Appl. Set-Valued Anal. Optim. 5 (2023), No. 1, pp. 121-135
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.5.2023.1.08

CONNECTEDNESS PROPERTIES OF EFFICIENT AND MINIMAL SETS TO VECTOR OPTIMIZATION PROBLEMS

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Abstract. In this paper, we investigate the approximate efficient solution and minimal sets of vector optimization problems. Firstly, we propose a new scalar function through the Gerstewitz function and discuss some of its properties. Secondly, we apply this scalar function to address the connectedness of the approximate efficient solution sets. Finally, the connectedness of the minimal sets is studied based on the above results.
Keywords. Connectedness; Efficient and minimal sets; Scalarization method; Vector optimization problems.

2020 Mathematics Subject Classification. 49K40, 90C31.

1. INTRODUCTION

Vector optimization is one of the most developed areas of Applied Mathematics. It has great influence on many fields, such as Economics, Engineering, Medicine, etc. The structure of solution sets is the most common topic for optimization models in which allowing to move continuously between one optimal solution to another along optimal alternatives plays a vital role [1, 2]. This possibility is assured if the efficient set is arcwise connected or at least connected, and in the theory of differential equations, it is also known as the Kneser condition; see, e.g., [3, 4]. Therefore, connectedness conditions in optimization have received an increasing attention from many researchers recently such as for vector optimization problems [5, 6, 7], for vector equilibrium problems [8, 9, 10, 11], for vector variational inequalities [12, 13, 14], and for many different types of minimal solution sets in the class of set optimization problems [15, 16].

Now let us present a brief overview of connectedness conditions for vector optimization problems (VOP). In [17], Gong used the convexity conditions of constraint sets and objective

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Received January 22, 2023; Accepted March 13, 2023

functions to study the connectedness for nondominate sets and efficient sets of vector optimization problems. Then, motivated by this work, Sun [18] replaced the convex conditions of objective functions by strict quasiconvexity conditions to discuss sufficient conditions of the connectedness for the efficient sets of multiobjective optimization problems. After that, in [5], by applying the linear scalar method, Han and Huang generalized the results of [18] to (VOP) in normed spaces, which, in company with nonlinear scalars is one of the best methods to study (VOP). Later on, Han and Huang [16] used the strictly natural quasiconvexity, a slightly reduced version of the strictly quasiconvexity, to consider the connectedness of efficient sets of (VOP) in normed spaces via linear scalarization method. Recently, Anh et al. [7] introduced concepts related to generalized convexlikeness of a vector mapping and a nonlinear scalarization function concerning the Hiriart-Urruty oriented distance, and then by employing these results, the authors formulated connectedness conditions of efficient and nondominate sets of nonconvex vector optimization problems.

Motivated by above observations, in this paper, we aim to study the connectedness of the approximate efficient solution and approximate minimal sets to vector optimization problems without using any convexity conditions. Precisely, based on the Gerstewitz function, we introduce a nonlinear scalar function and discuss its properties, including generalized convexity, extended connectedness and continuity. Then, by using these results we investigate connectedness conditions for approximate efficient solution sets of nonconvex vector optimization problems via the nonlinear scalarization function. Finally, we use the above results to build the connectedness for approximate minimal sets to such problems.

2. Preliminaries

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces, and let $\mathscr{C}$ be a pointed, closed, and convex cone with nonempty interior ($\text{int}\mathscr{C} \neq \emptyset$) in $\mathbb{Y}$. We first recall some notions needed in the sequel.

Definition 2.1. Let $\mathscr{X}$ be a nonempty subset of $\mathbb{X}$.

(a) [19, page 10]) For each $x_1, x_2 \in \mathbb{X}$, the set $\mathscr{S}_{x_1,x_2} := \bigcup_{t\in[0,1]} \mathscr{L}_{x_1,x_2}(t)$ is called *a line segment* between x_1 and x_2, where $\mathscr{L}_{x_1,x_2}(t) = (1-t)x_1 + tx_2$. Then, $\mathscr{X}$ is said to be *convex* if $\mathscr{S}_{x_1,x_2} \subset \mathscr{X}$ for all $x_1, x_2 \in \mathscr{X}$.

(b) [20, Definition 2.1] For each pair of given points $x_1, x_2 \in \mathbb{X}$, let $\mathscr{A}_{x_1,x_2} : [0,1] \to \mathbb{X}$ be a continuous vector-valued mapping such that $\mathscr{A}_{x_1,x_2}(0) = x_1$ and $\mathscr{A}_{x_1,x_2}(1) = x_2$. Then, $\mathscr{A}_{x_1,x_2}$ is called *an arc* on $\mathbb{X}$ with endpoints x_1, x_2. The set $\mathscr{X}$ is said to be *arcwise connected* if, for each pair of points x_1, x_2 in $\mathscr{X}$, there is an arc $\mathscr{A}_{x_1,x_2}$ on $\mathscr{X}$.

(c) [21, page 540] The set $\mathscr{X}$ is said to be *separated* if there are two open subsets $\mathscr{U}, \mathscr{V}$ of $\mathbb{X}$ such that $\mathscr{X} \cap \mathscr{U} \neq \emptyset$, $\mathscr{X} \cap \mathscr{V} \neq \emptyset$, $\mathscr{U} \cap \mathscr{V} = \emptyset$ and $\mathscr{X} \subset \mathscr{U} \cup \mathscr{V}$. The set $\mathscr{X}$ is said to be *connected* if it is not separated.

Definition 2.2. A vector-valued mapping $f : \mathbb{X} \to \mathbb{Y}$ is said to be

(a) [22, Definition 6.1] *segmented $\mathscr{C}$-convex ($\mathscr{C}$-convex)* on a convex subset $\mathscr{X}$ of $\mathbb{X}$ if, for $x_1, x_2 \in \mathscr{X}$ and $t \in [0,1]$,

$$f(\mathscr{L}_{x_1,x_2}(t)) \in (1-t)f(x_1) + tf(x_2) - \mathscr{C};$$

(b) [20, Definition 2.2] *arcwise connected $\mathscr{C}$-convex* on an arcwise connected subset $\mathscr{X}$ of $\mathbb{X}$ if, for $x_1, x_2 \in \mathscr{X}$, there exists an arc $\mathscr{A}_{x_1,x_2}$ on $\mathscr{X}$ such that, for all $t \in [0,1]$,

$$f(\mathscr{A}_{x_1,x_2}(t)) \in (1-t)f(x_1) + tf(x_2) - \mathscr{C};$$

(c) [11, Definition 3.2] *connected $\mathscr{C}$-convex* on a connected subset $\mathscr{X}$ of $\mathbb{X}$ if, for $x_1, x_2 \in \mathscr{X}$, there exists a connected set $\mathscr{K}_{x_1,x_2} \subset \mathscr{X}$ containing x_1, x_2 such that the set

$$\bigcup_{t\in[0,1]} \{x \in \mathscr{K}_{x_1,x_2} : f(x) \in (1-t)f(x_1) + tf(x_2) - \mathscr{C}\}$$

is connected.

Definition 2.3. A vector-valued mapping $f : \mathbb{X} \to \mathbb{Y}$ is said to be

(a) [23, Definition 2.1] *naturally quasisegmented $\mathscr{C}$-convex (naturally $\mathscr{C}$-quasiconvex)* on a convex subset $\mathscr{X}$ of $\mathbb{X}$ if, for $x_1, x_2 \in \mathscr{X}$ and $t \in [0,1]$, there exists $s \in [0,1]$ such that

$$f(\mathscr{L}_{x_1,x_2}(t)) \in (1-s)f(x_1) + sf(x_2) - \mathscr{C};$$

(b) [11, Definition 3.3] *naturally quasiarcwise connected $\mathscr{C}$-convex* on an arcwise connected subset $\mathscr{X}$ of $\mathbb{X}$ if, for $x_1, x_2 \in \mathscr{X}$, there exists an arc $\mathscr{A}_{x_1,x_2}$ on $\mathscr{X}$ such that, for each $t \in [0,1]$, we can find some $s \in [0,1]$,

$$f(\mathscr{A}_{x_1,x_2}(t)) \in (1-s)f(x_1) + sf(x_2) - \mathscr{C};$$

(c) [11, Definition 3.3] *naturally quasiconnected $\mathscr{C}$-convex* on a connected subset $\mathscr{X}$ of $\mathbb{X}$ if, for $x_1, x_2 \in \mathscr{X}$, there exists a connected subset $\mathscr{K}_{x_1,x_2} \subset \mathscr{X}$ containing x_1, x_2 such that for all $x \in \mathscr{K}_{x_1,x_2}$, we can find some $s \in [0,1]$,

$$f(x) \in (1-s)f(x_1) + sf(x_2) - \mathscr{C}.$$

Remark 2.1. In view of Definitions 2.2 and 2.3, the following statements hold true.

(a) If f is segmented $\mathscr{C}$-convex (arcwise connected $\mathscr{C}$-convex, respectively), then f is arcwise connected $\mathscr{C}$-convex (connected $\mathscr{C}$-convex, respectively).

(b) If f is naturally quasisegmented $\mathscr{C}$-convex (naturally quasiarcwise connected $\mathscr{C}$-convex, respectively), then f is naturally quasiarcwise connected $\mathscr{C}$-convex (naturally quasiconnected $\mathscr{C}$-convex, respectively).

(c) If f is is segmented $\mathscr{C}$-convex (arcwise connected $\mathscr{C}$-convex, connected $\mathscr{C}$-convex, respectively), then f is naturally quasisegmented $\mathscr{C}$-convex (naturally quasiarcwise connected $\mathscr{C}$-convex, naturally quasiconnected $\mathscr{C}$-convex, respectively).

Remark 2.2. (a) A function f is segmented $\mathscr{C}$-concave (arcwise connected $\mathscr{C}$-concave, connected $\mathscr{C}$-concave, respectively) if $-f$ is segmented $\mathscr{C}$-convex (arcwise connected $\mathscr{C}$-convex, connected $\mathscr{C}$-convex, respectively).

(b) A function f is naturally quasisegmented $\mathscr{C}$-concave (naturally quasiarcwise connected $\mathscr{C}$-concave, naturally quasiconnected $\mathscr{C}$-concave, respectively) if $-f$ is naturally quasisegmented $\mathscr{C}$-convex (naturally quasiarcwise connected $\mathscr{C}$-convex, naturally quasiconnected $\mathscr{C}$-convex, respectively).

Definition 2.4. [24, Definition 2.5.1] A set-valued mapping $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be

(a) *upper semicontinuous* (usc) at $x_0 \in \mathbb{X}$ if, for any neighborhood $\mathscr{V}$ of $F(x_0)$, there is some neighborhood $\mathscr{U}$ of x_0 such that $F(\mathscr{U}) \subset \mathscr{V}$;

(b) *lower semicontinuous* (lsc) at $x_0 \in \mathbb{X}$ if, for any open subset $\mathscr{V}$ of $\mathbb{Y}$ with $F(x_0) \cap \mathscr{V} \neq \emptyset$, there is some neighborhood $\mathscr{U}$ of x_0 such that $F(x) \cap \mathscr{V} \neq \emptyset$ for all $x \in \mathscr{U}$;
(c) *continuous* at $x_0 \in \mathbb{X}$ if it is both usc and lsc at x_0.

Definition 2.5. [22, Definition 5.1] A vector-valued mapping $f : \mathbb{X} \to \mathbb{Y}$ is said to be
(a) $\mathscr{C}$*-lower semicontinuous* ($\mathscr{C}$-lsc) at $x_0 \in \mathbb{X}$ if, for any neighborhood $\mathscr{V}$ of $f(x_0)$, there exists some neighborhood $\mathscr{U}$ of x_0 such that $f(x) \in \mathscr{V} + \mathscr{C}$ for all $x \in \mathscr{U}$;
(b) $\mathscr{C}$*-upper semicontinuous* ($\mathscr{C}$-usc) at $x_0 \in \mathbb{X}$ if $-f$ is $\mathscr{C}$-lsc at x_0;
(c) $\mathscr{C}$*-continuous* at $x_0 \in \mathbb{X}$ if it is both $\mathscr{C}$-usc and $\mathscr{C}$-lsc at x_0.

In the following, a mapping is considered to satisfy a given property on $\mathscr{X}$ if it holds this property at every point of $\mathscr{X}$, and if $\mathscr{X} = \mathbb{X}$, then we omit "on $\mathbb{X}$" in the statement.

Lemma 2.1. [2, Theorem 3.1] *Assume that $\mathscr{X}$ is a connected subset of $\mathbb{X}$, and a set-valued mapping $W : \mathbb{X} \rightrightarrows \mathbb{Y}$ is lower semicontinuous (upper semicontinuous) with connected values on $\mathscr{X}$. Then, $W(\mathscr{X})$ is connected.*

Lemma 2.2. [1, Theorem 4.3.2] *Let $f : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. Then, $f(\mathscr{X})$ is connected if $\mathscr{X}$ is a nonempty connected subset of $\mathbb{X}$.*

In order to finalize this section, we consider the following result played an important role in our analysis.

Lemma 2.3. *Let $\mathscr{A}, \mathscr{B}$ be nonempty subsets of $\mathbb{X}$ with $\mathscr{A} \subset \mathscr{B} \subset \operatorname{cl}\mathscr{A}$. Then, $\mathscr{B}$ is connected if $\mathscr{A}$ is connected.*

Proof. Suppose on the contrary that $\mathscr{B}$ is not connected. Then, there exist open subsets $\mathscr{U}, \mathscr{V}$ of $\mathbb{X}$ such that

$$\mathscr{U} \cap \mathscr{B} \neq \emptyset, \quad \mathscr{V} \cap \mathscr{B} \neq \emptyset, \tag{2.1}$$

$$\mathscr{U} \cap \mathscr{V} \cap \mathscr{B} = \emptyset, \quad \mathscr{U} \cup \mathscr{V} \supset \mathscr{B}. \tag{2.2}$$

If $\mathscr{U} \cap \mathscr{A} = \emptyset$ then, by $\mathscr{B} \subset \operatorname{cl}\mathscr{A} \subset \mathbb{X} \setminus \mathscr{U}$, we obtain $\mathscr{U} \cap \mathscr{B} = \emptyset$ which contradicts (2.1). Consequently

$$\mathscr{U} \cap \mathscr{A} \neq \emptyset. \tag{2.3}$$

By the same arguments, we also have

$$\mathscr{V} \cap \mathscr{A} \neq \emptyset. \tag{2.4}$$

From (2.2) and the assumption $\mathscr{A} \subset \mathscr{B}$, one has

$$\mathscr{U} \cap \mathscr{V} \cap \mathscr{A} = \emptyset, \quad \mathscr{U} \cup \mathscr{V} \supset \mathscr{A}.$$

This together with (2.3) and (2.4) yields that $\mathscr{A}$ is not connected, which is absurd. Hence, $\mathscr{B}$ is connected. □

3. A Nonlinear Scalarization Function

Definition 3.1. [24, Equation 2.23] Let e be given in $\operatorname{int}\mathscr{C}$. The nonlinear scalarization function $\varphi_{e,\mathscr{C}} : \mathbb{Y} \to \mathbb{R}$ is defined by

$$\varphi_{e,\mathscr{C}}(y) := \inf\{\lambda \in \mathbb{R} : y \in \lambda e - \mathscr{C}\}, \quad \forall y \in \mathbb{Y}.$$

We now recall important properties of the above function as follows.

Lemma 3.1. [24, Theorem 2.3.1] *The following statements hold true.*

(a) $\varphi_{e,\mathscr{C}}$ *is continuous*;
(b) $\varphi_{e,\mathscr{C}}$ *is convex if* $\mathscr{C}$ *is convex*;
(c) *for all* $\lambda \in \mathbb{R}$, $\varphi_{e,\mathscr{C}}(y) < \lambda$ *if and only if* $y \in \lambda e - \operatorname{int}\mathscr{C}$;
(d) *for all* $\lambda \in \mathbb{R}$, $\varphi_{e,\mathscr{C}}(y) = \lambda$ *if and only if* $y \in \lambda e - \operatorname{bd}\mathscr{C}$;
(e) *for all* $\lambda \in \mathbb{R}, y \in \mathbb{Y}$, $\varphi_{e,\mathscr{C}}(y+\lambda e) = \varphi_{e,\mathscr{C}}(y) + \lambda$,

where $\operatorname{int}\mathscr{C}$ *and* $\operatorname{bd}\mathscr{C}$ *stand for the interior and the boundary of* $\mathscr{C}$, *respectively.*

Lemma 3.2. [24, Theorem 2.3.1] *Let* $y_1, y_2 \in \mathbb{Y}$. *Then,*

(a) $y_1 \in y_2 - \mathscr{C}$ *implies that* $\varphi_{e,\mathscr{C}}(y_1) \leq \varphi_{e,\mathscr{C}}(y_2)$;
(b) $y_1 \in y_2 - \operatorname{int}\mathscr{C}$ *implies that* $\varphi_{e,\mathscr{C}}(y_1) < \varphi_{e,\mathscr{C}}(y_2)$.

In the remaining part of this section, based on the Gerstewitz function [25], a new nonlinear scalarization function along with its properties is introduced and discussed, which are used to scalarize vector optimization problems in the next section.

Let $f : \mathbb{X} \to \mathbb{Y}$ be a vector-valued mapping. We consider the function $\xi : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ defined by

$$\xi(z,x) := \varphi_{e,\mathscr{C}}(f(z) - f(x)), \quad \forall x, z \in \mathbb{X}. \tag{3.1}$$

Example 3.1. Let $\mathbb{X} = \mathbb{R}, \mathbb{Y} = \mathbb{R}^2, \mathscr{X} =]0,1[, \mathscr{C} = \mathbb{R}^2_+, e = (1,1)$ and $f(x) = (x, x^2)$. Then, f is $\mathbb{R}^2_+$-convex. Indeed, for all $x_1, x_2 \in \mathscr{X}$ and $t \in [0,1]$, one has

$$\begin{aligned} f((1-t)x_1 + tx_2) &= ((1-t)x_1 + tx_2, [(1-t)x_1 + tx_2]^2) \\ &\in (1-t)(x_1, x_1^2) + t(x_2, x_2^2) - \mathbb{R}^2_+ \\ &\in (1-t)f(x_1) + tf(x_2) - \mathbb{R}^2_+. \end{aligned}$$

Taking arbitrarily $x, z \in \mathscr{X}$ with $f(z) - f(x) \in \lambda e - \mathbb{R}^2_+$, we have

$$\begin{cases} z - x \leq \lambda, \\ z^2 - x^2 \leq \lambda. \end{cases} \tag{3.2}$$

Combining (3.1) and (3.2), we gain

$$\begin{aligned} \xi(z,x) &= \varphi_{e,\mathscr{C}}(f(z) - f(x)) \\ &= \inf\{\lambda \in \mathbb{R} : f(z) - f(x) \in \lambda e - \mathbb{R}^2_+\} \\ &= \max\{z - x, z^2 - x^2\} \\ &= \begin{cases} z - x, & \text{if } 0 < z + x \leq 1, z \geq x, \\ z - x, & \text{if } 1 \leq z + x < 2, z \leq x, \\ z^2 - x^2, & \text{if } 0 < z + x \leq 1, z \leq x, \\ z^2 - x^2, & \text{if } 1 \leq z + x < 2, z \geq x. \end{cases} \end{aligned}$$

Then, ξ is twice differentiable with non-negative in the first variable on $\mathscr{X}$, and hence $\xi(\cdot, x)$ is segmented $\mathbb{R}_+$-convex on $\mathscr{X}$.

Example 3.2. Let $\mathbb{X} = \mathscr{X} = \mathbb{Y} = \mathbb{R}, \mathscr{C} = \mathbb{R}_+, e = 1$, and $f(x) = x^3$. Clearly, f is not segmented $\mathbb{R}_+$-convex on $\mathbb{R}$, but it is arcwise connected $\mathscr{C}$-convex. Indeed, for every $z_1, z_2 \in \mathscr{X}$, we consider an arc $\mathscr{A}_{z_1,z_2}$on $\mathscr{X}$ defined by

$$\mathscr{A}_{z_1,z_2}(t) = \sqrt[3]{(1-t)z_1^3 + tz_2^3}, \quad \forall t \in [0,1].$$

Then, for all $t \in [0,1]$, $f(\mathscr{A}_{z_1,z_2}(t)) = (1-t)z_1^3 + tz_2^3 = (1-t)f(z_1) + tf(z_2)$, and consequently f is arcwise connected $\mathbb{R}_+$-convex on $\mathscr{X}$.

Now we demonstrate that $\xi(\cdot, x)$ is arcwise connected $\mathbb{R}_+$-convex. If $f(\mathscr{A}_{z_1,z_2}(t)) - f(x) \in \lambda e - \mathbb{R}_+$ for some $x \in \mathscr{X}$, $t \in [0,1]$ and $\lambda \in \mathbb{R}$, then $(1-t)z_1^3 + tz_2^3 - x^3 \leq \lambda$. Consequently,

$$\begin{aligned}
\xi(\mathscr{A}_{z_1,z_2}(t), x) &= \varphi_{e,\mathscr{C}}(f(\mathscr{A}_{z_1,z_2}(t)) - f(x)) \\
&= \inf\{\lambda \in \mathbb{R} : f(\mathscr{A}_{z_1,z_2}(t)) - f(x) \in \lambda e - \mathbb{R}_+\} \\
&= (1-t)z_1^3 + tz_2^3 - x^3 \\
&= (1-t)(z_1^3 - x^3) + t(z_2^3 - x^3) \\
&= (1-t)\xi(z_1, x) + t\xi(z_2, x).
\end{aligned}$$

Thus ξ is arcwise connected $\mathbb{R}_+$-convex in the first variable on $\mathscr{X}$.

The following results give us sufficient conditions for convexity and connectedness properties of the function ξ in the first variable discussed as in above examples.

Lemma 3.3. *Let $f : \mathbb{X} \to \mathbb{Y}$ be a vector valued mapping and $\mathscr{X}$ be a nonempty and convex subset of $\mathbb{X}$. Then,*

(a) *ξ is segmented $\mathbb{R}_+$-convex in the first variable on $\mathscr{X}$ if f is segmented $\mathscr{C}$-convex on $\mathscr{X}$;*
(b) *ξ is naturally quasisegmented $\mathbb{R}_+$-convex in the first variable on $\mathscr{X}$ if f is naturally quasisegmented $\mathscr{C}$-convex on $\mathscr{X}$.*

Proof. We prove only Statement (a), and the other case can be discussed similarly. Since f is segmented $\mathscr{C}$-convex on $\mathscr{X}$, for all $z_1, z_2 \in \mathscr{X}$ and $t \in [0,1]$, one has

$$f((1-t)z_1 + tz_2) \in (1-t)f(z_1) + tf(z_2) - \mathscr{C}.$$

Consequently,

$$f((1-t)z_1 + tz_2) - f(x) \in (1-t)[f(z_1) - f(x)] + t[f(z_2) - f(x)] - \mathscr{C}, \quad \forall x \in \mathscr{X}.$$

This together with Lemmas 3.1 (b) and 3.2 (a) imply that

$$\begin{aligned}
\xi((1-t)z_1 + tz_2, x) &\leq \varphi_{e,\mathscr{C}}((1-t)[f(z_1) - f(x)] + t[f(z_2) - f(x)]) \\
&\leq (1-t)\varphi_{e,\mathscr{C}}(f(z_1) - f(x)) + t\varphi_{e,\mathscr{C}}(f(z_2) - f(x)) \\
&\leq (1-t)\xi(z_1, x) + t\xi(z_2, x).
\end{aligned}$$

Thus, ξ is segmented $\mathbb{R}_+$-convex in the first variable on $\mathscr{X}$. □

Lemma 3.4. *Let $f : \mathbb{X} \to \mathbb{Y}$ be a vector valued mapping and $\mathscr{X}$ be a nonempty and convex subset of $\mathbb{X}$. Then,*

(a) *ξ is arcwise connected $\mathbb{R}_+$-convex in the first variable on $\mathscr{X}$ if f is arcwise connected $\mathscr{C}$-convex on $\mathscr{X}$;*

(b) ξ *is naturally quasiarcwise connected* $\mathbb{R}_+$*-convex in the first variable on* $\mathscr{X}$ *if* f *is naturally quasiarcwise connected* $\mathscr{C}$*-convex on* $\mathscr{X}$;
(c) ξ *is connected* $\mathbb{R}_+$*-convex in the first variable on* $\mathscr{X}$ *if* f *is connected* $\mathscr{C}$*-convex on* $\mathscr{X}$;
(d) ξ *is naturally quasiconnected* $\mathbb{R}_+$*-convex in the first variable on* $\mathscr{X}$ *if* f *is naturally quasiconnected* $\mathscr{C}$*-convex on* $\mathscr{X}$.

Proof. Because of the similarity of the techniques, we only prove the case (a).

Due to the arcwise connected $\mathscr{C}$-convexity of f, for all $x_1, x_2 \in \mathscr{X}$, there is an arc $\mathscr{A}_{x_1,x_2}$ on $\mathscr{X}$ such that

$$f(\mathscr{A}_{x_1,x_2}(t)) \in (1-t)f(x_1) + tf(x_2) - \mathscr{C}, \quad \forall t \in [0,1].$$

We obtain

$$f(\mathscr{A}_{x_1,x_2}(t)) - f(z) \in (1-t)[f(x_1) - f(z)] + t[f(x_2) - f(z)], \quad \forall z \in \mathscr{X}.$$

Combining this with Lemmas 3.1 (b) and 3.2 (a), one has

$$\begin{aligned}\xi(\mathscr{A}_{x_1,x_2}(t), z) &\le \varphi_{e,\mathscr{C}}((1-t)[f(x_1) - f(z)] + t[f(x_2) - f(z)]) \\ &\le (1-t)\varphi_{e,\mathscr{C}}(f(x_1) - f(z)) + t\varphi_{e,\mathscr{C}}(f(x_2) - f(z)) \\ &\le (1-t)\xi(x_1, z) + t\xi(x_2, z).\end{aligned}$$

Hence, ξ is arcwise connected $\mathbb{R}_+$-convex in the first variable on $\mathscr{X}$. □

By using the same arguments as in the proofs for Lemmas 3.3 and 3.4, we also obtain the results of convexity and connectedness of the function ξ in the second variable shown in the following results.

Lemma 3.5. *Let* $f : \mathbb{X} \to \mathbb{Y}$ *be a vector valued mapping and* $\mathscr{X}$ *be a nonempty and convex subset of* $\mathbb{X}$*. Then*

(a) ξ *is segmented* $\mathbb{R}_+$*-convex in the second variable on* $\mathscr{X}$ *if* f *is segmented* $\mathscr{C}$*-concave on* $\mathscr{X}$;
(b) ξ *is naturally quasisegmented* $\mathbb{R}_+$*-convex in the second variable on* $\mathscr{X}$ *if* f *is naturally quasisegmented* $\mathscr{C}$*-concave on* $\mathscr{X}$.

Lemma 3.6. *Let* $f : \mathbb{X} \to \mathbb{Y}$ *be a vector valued mapping and* $\mathscr{X}$ *be a nonempty and convex subset of* $\mathbb{X}$*. Then,*

(a) ξ *is arcwise connected* $\mathbb{R}_+$*-convex in the second variable on* $\mathscr{X}$ *if* f *is arcwise connected* $\mathscr{C}$*-concave on* $\mathscr{X}$;
(b) ξ *is naturally quasiarcwise connected* $\mathbb{R}_+$*-convex in the second variable on* $\mathscr{X}$ *if* f *is naturally quasiarcwise connected* $\mathscr{C}$*-concave on* $\mathscr{X}$;
(c) ξ *is connected* $\mathbb{R}_+$*-convex in the second variable on* $\mathscr{X}$ *if* f *is connected* $\mathscr{C}$*-concave on* $\mathscr{X}$;
(d) ξ *is naturally quasiconnected* $\mathbb{R}_+$*-convex in the second variable on* $\mathscr{X}$ *if* f *is naturally quasiconnected* $\mathscr{C}$*-concave on* $\mathscr{X}$.

We will finalize this section with discussing the semicontinuity and continuity properties of the function ξ.

Lemma 3.7. *Let* $\mathbb{W}$ *be a normed space and a vector-valued mapping* $g : \mathbb{W} \to \mathbb{Y}$*. Then,* $\phi := \varphi_{e,\mathscr{C}} \circ g$ *is*

(a) *lower semicontinuous at* $x_0 \in \mathbb{W}$ *if* g *is* $\mathscr{C}$*-lower semicontinuous at* x_0;

(b) *upper semicontinuous at* $x_0 \in \mathbb{W}$ *if* g *is* $\mathscr{C}$*-upper semicontinuous at* x_0;
(c) *continuous at* $x_0 \in \mathbb{W}$ *if* g *is* $\mathscr{C}$*-continuous at* x_0.

Proof. Due to the continuity of $\varphi_{e,\mathscr{C}}$, for all $\mu > 0$, there is a neighborhood $\mathscr{U}$ of $g(x_0)$ such that

$$\varphi_{e,\mathscr{C}}(y) \geq \varphi_{e,\mathscr{C}}(g(x_0)) - \mu, \quad \forall y \in \mathscr{U}. \tag{3.3}$$

Since g is $\mathscr{C}$-lower semicontinuous at x_0, there exists a neighborhood $\mathscr{V}$ of x_0 such that $g(x) \in \mathscr{U} + \mathscr{C}$ for all $x \in \mathscr{V}$. Then, for each $x \in \mathscr{V}$, there is $y_0 \in \mathscr{U}$ such that $g(x) \in y_0 + \mathscr{C}$. Consequently, $y_0 \in g(x) - \mathscr{C}$. Applying Lemma 3.2 (a), one has $\varphi_{e,\mathscr{C}}(g(x)) \geq \varphi_{e,\mathscr{C}}(y_0)$. This together with (3.3) implies that $\varphi_{e,\mathscr{C}}(g(x)) \geq \varphi_{e,\mathscr{C}}(g(x_0)) - \mu$ for all $x \in \mathscr{V}$. Thus, $\phi(x) \geq \phi(x_0) - \mu$, or equivalently ϕ is lower semicontinuous at x_0. By the same discussions, we also obtain the remaining results of this lemma. □

Lemma 3.8. *Let* $f : \mathbb{X} \to \mathbb{Y}$ *be a vector-valued mapping and* $\mathscr{X}$ *be a nonempty subset of* $\mathbb{X}$. *Then, the following statements hold:*

(a) ξ *is lower semicontinuous in the first variable and upper semicontinuous in the second variable on* $\mathscr{X}$ *if* f *is* $\mathscr{C}$*-lower semicontinuous on* $\mathscr{X}$;
(b) ξ *is upper semicontinuous in the first variable and lower semicontinuous in the second variable on* $\mathscr{X}$ *if* f *is* $\mathscr{C}$*-upper semicontinuous on* $\mathscr{X}$;
(c) ξ *is continuous on* $\mathscr{X} \times \mathscr{X}$ *if* f *is* $\mathscr{C}$*-continuous on* $\mathscr{X}$.

Proof. For given vectors $\bar{x}, \bar{z} \in \mathscr{X}$, we define vector-valued mappings $\eta_1, \eta_2 : \mathbb{X} \to \mathbb{Y}$ as follows

$$\eta_1(z) := f(z) - f(\bar{x}), \quad \eta_2(x) := f(\bar{z}) - f(x).$$

Then, η_1 is $\mathscr{C}$-lower semicontinuous on $\mathscr{X}$ and η_2 is $\mathscr{C}$-upper semicontinuous on $\mathscr{X}$ as f is $\mathscr{C}$-lower semicontinuous on $\mathscr{X}$. Combining this with (3.1) and Lemma 3.7 (a), we derive that $\xi(\cdot, \bar{x}) = \varphi_{e,\mathscr{C}}(\eta_1(\cdot))$ is lower semicontinuous on $\mathscr{X}$ and $\xi(\bar{z}, \cdot) = \varphi_{e,\mathscr{C}}(\eta_2(\cdot))$ is upper semicontinuous on $\mathscr{X}$. By employing the above arguments, we also get Statement (b). For Statement (c), we set the vector-valued mappings $g_1(z,x) = f(z)$ and $g_2(z,x) = f(x)$ for all $x, z \in \mathbb{X}$. A vector-valued mapping $\psi : \mathbb{X} \times \mathbb{X} \to \mathbb{Y}$ is defined by

$$\psi(z,x) = g_1(z,x) - g_2(z,x).$$

Since f is $\mathscr{C}$-continuous on $\mathscr{X}$, g is $\mathscr{C}$-continuous on $\mathscr{X} \times \mathscr{X}$. Applying Lemma 3.7 (c), we gain the continuity of the function ξ. □

4. Connectedness Properties of Efficient and Minimal Sets to Vector Optimization Problems

Let $\mathbb{X}, \mathbb{Y}, \mathscr{C}$ be defined as in Section 2, $\mathscr{X}$ be a nonempty subset of $\mathbb{X}$, and $f : \mathbb{X} \to \mathbb{Y}$ be a vector-valued mapping. The aim of this section is to study connectedness properties of efficient and minimal sets to the following vector optimization problem:

$$\text{(VOP)} \quad \min \quad f(x) \quad \text{subject to} \quad x \in \mathscr{X}.$$

Motivated by [17], for each $\varepsilon > 0$, we define ε-minimal and ε-efficient points of (VOP) as follows.

Definition 4.1. A vector $y_0 \in f(\mathscr{X})$ is called

(a) an ε-minimal point of (VOP), written as $y_0 \in \mathrm{Min}(f,\mathscr{X},\varepsilon)$, if

$$(f(\mathscr{X}) - y_0 + \varepsilon e) \cap (-\mathscr{C} \setminus \{0\}) = \emptyset;$$

(b) an ε-weakly minimal point of (VOP), written as $y_0 \in \mathrm{WMin}(f,\mathscr{X},\varepsilon)$, if

$$(f(\mathscr{X}) - y_0 + \varepsilon e) \cap (-\operatorname{int}\mathscr{C}) = \emptyset.$$

Definition 4.2. A vector $x_0 \in \mathscr{X}$ is called

(a) an ε-efficient solution of (VOP), written as $x_0 \in \mathrm{Eff}(f,\mathscr{X},\varepsilon)$, if $f(x_0) \in \mathrm{Min}(f,\mathscr{X},\varepsilon)$, or equivalently

$$\mathrm{Min}(f,\mathscr{X},\varepsilon) = f(\mathrm{Eff}(f,\mathscr{X},\varepsilon));$$

(b) an ε-weakly efficient solution of (VOP), written as $x_0 \in \mathrm{WEff}(f,\mathscr{X},\varepsilon)$, if $f(x_0)$ is an element of $\mathrm{WMin}(f,\mathscr{X},\varepsilon)$, or equivalently

$$\mathrm{WMin}(f,\mathscr{X},\varepsilon) = f(\mathrm{WEff}(f,\mathscr{X},\varepsilon)).$$

For each $a \in \mathscr{X}$ and $\varepsilon > 0$, we set

$$E(a,\varepsilon) = \{x \in \mathscr{X} : \xi(z,a) + \varepsilon > \xi(x,a) \text{ for all } z \in \mathscr{X}\},$$

and

$$W(a,\varepsilon) = \{x \in \mathscr{X} : \xi(z,a) + \varepsilon \geq \xi(x,a) \text{ for all } z \in \mathscr{X}\}.$$

Remark 4.1. For all $a \in \mathscr{X}$ and $0 < \varepsilon_1 < \varepsilon_2$, one has $E(a,\varepsilon_1) \subset W(a,\varepsilon_1) \subset E(a,\varepsilon_2)$.

Lemma 4.1. *Let $\varepsilon > 0$ be given. If $\mathscr{X}$ is connected and f is naturally quasiconnected $\mathscr{C}$-convex on $\mathscr{X}$, then $W(a,\varepsilon)$ and $E(a,\varepsilon)$ are connected for all $a \in \mathscr{X}$.*

Proof. We only prove the first statement, and the other case can be discussed similarly. Taking arbitrarily $x_1, x_2 \in W(a,\varepsilon)$, one has

$$\max\{\xi(x_1,a), \xi(x_2,a)\} \leq \xi(z,a) + \varepsilon, \quad \forall z \in \mathscr{X}. \tag{4.1}$$

Since f is naturally quasiconnected $\mathscr{C}$-convex on $\mathscr{X}$, by Lemma 3.4 (d), there exists a connected set $\mathscr{K}_{x_1,x_2} \subset \mathscr{X}$ containing two points x_1, x_2 such that, for each $\bar{x} \in \mathscr{K}_{x_1,x_2}$, there is $s \in [0,1]$ satisfying

$$\xi(\bar{x},a) \leq (1-s)\xi(x_1,a) + s\xi(x_2,a).$$

This together with (4.1) implies that

$$\xi(\bar{x},a) \leq \xi(z,a) + \varepsilon, \quad \forall z \in \mathscr{X}, \bar{x} \in \mathscr{K}_{x_1,x_2}.$$

Therefore, $\mathscr{K}_{x_1,x_2} \subset W(a,\varepsilon)$, and so $W(a,\varepsilon)$ is connected. □

In the upcoming, we yield sufficient conditions for the connectedness property of weakly efficient solution sets to the problem (VOP).

Theorem 4.1. *Let $\varepsilon > 0$ be given. Assume that*

(i) *$\mathscr{X}$ is connected and compact;*
(ii) *f is naturally quasiconnected $\mathscr{C}$-convex as well as $\mathscr{C}$-continuous on $\mathscr{X}$.*

Then, $\mathrm{WEff}(f,\mathscr{X},\varepsilon)$ *is connected.*

Proof. The proof is divided into four steps.

Step 1. *$W(a,\varepsilon)$ is nonempty for any $a \in \mathscr{X}$.*

Since f is $\mathscr{C}$-lower semicontinuous on $\mathscr{X}$, Lemma 3.8 (a) yields that $\xi(\cdot,a)$ is lower semicontinuous on $\mathscr{X}$, so the function $\xi(\cdot,a)$ attains the minimal values over the compact subset $\mathscr{X}$ of $\mathbb{X}$, that is, the subset $\{x \in \mathscr{X} : \xi(z,a) \geq \xi(x,a)$ for all $z \in \mathscr{X}\}$ of $W(a,\varepsilon)$ is nonempty. Hence $W(a,\varepsilon)$ is also a nonempty subset of $\mathbb{X}$.

Step 2. *The mapping $W(\cdot,\varepsilon) : \mathscr{X} \rightrightarrows \mathscr{X}$ is upper semicontinuous on $\mathscr{X}$.*

Suppose on the contrary that we can find a vector $\hat{a}$ such that W is not usc at $\hat{a}$. Then, there is an open neighborhood $\mathscr{U}$ of $W(\hat{a},\varepsilon)$ and a sequence $\{a_n\}$ converging to $\hat{a}$ such that, for each n, there exists $\hat{x}_n \in W(\hat{a}_n,\varepsilon) \setminus \mathscr{U}$. Since $\mathscr{X}$ is compact, we can assume that $\{\hat{x}_n\}$ converges to some vector $\hat{x} \in \mathscr{X}$. If $\hat{x} \notin W(\hat{a},\varepsilon)$, then there is $\hat{z} \in \mathscr{X}$ such that

$$\xi(\hat{z},\hat{a}) + \varepsilon < \xi(\hat{x},\hat{a}). \tag{4.2}$$

Since $\hat{x}_n \in W(\hat{a}_n,\varepsilon)$, we have $\xi(\hat{z},\hat{a}_n) + \varepsilon \geq \xi(\hat{x}_n,\hat{a}_n)$. Combining this with the $\mathscr{C}$-continuity of f and Lemma 3.8 (c), we obtain $\xi(\hat{z},\hat{a}) + \varepsilon \geq \xi(\hat{x},\hat{a})$. This contradicts (4.2), and hence $\hat{x}$ belongs to $W(\hat{a},\varepsilon)$, which is absurd as $\hat{x}_n \notin \mathscr{U}$ for all n. Therefore, W is upper semicontinuous in the first variable on $\mathscr{X}$.

Step 3. *We demonstrate that* $\mathrm{WEff}(f,\mathscr{X},\varepsilon) = \bigcup_{a\in\mathscr{X}} W(a,\varepsilon)$.

($\subset$) Let $\bar{x} \in \mathrm{WEff}(f,\mathscr{X},\varepsilon)$ be arbitrary. Then, for any $x \in \mathscr{X}$, one has $f(x) - f(\bar{x}) + \varepsilon e \notin -\operatorname{int}\mathscr{C}$. Combining this with Lemma 3.1 (c), we conclude that

$$\xi(x,\bar{x}) + \varepsilon \geq 0 = \xi(\bar{x},\bar{x}), \quad \forall x \in \mathscr{X},$$

and consequently $\bar{x} \in W(\bar{x},\varepsilon)$.

($\supset$) Taking arbitrarily $\bar{x} \in \cup_{a\in\mathscr{X}} W(a,\varepsilon)$, there exists $a_0 \in \mathscr{X}$ such that $\bar{x} \in W(a_0,\varepsilon)$. Then,

$$\xi(z,a_0) + \varepsilon \geq \xi(\bar{x},a_0), \quad \forall z \in \mathscr{X}. \tag{4.3}$$

If $\bar{x} \notin \mathrm{WEff}(f,\mathscr{X},\varepsilon)$, then we can find some vector $\hat{z} \in \mathscr{X}$ such that $f(\hat{z}) - f(\bar{x}) + \varepsilon e \in -\operatorname{int}\mathscr{C}$, which leads to

$$f(\hat{z}) - f(a_0) + \varepsilon e \in f(\bar{x}) - f(a_0) - \operatorname{int}\mathscr{C}.$$

Combining this with Lemmas 3.1 (e) and 3.2 (b), we obtain $\xi(\hat{z},a_0) + \varepsilon < \xi(\bar{x},a_0)$. This contradicts (4.3), and hence $\bar{x} \in \mathrm{WEff}(f,\mathscr{X},\varepsilon)$.

Step 4. *The set $\bigcup_{a\in\mathscr{X}} W(a,\varepsilon)$ is a connected subset of $\mathscr{X}$.*

It follows from Steps 1, 2, Condition (i) and Lemma 4.1 that all conditions of Lemma 2.1 hold true. Therefore, $\bigcup_{a\in\mathscr{X}} W(a,\varepsilon)$ is connected.

Combining Steps 3 and 4, we conclude that $\mathrm{WEff}(f,\mathscr{X},\varepsilon)$ is connected. □

Example 4.1. Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}^2$, $\mathscr{X} = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 \leq x_2 \leq 0.5x_1^2 + 2\}$, $\mathscr{C} = \mathbb{R}^2_+$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined by

$$f(x) = (|x|, 2x).$$

Then, $\mathscr{X}$ is compact but not convex, f is $\mathscr{C}$-continuous and naturally quasiconnected $\mathscr{C}$-convex on $\mathscr{X}$. Applying Theorem 4.1, we imply that $\mathrm{WEff}(f,\mathscr{X},\varepsilon)$ is connected.

To investigate the connectedness of the efficient solution set, we consider the following important relation.

Lemma 4.2. *Let $\varepsilon > 0$ and $\mathscr{X}$ be arcwise connected. Assume that f is naturally quasiarcwise connected $\mathscr{C}$-convex on $\mathscr{X}$. Then, $W(a,\varepsilon) \subset \operatorname{cl} E(a,\varepsilon)$.*

Proof. Let $\bar{x} \in W(a,\varepsilon)$ and $\hat{x} \in E(a,\varepsilon)$ be arbitrary. Then, we obtain

$$\xi(\bar{x},a) \leq \xi(z,a) + \varepsilon \text{ and } \xi(\hat{x},a) < \xi(z,a) + \varepsilon, \quad \forall z \in \mathscr{X}. \tag{4.4}$$

Since $\bar{x},\hat{x} \in \mathscr{X}$ and f is the naturally quasiarcwise connected $\mathscr{C}$-convex on $\mathscr{X}$, Lemma 3.4 (b) gives us to choose an arc $\mathscr{A}_{\bar{x},\hat{x}}(t) \subset \mathscr{X}$ such that, for each $t \in [0,1]$, there exists $s \in [0,1]$,

$$\xi(\mathscr{A}_{\bar{x},\hat{x}}(t),a) \leq (1-s)\xi(\bar{x},a) + s\xi(\hat{x},a).$$

This together with (4.4) would imply that

$$\begin{aligned} \xi(\mathscr{A}_{\bar{x},\hat{x}}(t),a) &< (1-s)(\xi(z,a)+\varepsilon) + s(\xi(z,a)+\varepsilon) \\ &< \xi(z,a) + \varepsilon, \end{aligned}$$

for all $z \in \mathscr{X}$. Consequently, $\mathscr{A}_{\bar{x},\hat{x}}(t) \in E(a,\varepsilon)$ for all $t \in [0,1]$. Since $\mathscr{A}_{\bar{x},\hat{x}}(t) \to \bar{x}$ when $t \to 0$, $\bar{x}$ belongs to $\operatorname{cl} E(a,\varepsilon)$. Therefore, $W(a,\varepsilon)$ is a subset of $\operatorname{cl} E(a,\varepsilon)$. □

In the next theorem, we present connectedness conditions for the approximate efficient solution set of the problem (VOP).

Theorem 4.2. *Let $\varepsilon > 0$ be given. Assume that*

(i) *$\mathscr{X}$ is arcwise connected and compact*;

(ii) *f is naturally quasiarcwise connected $\mathscr{C}$-convex as well as $\mathscr{C}$-continuous on $\mathscr{X}$.*

Then, $\operatorname{Eff}(f,\mathscr{X},\varepsilon)$ *is connected.*

Proof. By employing the same arguments in the proof of Theorem 4.1, we conclude that $E(x,\varepsilon)$ is nonempty for all $\varepsilon > 0$ and $x \in \mathscr{X}$. The rest of this proof is divided into three steps.

Step 1. *The mapping $E(\cdot,\varepsilon) : \mathscr{X} \rightrightarrows \mathscr{X}$ is lower semincontinuous on $\mathscr{X}$.*

If there is some $x_0 \in \mathscr{X}$ that makes $E(\cdot,\varepsilon)$ not lsc at x_0, then there are a vector $\bar{x} \in E(x_0,\varepsilon)$ and a sequence $\{x_n\} \subset \mathscr{X}$ converging to x_0 such that, for all $\bar{x}_n \in E(x_n,\varepsilon)$, the sequence $\{\bar{x}_n\}$ cannot converge to $\bar{x}$. So, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying $\bar{x} \notin E(x_{n_k},\varepsilon)$. Therefore, we can find some vector $\hat{z}_{n_k} \in \mathscr{X}$ such that

$$\xi(\bar{x},x_{n_k}) \geq \xi(\hat{z}_{n_k},x_{n_k}) + \varepsilon. \tag{4.5}$$

By the compactness of $\mathscr{X}$, we can assume that $\{\hat{z}_{n_k}\}$ converges to $\hat{z} \in \mathscr{X}$. Applying Lemma 3.8 (c) and the $\mathscr{C}$-continuity of f, we obtain the continuity of ξ, which together with (4.5) implies that

$$\xi(\bar{x},x_0) \geq \xi(\hat{z},x_0) + \varepsilon.$$

It is impossible as $\bar{x}$ belongs to $E(x_0,\varepsilon)$. Therefore, $E(\cdot,\varepsilon)$ is lower semicontinuous on $\mathscr{X}$. Combining this with Lemmas 2.1 and 4.1, we conclude that the set $\bigcup_{a\in\mathscr{X}} E(a,\varepsilon)$ is connected.

Step 2. *We show that*

$$\bigcup_{a\in\mathscr{X}} E(a,\varepsilon) \subset \operatorname{Eff}(f,\mathscr{X},\varepsilon) \subset \operatorname{cl}\left(\bigcup_{a\in\mathscr{X}} E(a,\varepsilon)\right). \tag{4.6}$$

It follows from Definition 4.2 and Step 3 in the proof of Theorem 4.1 that

$$\operatorname{Eff}(f,\mathscr{X},\varepsilon) \subset \bigcup_{a\in\mathscr{X}} W(a,\varepsilon).$$

Combining this with the naturally quasiarcwise connected $\mathscr{C}$-convexity of f and Lemma 4.2, we have

$$\mathrm{Eff}(f,\mathscr{X},\varepsilon) \subset \bigcup_{a\in\mathscr{X}} W(a,\varepsilon) \subset \bigcup_{a\in\mathscr{X}} \mathrm{cl}(E(a,\varepsilon)) \subset \mathrm{cl}\left(\bigcup_{a\in\mathscr{X}} E(a,\varepsilon)\right).$$

Now we prove that

$$\bigcup_{a\in\mathscr{X}} E(a,\varepsilon) \subset \mathrm{Eff}(f,\mathscr{X},\varepsilon).$$

Taking arbitrarily $\bar{x} \in \cup_{a\in\mathscr{X}} E(a,\varepsilon)$, there exists $a_0 \in \mathscr{X}$ such that $\bar{x} \in E(a_0,\varepsilon)$. Then,

$$\xi(z,a_0)+\varepsilon > \xi(\bar{x},a_0), \quad \forall z \in \mathscr{X}. \tag{4.7}$$

If $\bar{x} \notin \mathrm{Eff}(f,\mathscr{X},\varepsilon)$, then we can pick up a vector $\hat{z} \in \mathscr{X}$ such that

$$f(\hat{z})-f(\bar{x})+\varepsilon e \in -\mathscr{C}\setminus\{0\} \subset -\mathscr{C},$$

and consequently

$$f(\hat{z})-f(a_0)+\varepsilon e \in f(\bar{x})-f(a_0)-\mathscr{C}. \tag{4.8}$$

From (4.8), Lemmas 3.2 (a) and 3.1 (e), we obtain

$$\xi(\hat{z},a_0)+\varepsilon \leq \xi(\bar{x},a_0),$$

which contradicts (4.7). Therefore, $\bar{x} \in \mathrm{Eff}(f,\mathscr{X},\varepsilon)$, this leads to $\bigcup_{a\in\mathscr{X}} E(a,\varepsilon) \subset \mathrm{Eff}(f,\mathscr{X},\varepsilon)$.

Step 3. It follows from Steps 1, 2, and Lemma 2.3 that the ε-efficient solution set $\mathrm{Eff}(f,\mathscr{X},\varepsilon)$ is connected. □

Remark 4.2. In the studies of connectedness conditions of solution sets, the convex condition for constraint sets is considered as a key assumption [8, 17, 18, 26, 27]. However, the convexity property of solution sets to vector optimization models is not easy to achieve, and so the obtained results via this approach are difficult to apply to bilevel optimization models. Therefore, we have replaced this essential condition by connectedness conditions and have obtained results on connectedness properties of (weakly) approximate efficient solution sets to vector optimization problems as in the Theorems 4.1 and 4.2. Therefore, although the problem considered in this paper is a special case of the problem in [27], our results cannot be inferred from the mentioned study.

Example 4.2. Let $\mathbb{X}=\mathbb{Y}=\mathbb{R}^2$, $\mathscr{X}=\{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2 \leq 10\}$, $\mathscr{C}=\mathbb{R}^2_+$ and $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined by

$$f(x_1,x_2)=((x_1x_2)^2, x_1^2+x_2^2).$$

We now demonstrate that f is naturally quasiarcwise connected $\mathbb{R}^2_+$-convex, but it is not naturally quasisegmented $\mathbb{R}^2_+$-convex on $\mathscr{X}$.

$\star$ *f is not naturally quasisegmented $\mathbb{R}^2_+$-convex*: For $\hat{x}=(-1,-3)$, $\bar{x}=(-3,-1)$, and $t=0.5$, we have

$$f(0.5\hat{x}+0.5\bar{x})=f(-2,-2)=(16,8)\notin(9,10)-\mathbb{R}^2_+=(1-s)g(\hat{x})+sg(\bar{x})-\mathbb{R}^2_+, \quad \forall s\in[0,1].$$

Therefore, f is not naturally quasisegmented $\mathbb{R}^2_+$-convex on $\mathbb{R}^2$.

⋆ *f is naturally quasiarcwise connected* $\mathbb{R}^2_+$*-convex*: For each $\hat{x} = (\hat{x}_1, \hat{x}_2)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2)$ in $\mathbb{R}^2$, we consider an arc $\mathscr{A}_{\hat{x},\bar{x}} : [0,1] \to \mathbb{R}^2$ defined by

$$\mathscr{A}_{\hat{x},\bar{x}}(t) = \begin{cases} (1-2t)\hat{x}, & \text{if } 0 \leq t \leq 0.5, \\ (2t-1)\bar{x}, & \text{if } 0.5 < t \leq 1. \end{cases}$$

Now, we prove that for each $t \in [0,1]$, we can choose some $s \in [0,1]$ such that

$$f(\mathscr{A}_{\hat{x},\bar{x}}(t)) \in (1-s)f(\hat{x}) + sf(\bar{x}) - \mathbb{R}^2_+. \tag{4.9}$$

We consider two cases.
Case 1. If $t \in [0, 0.5]$, then

$$\begin{aligned} f(\mathscr{A}_{\hat{x},\bar{x}}(t)) &\in ((1-2t)^4(\hat{x}_1\hat{x}_2)^2, (1-2t)^2(\hat{x}_1^2 + \hat{x}_2^2)) - \mathbb{R}^2_+ \\ &\in ((\hat{x}_1\hat{x}_2)^2, \hat{x}_1^2 + \hat{x}_2^2) - \mathbb{R}^2_+ \\ &\in f(\hat{x}) - \mathbb{R}^2_+. \end{aligned}$$

Hence, (4.9) holds with $s = 0$.
Case 2. If $t \in]0.5, 1]$, then

$$\begin{aligned} f(\mathscr{A}_{\hat{x},\bar{x}}(t)) &\in ((2t-1)^4(\bar{x}_1\bar{x}_2)^2, (2t-1)^2(\bar{x}_1^2 + \bar{x}_2^2)) - \mathbb{R}^2_+ \\ &\in ((\bar{x}_1\bar{x}_2)^2, \bar{x}_1^2 + \bar{x}_2^2) - \mathbb{R}^2_+ \\ &\in f(\bar{x}) - \mathbb{R}^2_+, \end{aligned}$$

and consequently (4.9) satisfies with $s = 1$.

Moreover, $\mathscr{X}$ is compact and f is continuous on $\mathscr{X}$, by Theorem 4.2, the set $\mathrm{Eff}(f, \mathscr{X}, \varepsilon)$ is connected.

In the following result we provide connectedness conditions of minimal sets via the corresponding properties of efficient sets. More precisely, we have used Theorems 4.1 and 4.2 to obtain connectedness properties of $\mathrm{WEff}(f, \mathscr{X}, \varepsilon), \mathrm{Eff}(f, \mathscr{X}, \varepsilon)$, and then combining Lemma 2.2 and the fact that $\mathrm{WMin}(f, \mathscr{X}, \varepsilon) = f(\mathrm{WEff}(f, \mathscr{X}, \varepsilon))$ and $\mathrm{Min}(f, \mathscr{X}, \varepsilon) = f(\mathrm{Eff}(f, \mathscr{X}, \varepsilon))$, the connectedness properties of $\mathrm{WMin}(f, \mathscr{X}, \varepsilon)$ and $\mathrm{Min}(f, \mathscr{X}, \varepsilon)$ follow.

Corollary 4.1. *Let* $\varepsilon > 0$ *be given. Assume that*

(i) $\mathscr{X}$ *is compact and connected*;
(ii) f *is naturally quasiconnected* $\mathscr{C}$*-convex as well as continuous on* $\mathscr{X}$.

Then, $\mathrm{WMin}(f, \mathscr{X}, \varepsilon)$ *is connected.*

Corollary 4.2. *Let* $\varepsilon > 0$ *be given. Assume that*

(i) $\mathscr{X}$ *is compact and arcwise connected*;
(ii) f *is naturally quasiarcwise connected* $\mathscr{C}$*-convex as well as continuous on* $\mathscr{X}$.

Then, $\mathrm{Min}(f, \mathscr{X}, \varepsilon)$ *is connected.*

Remark 4.3. Minimal sets play an important role in vector optimization, and so they have received much attention from several mathematicians in various topics such as for existence conditions [3, 22, 28, 29, 30, 31], for optimal conditions [32, 33, 34, 35, 36] and for topology properties [3, 7, 17]. It is worthy noting that when we consider models of practical situations, the data is usually obtained from measuring or statistics, and so data errors are inevitable.

Therefore, recently there are many works devoted to approximate minimal sets of optimization models, see e.g [6, 27] and the references therein. To the best of our knowledge, there have no works on studying the connectedness property of (weakly) approximate minimal sets to vector optimization problems, and hence Corollaries 4.1 and 4.2 are completely new.

Acknowledgements

The authors would like to thank the Editors and anonymous referees for their valuable remarks and suggestions which helped us to improve the paper. This study is a result of the project under Grant number B2023-TCT-05 supported by The Ministry of Education and Training of Viet Nam. The fourth author was funded by the project under Grant number TSV2023-81 supported by Can Tho University.

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