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# CONVERGENCE OF INEXACT ITERATES OF AN ALGORITHM BASED ON UNIONS OF NONEXPANSIVE MAPPINGS

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**Abstract.** In this paper, we obtain a generalization of a recent result of Tam (2018) which is for iterates of a set-valued paracontracting operators in a finite-dimensional space. Our result in this paper is obtained for operators in a metric space such that every its bounded and closed set is compact. We establish a convergence of inexact iterates of the set-valued operator by taking into account computational errors.
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## 1. INTRODUCTION

For more than sixty years now, there has been numerous research activities regarding the fixed point theory of nonlinear contractive and nonexpansive (that is, 1-Lipschitz) mappings; see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references cited therein. These activities mainly stem from the classical Banach's fixed point theorem [12] concerning the existence of a unique fixed point for a strict contraction in complete metric spaces. They also concern the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this research field including, in particular, the studies of feasibility, common fixed point problems, and monotone variational inequalities, which find important applications in mathematical analysis, optimization theory, and engineering, medical science and so on; see, e.g., [13, 14, 15, 16, 17]. In particular, in [18], it was considered a framework for the analysis of iterative algorithms which can be described in terms of a structured set-valued operator. More precisely, at each point in the ambient space, it is assumed that the value of the operator can be expressed as a finite union of values of single-valued paracontracting operators. For such algorithms, a convergence result was proved which generalizes a result obtained in [19].

In this paper, we obtain a generalization of a recent result of Tam [18] which was proved for iterates of a set-valued paracontracting operators in a finite-dimensional space. In this paper, our result is obtained for operators in a metric space such that every its bounded and closed set is compact. We establish a convergence of inexact iterates of the set-valued operator by taking into account computational errors.

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## 2. Preliminaries

Assume that $(X,\rho)$ is a metric space and $C \subset X$ is a nonempty and closed set. For each $x \in X$ and each $r > 0$, set

$$B(x,r) = \{y \in X : \rho(x,y) \le r\}.$$

Assume that, for each $x \in X$ and each $M > 0$, the set $C \cap B(x,M)$ is compact. For each $S : C \to C$, set

$$\mathrm{Fix}(S) = \{x \in C : S(x) = x\}.$$

Assume that $m \ge 1$ is an integer, $T_i : C \to C$, $i = 1,\dots,m$ are continuous operators, $\phi : C \to \{1,\dots,m\}$, and that the following assumptions hold:

(A1) For each $x \in C$, there exists $\delta > 0$ such that, for each $y \in B(x,\delta) \cap C$, $\phi(y) \subset \phi(x)$.

(A2) For each $j \in \{1,\dots,m\}$, each $z \in \mathrm{Fix}(T_j)$, each $y \in C$, and each $x \in C \setminus \mathrm{Fix}(T_j)$,

$$\rho(z, T_j(y)) \le \rho(z,y)$$

and

$$\rho(z, T_j(x)) < \rho(z,x).$$

Note that (A2) holds for many nonlinear mappings, such as [10, 11], including the projections on closed and convex sets in a Hilbert space. Define $T(x) = \{T_i(x) : i \in \phi(x)\}$ for each $x \in C$ and $F(T) = \{x \in C : x \in T(x)\}$, and $\bar{F}(T) = \{x \in C : T(x) = \{x\}\}$. From now on, we always assume that $\cap_{j=1}^{m} \mathrm{Fix}(T_j) \neq \emptyset$.

## 3. The First Result

In this section, we prove the following result which demonstrates that a sequence of inexact iterates of our set-valued mappings is bounded if computational errors are summable.

**Theorem 3.1.** *Assume that $\{x_t\}_{t=0}^{\infty} \subset C$ satisfies*

$$\sum_{t=0}^{\infty} \min\{\rho(x_{t+1}, T_i(x_t)) : i \in \phi(x_t)\} < \infty. \tag{3.1}$$

*Then the sequence $\{x_t\}_{t=0}^{\infty} \subset C$ is bounded.*

*Proof.* Since $\cap_{j=1}^{m} \mathrm{Fix}(T_j) \neq \emptyset$, one can fix $z \in \cap_{j=1}^{m} \mathrm{Fix}(T_j)$. For each integer $t \ge 0$, one has

$$\Delta_t = \min\{\rho(x_{t+1}, T_i(x_t)) : i \in \phi(x_t)\}. \tag{3.2}$$

It follows from (3.1) and (3.2) that $\sum_{t=0}^{\infty} \Delta_t < \infty$. Let $t \ge 0$ be an integer. By (3.2), there exists $j \in \phi(x_t)$ such that

$$\rho(x_{t+1}, T_j(x_t)) = \Delta_t. \tag{3.3}$$

By (A2) and (3.3), one has

$$\rho(z, x_{t+1}) \le \rho(z, T_j(x_t)) + \Delta_t \le \rho(z, x_t) + \Delta_t,$$

which implies that, for every integer $t \ge 0$,

$$\begin{aligned}\rho(z, x_t) &\le \rho(z, x_0) + \sum_{j=0}^{t} \Delta_j \\ &\le \rho(z, x_0) + \sum_{j=0}^{\infty} \Delta_j.\end{aligned}$$

Theorem 3.1 is proved. □

## 4. The Second Result

In this section, we prove the following result which demonstrates the convergence of a sequence of inexact iterates of our set-valued mappings to the set of its fixed points if computational errors are summable. For the results of this type, we refer to [10, 11].

**Theorem 4.1.** *Assume that* $\{x_t\}_{t=0}^{\infty} \subset C$ *satisfies*

$$\sum_{t=0}^{\infty} \min\{\rho(x_{t+1}, T_i(x_t)) : i \in \phi(x_t)\} < \infty \tag{4.1}$$

*and that a subsequence* $\{x_{t_s}\}_{s=1}^{\infty}$ *converges. Then*

$$\lim_{s\to\infty} x_{t_s} \in T(\lim_{s\to\infty} x_{t_s}).$$

*Proof.* Since $\cap_{j=1}^{m}\mathrm{Fix}(T_j) \neq \emptyset$, one sees that there exists $z \in \cap_{j=1}^{m}\mathrm{Fix}(T_j)$. For each integer $t \geq 0$, set

$$\Delta(t) = \min\{\rho(x_{t+1}, T_i(x_t)) : i \in \phi(x_t)\}. \tag{4.2}$$

Set $\xi = \lim_{s\to\infty} x_{t_s}$. We prove that $\xi \in T(\xi)$. Assume the contrary. Then, for each $i \in \phi(\xi)$, $\rho(\xi, T_i(\xi)) > 0$. Assumption (A2) implies that

$$\rho(z, T_i(\xi)) < \rho(z, \xi),\ i \in \phi(\xi). \tag{4.3}$$

By $\rho(\xi, T_i(\xi)) > 0$ and (4.3), there exists $\varepsilon_0 \in (0,1)$ such that

$$\rho(\xi, T_i(\xi)) > 4\varepsilon_0,\ i \in \phi(\xi)$$

and

$$\rho(z, T_i(\xi)) < \rho(\xi, z) - 4\varepsilon_0,\ i \in \phi(\xi). \tag{4.4}$$

Assumption (A1) implies that there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that $\phi(x) \subset \phi(\xi)$ for each $x \in B(\xi, \varepsilon_1) \cap C$. In view of (4.4), $\xi = \lim_{s\to\infty} x_{t_s}$, and $\phi(x) \subset \phi(\xi)$ for each $x \in B(\xi, \varepsilon_1) \cap C$, we may assume without loss of generality that, for every integer $s \geq 1$,

$$\rho(\xi, x_{t_s}) \leq \varepsilon_1/4,\ \phi(x_{t_s}) \subset \phi(\xi) \tag{4.5}$$

and

$$\rho(z, T_i(x_{t_s})) < \rho(z, x_{t_s}) - 3\varepsilon_0,\ i \in \phi(\xi).$$

Extracting a subsequence and re-indexing we may assume without loss of generality that

$$\phi(x_{t_1}) = \phi(x_{t_s}),\ s = 1, 2, \ldots.$$

By (4.1) and (4.2), there exists an integer $s_0 \geq 1$ such that

$$\sum_{t=t_{s_0}}^{\infty} \Delta(t) < \varepsilon_0/4. \tag{4.6}$$

Assume that $s \geq s_0$ is an integer. Then (4.6) holds. Clearly,

$$\rho(z, x_{t_s+1}) \leq \rho(z, T_i(x_{t_s})) + \rho(T_i(x_{t_s}), x_{t_s+1}),\ i \in \phi(\xi). \tag{4.7}$$

In view of (4.2), we find that there exists $i \in \phi(x_{t_s})$ such that

$$\Delta(t_s) = \rho(T_i(x_{t_s}), x_{t_s+1}),$$

which together with (4.5), (4.6), and (4.7) implies that

$$\rho(z, x_{t_s+1}) \leq \rho(z, x_{t_s}) - 3\varepsilon_0 + \Delta(t_s). \tag{4.8}$$

We prove by induction that, for each $p \in \{1, \ldots, t_{s+1} - t_s\}$,

$$\rho(z, x_{t_s+p}) \leq \rho(z, x_{t_s}) - 3\varepsilon_0 + \sum_{t=t_s}^{t_s+p-1} \Delta(t). \tag{4.9}$$

In view of (4.8), equation (4.9) holds for $p = 1$.

Assume that $p \in \{1, \ldots, t_{s+1} - t_s\}$ and (4.9) holds. It follows from (4.2) that there exists $i \in \phi(x_{t_s+p})$ such that

$$\Delta(t_s + p) = \rho(T_i(x_{t_s+p}), x_{t_s+p+1}). \tag{4.10}$$

Assumption (A2), $i \in \phi(x_{t_s+p})$, (4.9), and (4.10) imply that

$$\begin{aligned} \rho(z, x_{t_s+p+1}) &\leq \rho(z, T_i(x_{t_s+p})) + \rho(T_i(x_{t_s+p}), x_{t_s+p+1}) \\ &\leq \rho(z, x_{t_s+p}) + \Delta_{t_s+p} \\ &\leq \rho(z, x_{t_s}) - 3\varepsilon_0 + \sum_{t=t_s}^{t_s+p-1} \Delta(t) + \Delta(t_s + p) \end{aligned}$$

and (4.9) holds for $p+1$ too. Thus by induction we prove that (4.9) holds for all $p \in \{1, \ldots, t_{s+1} - t_s - 1\}$, which together with (4.6) implies that

$$\begin{aligned} \rho(z, x_{t_{s+1}}) &\leq \rho(z, x_{t_s}) - 3\varepsilon_0 + \sum_{t=t_s}^{t_{s+1}-t_s-1} \Delta(t) \\ &\leq \rho(z, x_{t_s}) - \varepsilon_0. \end{aligned}$$

Then, for every integer $s \geq s_0$, $\rho(z, x_{t_{s+1}}) < \rho(z, x_{t_s}) - \varepsilon_0$, which implies that, for every integer $Q > 1$,

$$\begin{aligned} \rho(z, x_{t_{s_0}}) &\geq \rho(z, x_{t_{s_0}}) - \rho(z, x_{t_{s_0+Q}}) \\ &= \sum_{i=0}^{Q-1} (\rho(z, x_{t_{s_0+i}}) - \rho(z, x_{t_{s_0+i}})) \\ &\geq Q\varepsilon \to \infty \end{aligned}$$

as $Q \to \infty$. The contradiction we have reached proves Theorem 4.1. □

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