma \to \mathbb{R}$ is assumed to be a locally Lipschitz function on Γ, unless otherwise specified.

Definition 2.4. The Clarke generalized directional derivative of Ψ at $z \in \Gamma$ in the direction of a vector $\nu \in T_zH$, denoted by $\Psi^\circ(z;\nu)$, is defined as
$$\Psi^\circ(z;\nu) := \limsup_{\substack{y \to z \\ \lambda \downarrow 0}} \frac{\Psi(exp_y \lambda (d\,exp_z)_{exp_z^{-1}y}\nu) - \Psi(y)}{\lambda},$$
where $(d\,exp_z)_{exp_z^{-1}y}$ is the differential of exponential mapping at $exp_z^{-1}y$.

Definition 2.5. The Clarke generalized subdifferential of Ψ at $z \in \Gamma$, denoted by $\partial_c\Psi(z)$, is the subset of T_zH^* defined by $\partial_c\Psi(z) := \{\xi \in T_zH^* : \Psi^\circ(z;\nu) \geq \langle \xi, \nu \rangle, \forall \nu \in T_zH\}$.

Lemma 2.1. *Let H be a Riemannian manifold, and let $\Psi : H \to \mathbb{R}$ be a Lipschitz function near $z \in H$. Then,*

(i) *$\partial_c \Psi(z)$ is a nonempty weak*-compact subset of $T_z H^*$, and $\|\xi\| \leq L_z$ for all $\xi \in \partial_c \Psi(z)$, where L_z is the Lipschitz constant of Ψ in the neighbourhood of z.*

(ii) *If $\{z_j\}$ and $\{\xi_j\}$ are sequences in H and TH^*, respectively such that $\xi_j \in \partial_c \Psi(z_j)$ for each j, and if $\{z_j\}$ converges to z, and ξ is a weak*-cluster point of the sequence $\{P^{z_j}_{z,\Omega} \xi_j\}$, then $\xi \in \partial_c \Psi(z)$.*

Moreover, we recall geodesic approximately convex functions and geodesic α-convex functions on a Hadamard manifold from [37].

Definition 2.6. Let $\Psi : \Gamma \to \mathbb{R}$ be a locally Lipschitz function. Then

(i) Ψ is said to be (geodesic strictly approximately convex) geodesic approximately convex at $\bar{z} \in \Gamma$ if, for all $\alpha > 0$, there exists $\delta > 0$, such that, for all $z, y \in B(\bar{z}, \delta) \cap \Gamma$,

$$\Psi(\Omega(\lambda))(<) \leq \lambda \Psi(y) + (1-\lambda)\Psi(z) + \alpha\lambda(1-\lambda) \left\| \exp_z^{-1} y \right\|, \forall \lambda \in [0,1];$$

(ii) the function Ψ is said to be (geodesic strictly α-convex) geodesic α-convex on Γ if, for all $z, y \in \Gamma$,

$$\Psi(\Omega(\lambda))(<) \leq \lambda \Psi(y) + (1-\lambda)\Psi(z) + \alpha\lambda(1-\lambda) \left\| \exp_z^{-1} y \right\|, \forall \lambda \in [0,1],$$

where $\Omega(\lambda) := \exp_z(\lambda \exp_z^{-1} y), \lambda \in [0,1]$.

Theorem 2.1. [37] *Let $\Psi : \Gamma \to \mathbb{R}$ be a locally Lipschitz function. Then Ψ is geodesic approximately convex at $\bar{z} \in \Gamma$ if and only if, for every $\alpha > 0$ there exists $\delta > 0$ such that, for any $z, y \in B(\bar{z}, \delta) \cap \Gamma$ and $\xi \in \partial_c \Psi(z)$,*

$$\Psi(y) - \Psi(z) \geq \left\langle \xi, exp_z^{-1} y \right\rangle - \alpha \left\| exp_z^{-1} y \right\|. \tag{2.1}$$

Theorem 2.2. [37] *Let $\Psi : \Gamma \to \mathbb{R}$ be a locally Lipschitz function. Then Ψ is geodesic α-convex at $\bar{z} \in \Gamma$ if and only if, for $\alpha > 0$ and for every $\xi \in \partial_c \Psi(\bar{z})$,*

$$\Psi(y) - \Psi(\bar{z}) \geq \left\langle \xi, exp_{\bar{z}}^{-1} y \right\rangle - \alpha \left\| exp_{\bar{z}}^{-1} y \right\|, \forall y \in \Gamma.$$

The following theorem from [37] reveals the essential relationship between a geodesic approximately convex function and geodesic submonotonicity of its Clarke subdifferential on Hadamard manifolds.

Theorem 2.3. *Let $\Psi : \Gamma \to \mathbb{R}$ be a locally Lipschitz function on Γ. Then, Ψ is geodesic approximately convex at $\bar{z} \in \Gamma$ if and only if $\partial_c \Psi$ is geodesic submonotone at $\bar{z} \in \Gamma$.*

Theorem 2.4. *Let $\Psi : \Gamma \to \mathbb{R}$ be a locally Lipschitz function on Γ. Then Ψ is geodesic α-convex on Γ if and only if $\partial_c \Psi$ is geodesic 2α-monotone on Γ.*

Definition 2.7. An interval-valued function $\Psi : \Gamma \to \mathscr{I}$ is geodesic approximately LU-convex at $\bar{y} \in \Gamma$ if the real-valued functions Ψ^L and Ψ^U are geodesic approximately convex at $\bar{y} \in \Gamma$.

Definition 2.8. An interval-valued function $\Psi : \Gamma \to \mathscr{I}$ is geodesic LU-α-convex at $\bar{y} \in \Gamma$ if the real-valued functions Ψ^L and Ψ^U are geodesic α-convex at $\bar{y} \in \Gamma$.

We consider the following nonsmooth interval-valued multiobjective optimization problem:

$$\textbf{(NIVMOP)} \quad \min \Psi(z) = (\Psi_1(z), \Psi_2(z), ..., \Psi_p(z))$$
$$\text{subject to } z \in \Gamma,$$

where $\Psi_j : \Gamma \to \mathscr{I}$, $j \in J := \{1,2,...,p\}$ and Ψ_j^L and Ψ_j^U, $j \in J$ are locally Lipschitz functions on Γ.

The concepts of LU-efficient solutions and weakly LU-efficient solutions for (NIVMOP) are defined as follows:

Definition 2.9. A point $\bar{z} \in \Gamma$ is said to be an LU-efficient solution (local LU-efficient solution) of (NIVMOP), if there does not exist any $y \in \Gamma$, $(y \in B(\bar{z}, \delta) \cap \Gamma,\ \delta > 0)$, such that $\Psi(y) \prec_{LU} \Psi(\bar{z})$.

Definition 2.10. A point $\bar{z} \in \Gamma$ is said to be weakly LU-efficient solution (local weakly LU-efficient solution) of (NIVMOP), if there does not exist any $y \in \Gamma$, $(y \in B(\bar{z}, \delta) \cap \Gamma,\ \delta > 0)$, such that $\Psi_j(y) \prec_{LU} \Psi_j(\bar{z})$ for all $j \in J$.

We formulate the following approximate vector variational inequalities in terms of Clarke subdifferential on a Hadamard manifold:

(ALMVVIP) Find a point $\bar{z} \in \Gamma$ such that there exist a scalar β and $\delta > 0$ satisfying

$$\left(\left\langle \zeta_1^L, exp_y^{-1}\bar{z}\right\rangle, \dots, \left\langle \zeta_p^L, exp_y^{-1}\bar{z}\right\rangle\right) \ngeq \beta \left\|exp_y^{-1}\bar{z}\right\| e,$$
$$\left(\left\langle \zeta_1^U, exp_y^{-1}\bar{z}\right\rangle, \dots, \left\langle \zeta_p^U, exp_y^{-1}\bar{z}\right\rangle\right) \ngeq \beta \left\|exp_y^{-1}\bar{z}\right\| e,$$

for all $y \in B(\bar{z}, \delta) \cap \Gamma$, $\zeta_j^L \in \partial_c \Psi_j^L(y)$, and $\zeta_j^U \in \partial_c \Psi_j^U(y)$, $j \in J$;

(AMVVIP) Find a point $\bar{z} \in \Gamma$ such that there exists a scalar β satisfying

$$\left(\left\langle \zeta_1^L, exp_y^{-1}\bar{z}\right\rangle, \dots, \left\langle \zeta_p^L, exp_y^{-1}\bar{z}\right\rangle\right) \ngeq \beta \left\|exp_y^{-1}\bar{z}\right\| e,$$
$$\left(\left\langle \zeta_1^U, exp_y^{-1}\bar{z}\right\rangle, \dots, \left\langle \zeta_p^U, exp_y^{-1}\bar{z}\right\rangle\right) \ngeq \beta \left\|exp_y^{-1}\bar{z}\right\| e,$$

for all $y \in \Gamma$, $\zeta_j^L \in \partial_c \Psi_j^L(y)$, and $\zeta_j^U \in \partial_c \Psi_j^U(y)$, $j \in J$;

(ALSVVIP) Find a point $\bar{z} \in \Gamma$ such that there exist $\delta > 0$, $\beta > 0$, $\xi_j^L \in \partial_c \Psi_j^L(\bar{z})$, and $\xi_j^U \in \partial_c \Psi_j^U(\bar{z})$, $j \in J$ satisfying

$$\left(\left\langle \xi_1^L, exp_{\bar{z}}^{-1}y\right\rangle, \dots, \left\langle \xi_p^L, exp_{\bar{z}}^{-1}y\right\rangle\right) \nleq \beta \left\|exp_{\bar{z}}^{-1}y\right\| e,$$
$$\left(\left\langle \xi_1^U, exp_{\bar{z}}^{-1}y\right\rangle, \dots, \left\langle \xi_p^U, exp_{\bar{z}}^{-1}y\right\rangle\right) \nleq \beta \left\|exp_{\bar{z}}^{-1}y\right\| e,$$

for any $y \in B(\bar{z}, \delta) \cap \Gamma$;

(ASVVIP) Find a point $\bar{z} \in \Gamma$ such that there exist $\beta > 0$, $\xi_j^L \in \partial_c \Psi_j^L(\bar{z})$, and $\xi_j^U \in \partial_c \Psi_j^U(\bar{z})$, $j \in J$ satisfying

$$\left(\left\langle \xi_1^L, exp_{\bar{z}}^{-1}y\right\rangle, \dots, \left\langle \xi_p^L, exp_{\bar{z}}^{-1}y\right\rangle\right) \nleq \beta \left\|exp_{\bar{z}}^{-1}y\right\| e,$$
$$\left(\left\langle \xi_1^U, exp_{\bar{z}}^{-1}y\right\rangle, \dots, \left\langle \xi_p^U, exp_{\bar{z}}^{-1}y\right\rangle\right) \nleq \beta \left\|exp_{\bar{z}}^{-1}y\right\| e,$$

for all $y \in \Gamma$, where $e = \underbrace{(1,1,...,1)}_{p \text{ times}}$.

Remark 2.2. It is easy to prove that

(i) if $\bar{z}$ is a solution to (AMVVIP) with some constant β, then it is also the solution of the same problem for all parameters $\beta' \geq \beta$. Hence,

$$\begin{matrix}\text{solution set of} \\ \text{(AMVVIP) for } \beta>0\end{matrix} \subseteq \begin{matrix}\text{solution set of} \\ \text{(MVVIP)}\end{matrix} \subseteq \begin{matrix}\text{solution set of} \\ \text{(AMVVIP)for } \beta<0\end{matrix}$$

(ii) If $\beta = 0$, then (AMVVIP) and (ASVVIP) reduce to the Minty and Stampacchia vector variational inequalities (MVVIP) and (SVVIP), respectively, which were considered by Chen and Huang [33], respectively.

(iii) If $H = \mathbb{R}^n$, then $exp_z^{-1}y = y - z$. Moreover, for real-valued functions, when $\beta = 0$, (ASVVIP) reduces to the Stampacchia vector variational inequality (VVIP) which was considered by Mishra and Upadhyay [29] and Upadhyay et al. [30].

3. Relationship between Vector Variational Inequalities and Nonsmooth Interval-valued Multiobjective Optimization Problems

This section is devoted to the certain relations between approximate vector variational inequality problems (AMVVIP), (ALMVVIP), (ASVVIP), (ALSVVIP), and the nonsmooth interval-valued multiobjective optimization problem (NIVMOP), by using the tool of Clarke subdifferentials and the notion of LU-efficiency, as well as the notions of geodesic approximate LU-convexity and geodesic LU-α-convexity on Hadamard manifolds.

The proof of the following theorem follows on the lines presented in [41, Theorem 3(c)].

Theorem 3.1. *Let $\Psi_j : \Gamma \to \mathscr{I}$, $j \in J$ be a geodesic approximately LU-convex function at $\bar{z}$ on Γ. If $\bar{z} \in \Gamma$ solves (ALMVVIP), then $\bar{z}$ is a local LU-efficient solution of (NIVMOP).*

Theorem 3.2. *Let $\Psi_j : \Gamma \to \mathscr{I}$, $j \in J$, be a geodesic approximately LU-convex function at $\bar{z} \in \Gamma$. If $\bar{z}$ is a local LU-efficient solution to the (NIVMOP), then $\bar{z}$ solves (ALMVVIP).*

Proof. Suppose that $\bar{z}$ is a local LU-efficient solution to (NIVMOP), but $\bar{z}$ is not a solution to (ALMVVIP). Hence, for any $\beta > 0$, there exists $\delta' > 0$ such that, for some $y \in B(\bar{z}, \delta') \cap \Gamma$,

$$\begin{aligned} &(\langle \zeta_1^L, \exp_y^{-1}\bar{z}\rangle, \ldots, \langle \zeta_p^L, \exp_y^{-1}\bar{z}\rangle) \geq \beta \left\|\exp_y^{-1}\bar{z}\right\| e,\ \forall \zeta_j^L \in \partial_c \Psi_j^L(y),\ j \in J, \\ &(\langle \zeta_1^U, \exp_y^{-1}\bar{z}\rangle, \ldots, \langle \zeta_p^U, \exp_y^{-1}\bar{z}\rangle) \geq \beta \left\|\exp_y^{-1}\bar{z}\right\| e,\ \forall \zeta_j^U \in \partial_c \Psi_j^U(y),\ j \in J. \end{aligned} \tag{3.1}$$

Observe that each Ψ_j is geodesic approximately LU-convex at $\bar{z}$, Therefore, in particular, for $\beta > 0$, there exists $\delta_j > 0$ such that, for every $y \in B(\bar{z}, \delta_j) \cap \Gamma$, $\zeta_j^L \in \partial_c \Psi_j^L(y)$, and $\zeta_j^U \in \partial_c \Psi_j^U(y)$,

$$\begin{aligned} \Psi_j^L(\bar{z}) - \Psi_j^L(y) &\geq \langle \zeta_j^L, \exp_y^{-1}\bar{z}\rangle - \beta \left\|\exp_y^{-1}\bar{z}\right\|,\ \forall j \in J, \\ \Psi_j^U(\bar{z}) - \Psi_j^U(y) &\geq \langle \zeta_j^U, \exp_y^{-1}\bar{z}\rangle - \beta \left\|\exp_y^{-1}\bar{z}\right\|,\ \forall j \in J. \end{aligned}$$

Define $\delta'' = min\{\delta_1, \delta_2, ..., \delta_p\}$. Then, for any $y \in B(\bar{z}, \delta'') \cap \Gamma$, $\zeta_j^L \in \partial_c \Psi_j^L(y)$, and $\zeta_j^U \in \partial_c \Psi_j^U(y)$,

$$\begin{aligned} \Psi_j^L(\bar{z}) - \Psi_j^L(y) &\geq \left\langle \zeta_j^L, \exp_y^{-1}\bar{z}\right\rangle - \beta \left\|\exp_y^{-1}\bar{z}\right\|,\ \forall j \in J, \\ \Psi_j^U(\bar{z}) - \Psi_j^U(y) &\geq \left\langle \zeta_j^U, \exp_y^{-1}\bar{z}\right\rangle - \beta \left\|\exp_y^{-1}\bar{z}\right\|,\ \forall j \in J. \end{aligned} \tag{3.2}$$

Set $\delta = \min\{\delta', \delta''\}$. From (3.1) and (3.2), we can find $\delta > 0$ such that, for every $y \in B(\bar{z}, \delta) \cap \Gamma$, it holds $\Psi(y) \prec_{LU} \Psi(\bar{z})$. This contradiction leads to the consequence that $\bar{z}$ is a local LU-efficient solution to (NIVMOP). □

Theorem 3.3. *Let $\Psi_j : \Gamma \to \mathscr{I}$, $j \in J$, be a geodesic LU-α_j-convex function on Γ. Then $\bar{z}$ is an LU-efficient solution to (NIVMOP) if and only if $\bar{z}$ solves (AMVVIP).*

Proof. The proof follows the lines of the proof of Theorems 3.1 and 3.2. □

Remark 3.1. Theorem 3.3 generalizes [33, Theorem 3.1] and [36, Theorem 3.1] for a more general problem, namely (NIVMOP), and for a more general class of functions, namely geodesic LU-α-convex functions.

The following theorem indicates that, under geodesic α-convexity assumption, a solution of (ASVVIP) solves (AMVVIP) too.

Theorem 3.4. *Let each $\Psi_j : \Gamma \to \mathscr{I}$, $j \in J$, be a geodesic LU-α_j-convex function on Γ. If $\bar{z}$ solves (ASVVIP), then $\bar{z}$ is a solution to (AMVVIP).*

Proof. Let $\bar{z}$ be a solution of (ASVVIP). Assume to the contrary that $\bar{z}$ is not a solution of (AMVVIP). Then, for every $\beta > 0$, there exist $y \in \Gamma$, $\zeta_j^L \in \partial_c \Psi_j^L(y)$, and $\zeta_j^U \in \partial_c \Psi_j^U(y)$ such that, for all $j \in J$,

$$\begin{aligned} &(\langle \zeta_1^L, exp_y^{-1}\bar{z}\rangle, \dots, \langle \zeta_p^L, exp_y^{-1}\bar{z}\rangle) \geq \beta \|exp_y^{-1}\bar{z}\| e, \\ &(\langle \zeta_1^U, exp_y^{-1}\bar{z}\rangle, \dots, \langle \zeta_p^U, exp_y^{-1}\bar{z}\rangle) \geq \beta \|exp_y^{-1}\bar{z}\| e. \end{aligned} \tag{3.3}$$

Since Ψ_j^L, Ψ_j^U $j \in J$, are geodesic α_j-convex at $\bar{z}$, then $\partial_c \Psi_j^L$ and $\partial_c \Psi_j^U$ are geodesic $2\alpha_j$-monotone at $\bar{z}$. Therefore, for all $\xi_j^L \in \partial_c \Psi_j^L(\bar{z}), \zeta_j^L \in \partial_c \Psi_j^L(y)$, $\xi_j^U \in \partial_c \Psi_j^U(\bar{z}), \zeta_j^U \in \partial_c \Psi_j^U(y)$, we have

$$\left\langle P_{1,\Omega}^0 \xi_j^L - \zeta_j^L, exp_y^{-1}\bar{z} \right\rangle \geq -2\alpha_j \left\|exp_y^{-1}\bar{z}\right\|, \ \forall j \in J,$$
$$\left\langle P_{1,\Omega}^0 \xi_j^U - \zeta_j^U, exp_y^{-1}\bar{z} \right\rangle \geq -2\alpha_j \left\|exp_y^{-1}\bar{z}\right\|, \ \forall j \in J.$$

In particular, setting $\beta = \max\{\alpha_1, ..., \alpha_p\}$, for all $\xi_j^L \in \partial_c \Psi_j^L(\bar{z})$, $\zeta_j^L \in \partial_c \Psi_j^L(y)$, and $\xi_j^U \in \partial_c \Psi_j^U(\bar{z}), \zeta_j^U \in \partial_c \Psi_j^U(y)$, we obtain

$$\begin{aligned} &\left\langle P_{1,\Omega}^0 \xi_j^L - \zeta_j^L, exp_y^{-1}\bar{z} \right\rangle \geq -2\beta \left\|exp_y^{-1}\bar{z}\right\|, \ \forall j \in J, \\ &\left\langle P_{1,\Omega}^0 \xi_j^U - \zeta_j^U, exp_y^{-1}\bar{z} \right\rangle \geq -2\beta \left\|exp_y^{-1}\bar{z}\right\|, \ \forall j \in J. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we see that

$$\left(\left\langle P_{1,\Omega}^0 \xi_1^L, exp_y^{-1}\bar{z} \right\rangle, \dots, \left\langle P_{1,\Omega}^0 \xi_p^L, exp_y^{-1}\bar{z} \right\rangle\right) \geq -\beta \left\|exp_y^{-1}\bar{z}\right\| e, \ \forall \xi_j^L \in \partial_c \Psi_j^L(\bar{z}), \ j \in J,$$
$$\left(\left\langle P_{1,\Omega}^0 \xi_1^U, exp_y^{-1}\bar{z} \right\rangle, \dots, \left\langle P_{1,\Omega}^0 \xi_p^U, exp_y^{-1}\bar{z} \right\rangle\right) \geq -\beta \left\|exp_y^{-1}\bar{z}\right\| e, \ \forall \xi_j^U \in \partial_c \Psi_j^U(\bar{z}), \ j \in J.$$

Hence,

$$(\left\langle P_{0,\Omega}^1 (P_{1,\Omega}^0 \xi_1^L), P_{0,\Omega}^1 exp_y^{-1}\bar{z} \right\rangle, \dots, \left\langle P_{0,\Omega}^1 (P_{1,\Omega}^0 \xi_p^L), P_{0,\Omega}^1 exp_y^{-1}\bar{z} \right\rangle) \geq -\beta \left\|exp_y^{-1}\bar{z}\right\| e,$$
$$(\left\langle P_{0,\Omega}^1 (P_{1,\Omega}^0 \xi_1^U), P_{0,\Omega}^1 exp_y^{-1}\bar{z} \right\rangle, \dots, \left\langle P_{0,\Omega}^1 (P_{1,\Omega}^0 \xi_p^U), P_{0,\Omega}^1 exp_y^{-1}\bar{z} \right\rangle) \geq -\beta \left\|exp_y^{-1}\bar{z}\right\| e,$$

for all $\xi_j^L \in \partial_c \Psi_j^L(\bar{z})$ and $\xi_j^U \in \partial_c \Psi_j^U(\bar{z})$, $j \in J$, or

$$(\langle \xi_1^L, -exp_{\bar{z}}^{-1} y\rangle, \dots, \langle \xi_p^L, -exp_{\bar{z}}^{-1} y\rangle) \geq -\beta \left\|exp_y^{-1}\bar{z}\right\| e, \ \forall \xi_j^L \in \partial_c \Psi_j^L(\bar{z}), \ j \in J,$$
$$(\langle \xi_1^U, -exp_{\bar{z}}^{-1} y\rangle, \dots, \langle \xi_p^U, -exp_{\bar{z}}^{-1} y\rangle) \geq -\beta \left\|exp_y^{-1}\bar{z}\right\| e, \ \forall \xi_j^U \in \partial_c \Psi_j^U(\bar{z}), \ j \in J,$$

that is,

$$\begin{aligned} &(\langle \xi_1^L, exp_{\bar{z}}^{-1} y\rangle, \dots, \langle \xi_p^L, exp_{\bar{z}}^{-1} y\rangle) \leq \beta \left\|exp_y^{-1}\bar{z}\right\| e, \ \forall \xi_j^L \in \partial_c \Psi_j^L(\bar{z}), \ j \in J, \\ &(\langle \xi_1^U, exp_{\bar{z}}^{-1} y\rangle, \dots, \langle \xi_p^U, exp_{\bar{z}}^{-1} y\rangle) \leq \beta \left\|exp_y^{-1}\bar{z}\right\| e, \ \forall \xi_j^U \in \partial_c \Psi_j^U(\bar{z}), \ j \in J. \end{aligned} \tag{3.5}$$

In view of $\left\|exp_y^{-1}\bar{z}\right\| = \left\|exp_{\bar{z}}^{-1}y\right\|$, (3.5) leads to

$$(\left\langle \xi_1^L, exp_{\bar{z}}^{-1}y\right\rangle, \ldots, \left\langle \xi_p^L, exp_{\bar{z}}^{-1}y\right\rangle) \leq \beta \left\|exp_{\bar{z}}^{-1}y\right\| e, \ \forall \xi_j^L \in \partial_c \Psi_j^L(\bar{z}), \ j \in J,$$
$$(\left\langle \xi_1^U, exp_{\bar{z}}^{-1}y\right\rangle, \ldots, \left\langle \xi_p^U, exp_{\bar{z}}^{-1}y\right\rangle) \leq \beta \left\|exp_{\bar{z}}^{-1}y\right\| e, \ \forall \xi_j^U \in \partial_c \Psi_j^U(\bar{z}), \ j \in J,$$

which contradicts that $\bar{z}$ is a solution of (ASVVIP). □

The following theorem delivers the important result that if the cost function is geodesic approximately LU-convex, then a solution of (ALSVVIP) is also a local LU-efficient solution to (NIVMOP).

Theorem 3.5. *Let $\Psi_j : \Gamma \to \mathscr{I}$, $j \in J$, be geodesic a approximately LU-convex function at $\bar{z} \in \Gamma$. If $\bar{z}$ solves (ALSVVIP), then $\bar{z}$ is a local LU-efficient solution to (NIVMOP). Moreover, $\bar{z}$ is a solution to (ALMVVIP).*

Proof. Since $\bar{z} \in \Gamma$ is a solution to (ALSVVIP), one sees that there exist $\beta > 0$, $\delta' > 0$, $\xi_j^L \in \partial_c \Psi_j^L(\bar{z})$, and $\xi_j^U \in \partial_c \Psi_j^U(\bar{z})$ such that, for all $y \in B(\bar{z}, \delta') \cap \Gamma$,

$$\begin{aligned} &(\left\langle \xi_1^L, exp_{\bar{z}}^{-1}y\right\rangle, \ldots, \left\langle \xi_p^L, exp_{\bar{z}}^{-1}y\right\rangle) \nleq \beta \|exp_{\bar{z}}^{-1}y\| e, \\ &(\left\langle \xi_1^U, exp_{\bar{z}}^{-1}y\right\rangle, \ldots, \left\langle \xi_p^U, exp_{\bar{z}}^{-1}y\right\rangle) \nleq \beta \|exp_{\bar{z}}^{-1}y\| e. \end{aligned} \tag{3.6}$$

Since each $\Psi_j, \ j \in J$ is geodesic approximately LU-convex at $\bar{z}$, one sees that, for $\beta > 0$, there exists $\delta_j > 0$, such that

$$\Psi_j^L(y) - \Psi_j^L(\bar{z}) \geq \left\langle \xi_j^L, exp_{\bar{z}}^{-1}y\right\rangle - \beta \|exp_{\bar{z}}^{-1}y\|, \ \forall j \in J,$$
$$\Psi_j^U(y) - \Psi_j^U(\bar{z}) \geq \left\langle \xi_j^U, exp_{\bar{z}}^{-1}y\right\rangle - \beta \|exp_{\bar{z}}^{-1}y\|, \ \forall j \in J,$$

for all $y \in B(\bar{z}, \delta_j) \cap \Gamma$. Set $\delta'' = min\{\delta_1, \delta_2, ..., \delta_p\}$. For all $y \in B(\bar{z}, \delta'') \cap \Gamma$, we have

$$\begin{aligned} \Psi_j^L(y) - \Psi_j^L(\bar{z}) &\geq \left\langle \xi_j^L, exp_{\bar{z}}^{-1}y\right\rangle - \beta \|exp_{\bar{z}}^{-1}y\|, \ \forall j \in J, \\ \Psi_j^U(y) - \Psi_j^U(\bar{z}) &\geq \left\langle \xi_j^U, exp_{\bar{z}}^{-1}y\right\rangle - \beta \|exp_{\bar{z}}^{-1}y\|, \ \forall j \in J. \end{aligned} \tag{3.7}$$

Let $\delta = \min\{\delta', \delta''\}$. From (3.6) and (3.7), there does not exist any $y \in B(\bar{z}, \delta) \cap \Gamma$ such that $\Psi(y) \prec_{LU} \Psi(\bar{z})$. This means that $\bar{z}$ is a local LU-efficient solution to (NIVMOP). Moreover, it from Theorem 3.2 admits that $\bar{z}$ is a solution to (ALMVVIP). □

Remark 3.2. The converse of Theorem 3.5 does not hold in general. To illustrate the fact, we consider the following interval-valued multiobjective optimization problem on Hadamard manifold:

$$(P) \qquad \min \Psi(z) = (\Psi_1(z), \Psi_2(z)),$$
$$\text{subject to } z \in \Gamma \subseteq H,$$

where $\Psi_1, \Psi_2 : \Gamma \to \mathscr{I}$ are interval-valued functions defined on $\Gamma = \{y : y = e^\lambda, \lambda \in [-1, 1]\}$ and $H = \{z \in \mathbb{R} : z > 0\}$ is the Riemannian manifold with Riemannian metric $g(z) = z^{-2}$ and sectional curvature $\kappa = 0$. It is clear that H is a Hadamard manifold and the set Γ is geodesic convex set.

The tangent plane at any point $z \in H$, denoted by T_zH, equals $\mathbb{R}$. The Riemannian distance function $d : H \times H \to \mathbb{R}$ is given by $d(z, y) = \|exp_z^{-1}y\| = |\ln \frac{z}{y}|$. The geodesic curve $\Omega : \mathbb{R} \to H$ starting from $\Omega(0) = z$ and with tangent unit vector $\Omega' = w \in T_zH$ of Ω at the starting point z is given by $\Omega(\lambda) = exp_z(\lambda w) = ze^{(\frac{w}{z})\lambda}$. The inverse of exponential map for any $z, y \in H$ is given by $exp_z^{-1} y = z\, ln\,(\frac{y}{z})$. Consider the functions $\Psi_1^L, \Psi_1^U, \Psi_2^L, \Psi_2^U : \Gamma \to \mathbb{R}$ given by

$$\Psi_1^L(z) = \begin{cases} z^3 - \frac{3}{8}, & z \geq \frac{1}{2} \\ z^2 - z, & z < \frac{1}{2} \end{cases} \quad \text{and} \quad \Psi_1^U(z) = \begin{cases} z^3 + z - \frac{7}{8}, & z \geq \frac{1}{2} \\ z^2 - 2z + \frac{1}{2}, & z < \frac{1}{2} \end{cases},$$

$$\Psi_2^L(z) = \begin{cases} z^2 + z, & z \geq \frac{1}{2} \\ -2z + \frac{7}{4}, & z < \frac{1}{2} \end{cases} \quad \text{and} \quad \Psi_2^U(z) = \begin{cases} 2z^2 + z, & z \geq \frac{1}{2} \\ z^2 - 2z + \frac{7}{4}, & z < \frac{1}{2} \end{cases}.$$

It is clear that the functions $\Psi_1^L, \Psi_1^U, \Psi_2^L$ and Ψ_2^U are locally Lipschitz continuous on Γ. The subdifferential of $\Psi_1^L, \Psi_1^U, \Psi_2^L, \Psi_2^U$ are given by

$$\partial_c \Psi_1^L(z) = \begin{cases} 3z^4, & z > \frac{1}{2} \\ [0, \frac{3}{16}], & z = \frac{1}{2} \\ 2z^3 - z^2, & z < \frac{1}{2} \end{cases} \quad \text{and} \quad \partial_c \Psi_1^U(z) = \begin{cases} 3z^4 + z^2, & z > \frac{1}{2} \\ [-\frac{1}{4}, \frac{7}{16}], & z = \frac{1}{2} \\ 2z^3 - 2z^2, & z < \frac{1}{2} \end{cases}.$$

Similarly, we have

$$\partial_c \Psi_2^L(z) = \begin{cases} 2z^3 + z^2, & z > \frac{1}{2} \\ [-\frac{1}{2}, \frac{1}{2}], & z = \frac{1}{2} \\ -2z^2, & z < \frac{1}{2} \end{cases} \quad \text{and} \quad \partial_c \Psi_2^U(z) = \begin{cases} 4z^3 + z^2, & z > \frac{1}{2} \\ [-\frac{1}{4}, \frac{3}{4}], & z = \frac{1}{2} \\ 2z^3 - 2z^2, & z < \frac{1}{2} \end{cases}.$$

We can verify that Ψ_1^L and Ψ_1^U are geodesic approximately LU-convex at $\bar{z} = \frac{1}{2}$. As for all $\alpha_1 > 0$, there exists $0 < \delta_1 < \frac{-1+\sqrt{1+1.5\alpha_1}}{3}$ such that condition (2.1) holds. Similarly, Ψ_2^L and Ψ_2^U are geodesic approximately LU-convex at $\bar{z} = \frac{1}{2}$, as for all $\alpha_2 > 0$, there exists $\delta_2 \in (0,1)$, such that the condition (2.1) holds. The point $\bar{z} = \frac{1}{2}$ is a local LU-efficient solution for (P) as for $\delta = \min\{\delta_1, \delta_2\} > 0$. the following does not hold $\Psi(y) \prec_{LU} \Psi(\bar{z})$ for all $y \in B(\bar{z}, \delta) \cap \Gamma$. However, we see that $\bar{z} = \frac{1}{2}$ is not a solution to (ALSVVIP) as there exists $\delta > 0$ and $\beta > 1$ such that

$$\left(\left\langle \xi_1^L, exp_{\bar{z}}^{-1} y\right\rangle, \left\langle \xi_2^L, exp_{\bar{z}}^{-1} y\right\rangle\right) \leq \beta \left\|exp_{\bar{z}}^{-1} y\right\| e,$$
$$\left(\left\langle \xi_1^U, exp_{\bar{z}}^{-1} y\right\rangle, \left\langle \xi_2^U, exp_{\bar{z}}^{-1} y\right\rangle\right) \leq \beta \left\|exp_{\bar{z}}^{-1} y\right\| e,$$

for every $y \in B(\bar{z}, \delta) \cap \Gamma$ and $\xi_j^L \in \partial_c \Psi_j^L(\bar{z})$, $\xi_j^U \in \partial_c \Psi_j^U(\bar{z})$, $j = 1, 2$.

Theorem 3.6. *Let $\Psi_j : \Gamma \to \mathscr{I}$, $j \in J$, be a geodesic LU-α_j-convex function on Γ. If $\bar{z} \in \Gamma$ is a solution to (ASVVIP), then it is an LU-efficient solution to (NIVMOP).*

Proof. The proof is similar to that of Theorem 3.5. □

We formulate the following weak forms of (ALMVVIP), (AMVVIP), (ALSVVIP), and (ASVVIP) in terms of Clarke subdifferentials on a Hadamard manifold:

(WALMVVIP) Find a point $\bar{z} \in \Gamma$ such that, for any $\beta > 0$, there exists $\delta > 0$ such that

$$\left(\left\langle \zeta_1^L, exp_y^{-1} \bar{z}\right\rangle, \ldots, \left\langle \zeta_p^L, exp_y^{-1} \bar{z}\right\rangle\right) \not> \beta \left\|exp_y^{-1} \bar{z}\right\| e,$$
$$\left(\left\langle \zeta_1^U, exp_y^{-1} \bar{z}\right\rangle, \ldots, \left\langle \zeta_p^U, exp_y^{-1} \bar{z}\right\rangle\right) \not> \beta \left\|exp_y^{-1} \bar{z}\right\| e,$$

for all $y \in B(\bar{z}, \delta) \cap \Gamma$, $\zeta_j^L \in \partial_c \Psi_j^L(y)$, and $\zeta_j^U \in \partial_c \Psi_j^U(y)$, $j \in J$;

(WAMVVIP) Find a point $\bar{z} \in \Gamma$ such that

$$\left(\left\langle \zeta_1^L, exp_y^{-1}\bar{z}\right\rangle, \ldots, \left\langle \zeta_p^L, exp_y^{-1}\bar{z}\right\rangle\right) \not> \beta \left\|exp_y^{-1}\bar{z}\right\| e,$$
$$\left(\left\langle \zeta_1^U, exp_y^{-1}\bar{z}\right\rangle, \ldots, \left\langle \zeta_p^U, exp_y^{-1}\bar{z}\right\rangle\right) \not> \beta \left\|exp_y^{-1}\bar{z}\right\| e,$$

for any $\beta > 0$, $y \in \Gamma$, $\zeta_j^L \in \partial_c \Psi_j^L(y)$, and $\zeta_j^U \in \partial_c \Psi_j^U(y)$, $j \in J$;

(WALSVVIP) Find a point $\bar{z} \in \Gamma$ such that there exist $\beta > 0$, $\delta > 0$, $\xi_j^L \in \partial_c \Psi_j^L(\bar{z})$, and $\xi_j^U \in \partial_c \Psi_j^U(\bar{z})$, satisfying

$$\left(\left\langle \xi_1^L, exp_{\bar{z}}^{-1}y\right\rangle, \ldots, \left\langle \xi_p^L, exp_{\bar{z}}^{-1}y\right\rangle\right) \not< \beta \left\|exp_{\bar{z}}^{-1}y\right\| e,$$
$$\left(\left\langle \xi_1^U, exp_{\bar{z}}^{-1}y\right\rangle, \ldots, \left\langle \xi_p^U, exp_{\bar{z}}^{-1}y\right\rangle\right) \not< \beta \left\|exp_{\bar{z}}^{-1}y\right\| e,$$

for all $y \in B(\bar{z}, \delta) \cap \Gamma$,;

(WASVVIP) Find a point $\bar{z} \in \Gamma$ such that there exist $\beta > 0$, $\xi_j^L \in \partial_c \Psi_j^L(\bar{z})$, and $\xi_j^U \in \partial_c \Psi_j^U(\bar{z})$ satisfying

$$\left(\left\langle \xi_1^L, exp_{\bar{z}}^{-1}y\right\rangle, \ldots, \left\langle \xi_p^L, exp_{\bar{z}}^{-1}y\right\rangle\right) \not< \beta \left\|exp_{\bar{z}}^{-1}y\right\| e,$$
$$\left(\left\langle \xi_1^U, exp_{\bar{z}}^{-1}y\right\rangle, \ldots, \left\langle \xi_p^U, exp_{\bar{z}}^{-1}y\right\rangle\right) \not< \beta \left\|exp_{\bar{z}}^{-1}y\right\| e,$$

for any $y \in \Gamma$, where $e = \underbrace{(1, 1, ..., 1)}_{p\,\text{times}}$.

Remark 3.3. If $\beta = 0$, then (WAMVVIP) and (WASVVIP) reduce to the Minty and Stampacchia vector variational inequalities (WMVVIP) and (WSVVIP), respectively, which were considered by Chen and Huang [33].

For weakly LU-efficient solutions of (NIVMOP), we have the following results.

Theorem 3.7. *Let $\Psi_j^L, \Psi_j^U : \Gamma \to \mathbb{R}$, $j \in J$, be geodesic strictly approximately convex functions on Γ. If $\bar{z}$ is a solution to (WALSVVIP), then it is also a local weakly LU-efficient solution to (NIVMOP).*

Proof. Suppose that $\bar{z} \in \Gamma$ is a solution to (WALSVVIP). Assume that $\bar{z} \in \Gamma$ is not a local weakly LU-efficient solution to (NIVMOP). Then, for all $\delta > 0$, there exists $y \in B(\bar{z}, \delta) \cap \Gamma$ such that $\Psi_j(y) \prec_{LU} \Psi_j(\bar{z})$ for all $j \in J$. Observe that Ψ_j^L, and Ψ_j^U, $j \in J$, are geodesic strictly approximately convex at $\bar{z}$. From Theorem 2.1 for any $\alpha_j > 0$, we find that $\delta > 0$ such that, for all $y \in B(\bar{z}, \delta) \cap \Gamma$, the inequalities are valid

$$\langle \xi_j^L, exp_{\bar{z}}^{-1}y\rangle - \alpha_j \left\|exp_{\bar{z}}^{-1}y\right\| < 0,\ \forall \xi_j^L \in \partial_c \Psi_j^L(\bar{z}),\ j \in J,$$
$$\langle \xi_j^U, exp_{\bar{z}}^{-1}y\rangle - \alpha_j \left\|exp_{\bar{z}}^{-1}y\right\| < 0,\ \forall \xi_j^U \in \partial_c \Psi_j^U(\bar{z}),\ j \in J.$$

In particular, setting $\beta = \max\left\{\alpha_1, \ldots, \alpha_p\right\}$, we can obtain some $y \in B(\bar{z}, \delta) \cap \Gamma$ such that

$$\left(\langle \xi_1^L, exp_{\bar{z}}^{-1}y\rangle, \ldots, \langle \xi_p^L, exp_{\bar{z}}^{-1}y\rangle\right) < \beta \left\|exp_{\bar{z}}^{-1}y\right\| e,\ \forall \xi_j^L \in \partial_c \Psi_j^L(\bar{z}),\ j \in J,$$
$$\left(\langle \xi_1^U, exp_{\bar{z}}^{-1}y\rangle, \ldots, \langle \xi_p^U, exp_{\bar{z}}^{-1}y\rangle\right) < \beta \left\|exp_{\bar{z}}^{-1}y\right\| e,\ \forall \xi_j^U \in \partial_c \Psi_j^U(\bar{z}),\ j \in J.$$

This contradicts the fact that $\bar{z}$ is a solution of (WALSVVIP). □

Theorem 3.8. *Let $\Psi_j^L, \Psi_j^U : \Gamma \to \mathbb{R}$, $j \in J$, be geodesic strictly α_j-convex functions on Γ. If $\bar{z}$ is a solution to (WASVVIP), then it is also a weakly LU-efficient solution to (NIVMOP).*

Proof. The proof is similar to that of Theorem 3.7. □

Theorem 3.9. *Let $\Psi_j : \Gamma \to \mathscr{I}$, $j \in J$, be a geodesic approximately LU-convex function at $\bar{z} \in \Gamma$. If $\bar{z} \in \Gamma$ is a local weakly LU-efficient solution to (NIVMOP), then $\bar{z}$ is also a solution to (WALMVVIP).*

Proof. Assume that $\bar{z}$ is a local weakly LU-efficient solution to (NIVMOP). Suppose to the contrary that $\bar{z}$ is not a solution of (WALMVVIP). Hence, for any β and all $\delta > 0$, there exists $y \in B(\bar{z}, \delta) \cap \Gamma$ such that

$$\left(\langle \zeta_1^L, exp_y^{-1}\bar{z}\rangle, \dots, \langle \zeta_p^L, exp_y^{-1}\bar{z}\rangle\right) > \beta \left\|exp_y^{-1}\bar{z}\right\| e,$$
$$\left(\langle \zeta_1^U, exp_y^{-1}\bar{z}\rangle, \dots, \langle \zeta_p^U, exp_y^{-1}\bar{z}\rangle\right) > \beta \left\|exp_y^{-1}\bar{z}\right\| e,$$

for all $\zeta_j^L \in \partial_c \Psi_j^L(y)$ and $\zeta_j^U \in \partial_c \Psi_j^U(y)$. This demonstrates that

$$\begin{aligned} &\langle \zeta_j^L, exp_y^{-1}\bar{z}\rangle - \beta \left\|exp_y^{-1}\bar{z}\right\| > 0,\ \forall \zeta_j^L \in \partial_c \Psi_j^L(y),\ j \in J,\\ &\langle \zeta_j^U, exp_y^{-1}\bar{z}\rangle - \beta \left\|exp_y^{-1}\bar{z}\right\| > 0,\ \forall \zeta_j^U \in \partial_c \Psi_j^U(y),\ j \in J. \end{aligned} \tag{3.8}$$

Recall that Ψ_j is geodesic approximately LU-convex at $\bar{z}$. From Definition 2.6, for any $\alpha_j > 0$, there exists $\delta > 0$ such that, for all $y \in B(\bar{z}, \delta) \cap \Gamma$, $\zeta_j^L \in \partial_c \Psi_j^L(y)$, and $\zeta_j^U \in \partial_c \Psi_j^U(y)$,

$$\langle \zeta_j^L, exp_y^{-1}\bar{z}\rangle - \alpha_j \left\|exp_y^{-1}\bar{z}\right\| \leq \Psi_j^L(\bar{z}) - \Psi_j^L(y),\ \forall j \in J,$$
$$\langle \zeta_j^U, exp_y^{-1}\bar{z}\rangle - \alpha_j \left\|exp_y^{-1}\bar{z}\right\| \leq \Psi_j^U(\bar{z}) - \Psi_j^U(y),\ \forall j \in J.$$

Let $\beta = \max\{\alpha_1, ..., \alpha_p\}$. Hence,

$$\begin{aligned} &\langle \zeta_j^L, exp_y^{-1}\bar{z}\rangle - \beta \left\|exp_y^{-1}\bar{z}\right\| \leq \Psi_j^L(\bar{z}) - \Psi_j^L(y),\ \forall j \in J,\\ &\langle \zeta_j^U, exp_y^{-1}\bar{z}\rangle - \beta \left\|exp_y^{-1}\bar{z}\right\| \leq \Psi_j^U(\bar{z}) - \Psi_j^U(y),\ \forall j \in J. \end{aligned} \tag{3.9}$$

By using (3.8) and (3.9), we find $y \in B(\bar{z}, \delta) \cap \Gamma$ such that $\Psi_j(y) \prec_{LU} \Psi_j(\bar{z})$ for all $j \in J$. This contradicts the fact that $\bar{z}$ is a local weakly LU-efficient solution of (NIVMOP). □

Theorem 3.10. *Let $\Psi_j : \Gamma \to \mathbb{R}$, $j \in J$, be a geodesic LU-α_j-convex function on Γ. If $\bar{z} \in \Gamma$ is a weakly LU-efficient solution to (NIVMOP), then $\bar{z}$ is also a solution to (WAMVVIP).*

Proof. The proof is similar to that of Theorem 3.9. □

Remark 3.4. Theorem 3.10 generalizes Chen and Huang [33, Theorem 3.5] and Jayswal et al. [36, Theorem 3.7] for a more general problem, namely (NIVMOP) and for a more general class of functions, namely the geodesic LU-α-convex function.

From Theorems 3.7 and 3.9, we have the following relationship between the solution sets of (WALSVVIP) and (WALMVVIP).

Theorem 3.11. *Let $\Psi_j^L, \Psi_j^U : \Gamma \to \mathbb{R}$, $j \in J$, be a geodesic strictly approximately convex function at $\bar{z} \in \Gamma$. If $\bar{z}$ solves (WALSVVIP), then $\bar{z}$ is a solution to (WALMVVIP).*

From Theorems 3.8 and 3.10, we have the following relationship between the solution of (WASVVIP) and (WAMVVIP).

Theorem 3.12. *Let Ψ_j^L, $\Psi_j^U : \Gamma \to \mathbb{R}$, $j \in J$, be a geodesic strictly α_j-convex function on Γ. If $\bar{z}$ solves (WASVVIP), then $\bar{z}$ is a solution to (WAMVVIP).*

4. Conclusions and the Future Research

In this paper, we considered the classes of approximate Minty and Stampacchia type vector variational inequalities (ALMVVIP), (AMVVIP), (ASVVIP), and (ALSVVIP) with their weaker forms, namely (WALMVVIP), (WAMVVIP), (WALSVVIP), and (WASVVIP). Under the assumption that the cost function is geodesic approximate LU-convex or geodesic LU-α-convex, we established the relations between the solutions of considered approximate variational inequalities (ALMVVIP), (AMVVIP), (ALSVVIP), and (ASVVIP), and LU-efficient solutions of nonsmooth interval-valued multiobjective optimization problem (NIVMOP). Furthermore, we also derived the equivalence among the solutions of the weak versions of considered approximate vector variational inequalities (WALMVVIP), (WAMVVIP), (WALSVVIP), (WASVVIP), and weakly LU-efficient solutions of considered nonsmooth interval-valued multiobjective problem (NIVMOP). The results established in this paper extend and generalize some earlier results of Giannessi [27], Lee and Lee [28], Osuna-Gomez et al. [42], and Yang [43] to the nonsmooth case as well as to a more general class of functions and the works of Chen and Huang [33], Chen and Fang [34], Jayswal et al. [36] and Upadhyay et al. [37] to a more general problem, that is, the nonsmooth interval-valued multiobjective optimization problem.

Acknowledgements
This project received the funding from the Natural Science Foundation of Guangxi Grant Nos. 2022AC21071 and 2021GXNSFFA196004, the NNSF of China Grant No. 12001478, and the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie grant agreement No. 823731 CONMECH. The First author was supported by Science and Engineering Research Board (SERB), Government of India, under Mathematical Research Impact Centric Support (MATRICS) scheme, through Grant No. MTR/2022/000925.

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