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ON THEOREMS OF ŠIŇAJOVÁ, RANKIN AND KUPERBERG CONCERNING SPHERICAL POINT CONFIGURATIONS

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Abstract. This note presents simple linear algebraic proofs of theorems due to Šiňajová, Rankin and Kuperberg concerning spherical point configurations. The common ingredient in these proofs is the use of spherical Euclidean distance matrices (EDMs) and the Perron-Frobenius theorem.

Keywords. Distance geometry; Dispersion problem, Euclidean distance matrices; Orthogonal representation of graphs; Sphere packing.

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1. Introduction

In this note, we are interested in four theorems on spherical point configurations. The first of these theorems is concerned with orthonormal representation of graphs. The notion of orthonormal representation of a graph was introduced by Lovász in his study of Shannon capacity of graphs [1]. For a detailed discussion of orthonormal representations see the recent book [2]. Parsons and Pisanski [3] introduced the following notion of orthonormal representation, which is slightly different from that of Lovász[1] Let G be a simple graph with nodes $1,\ldots,n$. An *orthonormal representation* of G is a mapping of the nodes of G to unit vectors $p^1,\ldots,p^n$ in Euclidean r-space $\mathbb{R}^r$ such that $(p^i)^T p^j$ is negative or zero depending on whether nodes i and j are adjacent or not. The smallest dimension r necessary for such a representation is denoted by $d(G)$. It is easy to see that $d(G) \geq \alpha(G)$, where $\alpha(G)$ is the independence number of G. Šiňajová proved the following.

Theorem 1.1 (Šiňajová [4]). *Let G be a simple graph on n nodes and let k be the number of its nontrivial connected components, i.e., those connected components with at least 2 nodes. Then $d(G) = n-k$.*

The remaining theorems are concerned with the dispersion problem. The *dispersion problem* is the problem of maximizing, over all n-point configurations on the unit $(r-1)$-sphere in $\mathbb{R}^r$, the minimum distance between any two points. The dispersion problem has applications in sphere packing and spherical designs [5]. Davenport and Hajós [6] and Rankin [7] provided solutions of this problem for the case $n = r+2$. Rankin [7], also, provided a solution for the case $n = 2r$. Before presenting Rankin's two theorems, we need the following definition. The

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[1] In Lovász's definition, the inner product $(p^i)^T p^j$ is unrestricted if nodes i and j are adjacent.

regular r-crosspolytope is the convex hull of the union of r mutually orthogonal line segments of length 2 and intersecting at their common midpoint. That is, the regular r-crosspolytope is the convex hull of $(\pm e^1, \pm e^2, \ldots, \pm e^r)$, where e^i is the ith standard unit vector in $\mathbb{R}^r$.

Theorem 1.2 (Rankin [7]). *Let p be an n-point configuration on the unit $(r-1)$-sphere in $\mathbb{R}^r$. If $n = r+2$, then two points of p are at a distance of at most $\sqrt{2}$ from each other.*

Theorem 1.3 (Rankin [7]). *Let p be an n-point configuration on the unit $(r-1)$-sphere in $\mathbb{R}^r$. If $n = 2r$ and the distance between any two points of p is $\geq \sqrt{2}$, then p is unique, up to a rigid motion, and the points of p are the vertices of the regular r-crosspolytope.*

Kuperberg [8] generalized Rankin's result to all n: $r+2 \leq n \leq 2r$.

Theorem 1.4 (Kuperberg [8]). *Let p be an n-point configuration on the unit sphere in $\mathbb{R}^r$ such that $2 \leq n-r \leq r$. If the minimum distance between any two points of p is at least $\sqrt{2}$, then $\mathbb{R}^r$ can be split into the orthogonal product $\prod_{i=1}^{n-r} L_i$ of $n-r$ subspaces of $\mathbb{R}^r$ such that L_i contains exactly r_i+1 points of p, where r_i is the dimension of L_i.*

In this note, we present simple linear algebraic proofs of Šiňajová, Rankin and Kuperberg's theorems based on spherical Euclidean distance matrices (EDMs) and the Perron-Frobenius theorem. These proofs are given in Sections 3, 4, and 5 respectively, while the necessary background material is given in Section 2.

Notations. We collect here the notation used in this note. e_n and E_n denote, respectively, the vector of all 1's in $\mathbb{R}^n$ and the matrix of all 1's of order n. I_n denotes the identity matrix of order n. e_n^i denotes the ith column of I_n. The subscript n, in e_n, E_n, I_n and e_n^i will be omitted if the dimension is clear from the context. For a matrix A, we denote the vector consisting of the diagonal entries of A by $\mathrm{diag}(A)$. Also, for a real symmetric matrix A, we denote by $\lambda_{\max}(A)$ and $m(\lambda_{\max}(A))$, respectively, the largest eigenvalue of A and its multiplicity. The zero vector or the zero matrix of the appropriate dimension is denoted by $\mathbf{0}$. PSD stands for positive semidefinite. Finally, $E(G)$ denotes the edge set of a simple graph G.

2. Preliminaries

In this section, we present the necessary background concerning EDMs and more specifically spherical EDMs. For a comprehensive treatment of EDMs, see the monograph [9].

An $n \times n$ matrix $D = (d_{ij})$ is said to be an EDM if there exist points $p^1, \ldots, p^n$ in some Euclidean space such that

$$d_{ij} = ||p^i - p^j||^2 \text{ for all } i, j = 1, \ldots, n,$$

where $||x||$ denotes the Euclidean norm of x, i.e., $||x|| = \sqrt{x^T x}$. $p^1, \ldots, p^n$ are called the *generating points* of D and the dimension of their affine span is called the *embedding dimension* of D. If the embedding dimension of an $n \times n$ EDM D is $n-1$, then we refer to D as the EDM of a simplex. For example, let E and I denote respectively the matrix of all 1's and the identity matrix. Then the EDM $D = \gamma(E - I)$, where γ is a positive scalar, is the EDM of a *regular simplex*. An EDM D is said to be *spherical* if its generating points lie on a sphere. A *unit* spherical EDM is a spherical EDM whose generating points lie on a sphere of radius $\rho = 1$.

Let e denote the vector of all 1's in $\mathbb{R}^n$ and let s be a vector in $\mathbb{R}^n$ such that $e^T s = 1$. The following theorem is a well-known characterization of EDMs [10, 11, 12, 13].

Theorem 2.1. *Let D be an $n\times n$ real symmetric matrix whose diagonal entries are all 0's. Then D is an EDM if and only if*

$$B = -\frac{1}{2}(I - es^T)D(I - se^T) \tag{2.1}$$

is positive semidefinite (PSD), in which case, the embedding dimension of D is given by rank(B).

That is, D is an EDM iff it is negative semidefinite on $e^{\perp}$, the orthogonal complement of e in $\mathbb{R}^n$. It can be easily shown that B as defined in Equation (2.1) is a Gram matrix of the generating points of D, or a Gram matrix of D for short.

Let B be a Gram matrix of an EDM D with rank r. Then B is PSD and hence $B = PP^T$ for some $n\times r$ matrix P. Consequently, $p^1,\ldots,p^n$, the generating points of D, are given by the rows of P. As a result, P is called a *configuration matrix* of D. It should be noted that Equation (2.1) implies that $Bs = \mathbf{0}$ and hence $P^Ts = \mathbf{0}$; that is

$$\sum_{i=1}^{n} s_i p^i = \mathbf{0}. \tag{2.2}$$

It is well known [12] that if D is a nonzero EDM, then e lies in the column space of D, i.e., there exists vector w such that $Dw = e$. It is also well known that if D is an $n\times n$ EDM of a simplex, i.e., if the embedding dimension of D is $n-1$, then D is spherical and nonsingular. Among the many different characterizations of spherical EDMs, the one given in the following theorem is the most relevant for our purpose.

Theorem 2.2 ([12, 14])**.** *Let D be an EDM and let $Dw = e$. Then D is spherical if and only if $e^Tw > 0$, in which case, ρ, the radius of the sphere containing the generating points of D, is given by $\rho = \left(\frac{1}{2e^Tw}\right)^{1/2}$.*

As an example, consider $D = \gamma(E_n - I_n)$, the EDM of a regular simplex. Then $w = e/(\gamma(n-1))$ and thus its generating points lie on a sphere of radius $\rho = \sqrt{\gamma(n-1)/(2n)}$. Consequently, the $n\times n$ unit spherical EDM of a regular simplex is given by

$$D = \frac{2n}{n-1}(E_n - I_n).$$

A vector x is *positive*, denoted by $x > \mathbf{0}$, if each of its entries is positive. Similarly, a matrix A is *positive* (*nonnegative*), denoted by $A > \mathbf{0}$ ($A \geq \mathbf{0}$), if each of its entries is > 0 (≥ 0). An $n\times n$ nonnegative matrix A is said to be *reducible* if A is the 1×1 zero matrix or if $n \geq 2$ and there exists a permutation matrix Q such that

$$QAQ^T = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices. It easily follows from the definition that if A is a nonnegative symmetric reducible matrix of order $n \geq 2$, then there exists a permutation matrix Q such that QAQ^T is a block diagonal matrix, of at least two blocks, such that each block is either irreducible or the 1×1 zero matrix. A nonnegative matrix that is not reducible is *irreducible*.

It is well known that an $n\times n$ nonnegative matrix A is irreducible if and only if $(I+A)^{n-1} > \mathbf{0}$. Moreover, if A is the adjacency matrix of a simple graph G, then A is irreducible if and only if G is connected. We will need the following fact from the celebrated Perron-Frobenius theorem:

If A is a nonnegative irreducible matrix, then the largest eigenvalue of A, $\lambda_{\max}(A)$, is positive with multiplicity $m(\lambda_{\max}(A)) = 1$ and the eigenvector associated with $\lambda_{\max}(A)$ is positive.

3. Proof of Šiňajová Theorem

A connected component of a graph G is said to be *nontrivial* if it consists of at least 2 nodes. In other words, isolated nodes are trivial connected components of G. Now let p^i and p^j be two unit vectors. Then, clearly, $(p^i)^T p^j = 0$ if and only if $||p^i - p^j||^2 = 2$ and $(p^i)^T p^j < 0$ if and only if $||p^i - p^j||^2 > 2$. As a result, Theorem 1.1 can be stated in the language of EDMs as follows.

Theorem 3.1 (Šiňajová [4]). *Let G be a simple graph on n nodes and let k be the number of its nontrivial connected components. Then there exists a unit spherical EDM $D = (d_{ij})$ of embedding dimension $r = n - k$ such that*

$$d_{ij} \begin{cases} > 2 \text{ iff } \{i, j\} \in E(G), \\ = 2 \text{ iff } \{i, j\} \notin E(G), \end{cases} \tag{3.1}$$

where $E(G)$ denotes the edge set of G. Furthermore, there does not exist a unit spherical EDM of embedding dimension $r \leq n - k - 1$ that satisfies (3.1).

Before proving Theorem 3.1, we first prove the following lemma.

Lemma 3.1. *Let D be an $n \times n$ unit spherical EDM of embedding dimension r and let $Dw = e$. Let $D = 2(E - I) + 2\Delta$. Then $\lambda_{\max}(\Delta) = 1$ and w is an eigenvector associated with $\lambda_{\max}(\Delta)$. Moreover, $r = n - m(\lambda_{\max}(\Delta))$, where $m(\lambda_{\max}(\Delta))$ denotes the multiplicity of $\lambda_{\max}(\Delta)$.*

Proof. By Theorem 2.2, $2e^T w = 1$. Thus by setting $s = 2w$ in Equation (2.1), it follows that the corresponding Gram matrix of D is $B = E - \frac{1}{2}D = I - \Delta$. Hence $\lambda_{\max}(\Delta) \leq 1$ since B is PSD. On the other hand, $Bw = \mathbf{0}$ implies that $\Delta w = w$. Hence $\lambda_{\max}(\Delta) \geq 1$ and consequently $\lambda_{\max}(\Delta) = 1$. As a result, $r = \text{rank}\,(B) = n - m(\lambda_{\max}(\Delta))$. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let A denote the adjacency matrix of G. Then there exists a permutation matrix Q such that

$$QAQ^T = \begin{bmatrix} A^1 & & & \\ & \ddots & & \\ & & A^k & \\ & & & \mathbf{0} \end{bmatrix},$$

where $A^1, \ldots, A^k$ denote the adjacency matrices of the nontrivial connected components of G. Hence, $A^1, \ldots, A^k$ are irreducible nonnegative matrices of orders ≥ 2. Therefore, by the Perron-Frobenius theorem, $m(\lambda_{\max}(A^1)) = \cdots = m(\lambda_{\max}(A^k)) = 1$. For $i = 1, \ldots, k$, let ξ^i denote the eigenvector of A^i associated with $\lambda_{\max}(A^i)$ and let $\Delta^i = A^i / \lambda_{\max}(A^i)$. Further, let

$$\Delta = \begin{bmatrix} \Delta^1 & & & \\ & \ddots & & \\ & & \Delta^k & \\ & & & \mathbf{0} \end{bmatrix}, \quad \xi = \begin{bmatrix} \xi^1 \\ \vdots \\ \xi^k \\ \mathbf{0} \end{bmatrix} \text{ and } w = \frac{\xi}{2e^T \xi}.$$

Then, obviously, $\Delta_{ij} > 0$ if and only if $\{i,j\} \in E(G)$ and $\Delta_{ij} = 0$ if and only if $i = j$ or $\{i,j\} \notin E(G)$. Also, it is equally obvious that $\lambda_{\max}(\Delta) = 1$, $m(\lambda_{\max}(\Delta)) = k$ and $\Delta w = w$.

Let $D = 2(E-I) + 2\Delta$. Then $Dw = e$ since $2e^T w = 1$. Now if we let $s = 2w$ in Equation (2.1), then

$$B = -\frac{1}{2}(I - es^T)D(I - se^T) = E - \frac{1}{2}D = I - \Delta$$

is PSD and of rank $n-k$. As a result, by Theorems 2.1 and 2.2, D is a unit spherical EDM of embedding dimension $r = n-k$ that satisfies (3.1).

To complete the proof, let r be the embedding dimension of any unit spherical EDM D that satisfies (3.1). Let $\Delta = D/2 + I - E$ and wlog assume that Δ is block diagonal. Thus Δ has k irreducible nonnegative diagonal blocks, each associated with a nontrivial connected component of G. Now it follows from Lemma 3.1 that $\lambda_{\max}(\Delta) = 1$ and $r = n - m(\lambda_{\max}(\Delta))$. Consequently, $r \le n-k$ since the contribution from each irreducible diagonal block of Δ to $m(\lambda_{\max}(\Delta))$ is at most 1. $\square$

4. Proof of Rankin's Theorems

Theorems 1.2 and 1.3 can be stated in the language of EDMs as follows.

Theorem 4.1 (Rankin [7])**.** *Let D be an $n\times n$ unit spherical EDM of embedding dimension r. If $n = r+2$, then at least one off-diagonal entry of D is ≤ 2.*

Theorem 4.2 (Rankin [7])**.** *Let D be an $n\times n$ unit spherical EDM of embedding dimension r. If $n = 2r$ and if each off-diagonal entry of D is ≥ 2, then there exists a permutation matrix Q such that*

$$QDQ^T = \begin{bmatrix} 4(E_2 - I_2) & 2E_2 & \cdots & 2E_2 \\ 2E_2 & 4(E_2 - I_2) & \cdots & 2E_2 \\ \vdots & \vdots & \ddots & \vdots \\ 2E_2 & \cdots & 2E_2 & 4(E_2 - I_2) \end{bmatrix}, \tag{4.1}$$

where E_2, I_2 are, respectively, the matrix of all 1's and the identity matrix of orders 2.

It should be noted that the RHS of Equation (4.1) is the EDM of the regular r-crosspolytope. As was mentioned earlier, Theorems 1.2 and 1.3 are special cases of Theorem 1.4 which we prove in the next section. However, in this section, we present an independent proof of Theorem 1.2 after we have proved the following lemma which will be needed in the sequel.

Lemma 4.1. *Let D be an $n\times n$ unit spherical EDM of embedding dimension r and assume that each off-diagonal entry of D is ≥ 2. Let $D = 2(E-I) + 2\Delta$ and let $Dw = e$. If Δ is irreducible, then $r = n-1$, i.e., D is the EDM of a simplex, and $w > \mathbf{0}$.*

Proof. Clearly, $\Delta \ge \mathbf{0}$. Thus, it follows from Lemma 3.1 and the Perron-Frobenius theorem that $\lambda_{\max}(\Delta) = 1$, $m(\lambda_{\max}(\Delta)) = 1$ and $w > \mathbf{0}$. Consequently, $r = \text{rank}\,(B) = n-1$. $\square$

Now Theorem 1.2 is an immediate corollary of Lemma 4.1.

Proof of Theorem 4.1. Let $\Delta = D/2 + I - E$ and assume, by way of contradiction, that each off-diagonal entry of D is > 2. Then each off-diagonal entry of Δ is > 0. Hence, $I + \Delta > 0$ and thus Δ is irreducible. Consequently, by Lemma 4.1, the embedding dimension of D is $r = n-1$, which contradicts the assumption that $r = n-2$. $\square$

5. Proof of Kuperberg's Theorem

Theorem 1.4 can be stated in the language of EDMs as follows.

Theorem 5.1 (Kuperberg [8])**.** *Let D be an $n \times n$ unit spherical EDM of embedding dimension r, where $2 \leq n-r \leq r$. If each off-diagonal entry of D is ≥ 2, then there exists a permutation matrix Q such that*

$$QDQ^T = \begin{bmatrix} D^1 & 2E & \cdots & 2E \\ 2E & D^2 & \cdots & 2E \\ \vdots & \vdots & \ddots & \vdots \\ 2E & \cdots & 2E & D^{n-r} \end{bmatrix},$$

where $D^1, \ldots, D^{n-r}$ are unit spherical EDMs of simplices; and E is the matrix of all 1's of the appropriate dimension.

Two remarks are in order here. First, as shown in [8], if $n = r+2$, then Theorem 5.1 reduces to Rankin's Theorem 4.1. This follows since if D has an off-diagonal entry < 2, then there is nothing to prove. On the other hand, if every off-diagonal entry of D is ≥ 2, then Theorem 5.1 implies that there is a permutation matrix Q such that $QDQ^T = \begin{bmatrix} D^1 & 2E \\ 2E & D^2 \end{bmatrix}$. Hence, at least one of the off-diagonal diagonal entries of D is 2 since $2E$ is a submatrix of QDQ^T.

Second, also, as shown in [8], if $n = 2r$, i.e., if $n-r = r$, then Theorem 5.1 reduces to Rankin's Theorem 4.2. This follows since in this case, each of the submatrices $D^1, \ldots, D^r$ in Theorem 5.1 is of order 2, and thus $D^1 = \cdots = D^r = 4(E_2 - I_2)$. Therefore, the configuration, in this case, is that of the regular r-crosspolytope since the matrix QDQ^T in Theorem 5.1 reduces to that in Theorem 4.2.

Before presenting the proof of Theorem 5.1, we need the following lemma which extends Lemma 4.1 to the case where Δ is padded with zero rows and columns.

Lemma 5.1. *Let D be an $n \times n$ unit spherical EDM of embedding dimension r and assume that each off-diagonal entry of D is ≥ 2. Let $D = 2(E-I) + 2\tilde{\Delta}$ and let $D\tilde{w} = e$. If $\tilde{\Delta} = \begin{bmatrix} \Delta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where Δ is irreducible, then $r = n-1$, i.e., D is the EDM of a simplex, and $\tilde{w} = \frac{1}{2e^T\xi} \begin{bmatrix} \xi \\ \mathbf{0} \end{bmatrix}$, where $\Delta\xi = \xi$ and $\xi > \mathbf{0}$.*

Proof. The proof is similar to that of Lemma 4.1. □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let $D = 2(E-I) + 2\Delta$ and thus $\Delta \geq \mathbf{0}$ and $\operatorname{diag}(\Delta) = \mathbf{0}$. Since the embedding dimension of D is r, it follows from Lemma 3.1 that $\lambda_{\max}(\Delta) = 1$ with multiplicity $m(\lambda_{\max}(\Delta)) = n-r \geq 2$. Therefore, by the Perron-Frobenius theorem, Δ is reducible and thus there exists a permutation matrix Q such that

$$Q\Delta Q^T = \begin{bmatrix} \Delta^1 & & \\ & \ddots & \\ & & \Delta^{n-r} \end{bmatrix} \text{ or } \begin{bmatrix} \Delta^1 & & & \\ & \ddots & & \\ & & \Delta^{n-r} & \\ & & & \mathbf{0} \end{bmatrix}, \tag{5.1}$$

where $\Delta^1, \ldots, \Delta^{n-r}$ are irreducible and thus $\lambda_{\max}(\Delta^1) = \cdots = \lambda_{\max}(\Delta^{n-r}) = 1$. For $i = 1, \ldots, n-r$, let ξ^i denote the eigenvector of Δ^i associated with $\lambda_{\max}(\Delta^i)$. Therefore, by the Perron-Frobenius theorem $\xi^i > \mathbf{0}$ since Δ^i is irreducible. Next, we consider the two cases of $Q\Delta Q^T$ in Equation (5.1) separately.

In the first case, all diagonal blocks of Δ are irreducible. Assume that, for $i = 1, \ldots, n-r$, Δ^i is of order n_i where $\sum_{i=1}^{n-r} n_i = n$. Then $n_i \geq 2$ since $\operatorname{diag}(\Delta^i) = \mathbf{0}$. Let $D^i = 2(E_{n_i} - I_{n_i}) + 2\Delta^i$ for $i = 1, \ldots, n-r$. Then $D^1, \ldots, D^{n-r}$ are EDMs since they are principal submatrices of D. Moreover, let $w^i = \xi^i/(2e_{n_i}^T \xi^i)$. Then $D^i w^i = e_{n_i}$ and $w^i > \mathbf{0}$. Consequently, $D^1, \ldots, D^{n-r}$ are unit spherical EDMs. Therefore, it follows from Lemma 4.1 that each of $D^1, \ldots, D^{n-r}$ is the EDM of a simplex. It is worth pointing out that Equation (2.2) implies that, for each $i = 1, \ldots, n-r$, the origin $\mathbf{0}$ lies in the relative interior [15] of the convex hull of the generating points of D^i since $w^i > \mathbf{0}$.

In the second case, let $\tilde{\Delta}^{n-r} = \begin{bmatrix} \Delta^{n-r} & \\ & \mathbf{0} \end{bmatrix}$. Then, similar to the first case, $D^1, \ldots, D^{n-r-1}$ are unit spherical EDMs of simplices and the origin $\mathbf{0}$ lies in the relative interior of the convex hull of the generating points of each of the EDMs $D^1, \ldots, D^{n-r-1}$. On the other hand, let $D^{n-r} = 2(E - I) + 2\tilde{\Delta}^{n-r}$ and let

$$\tilde{w}^{n-r} = \frac{1}{2e^T \xi^{n-r}} \begin{bmatrix} \xi^{n-r} \\ \mathbf{0} \end{bmatrix}.$$

Then $\tilde{\Delta}^{n-r}\tilde{w}^{n-r} = \tilde{w}^{n-r}$ and $D^{n-r}\tilde{w}^{n-r} = e$. Hence, D^{n-r} is a unit spherical EDM and hence, by Lemma 5.1, D^{n-r} is the EDM of a simplex. However, unlike $D^1, \ldots, D^{n-r-1}$, the origin lies on the relative boundary of the convex hull of the generating points of D^{n-r}. □

Finally, we should point out that in the second case of Equation (5.1), i.e., if $Q\Delta Q^T$ has, say s, zero rows (and columns), then we chose above to define $\tilde{\Delta}^{n-r}$ by appending these s zero rows and columns to Δ^{n-r}. In fact, we could have appended any number of these zero rows and columns to any of $\Delta^1, \ldots, \Delta^{n-r}$.

As an illustration of the theorems of Šiňajová and Kuperberg, consider the following example.

Example 5.1. Let G be the simple graph on the nodes $1, \ldots, 5$ and with edge set $E(G) = \{\,\{1,2\},\{3,4\}\,\}$. Hence, G has two nontrivial connected components and one isolated node. To illustrate Šiňajová's Theorem, let

$$\Delta = A = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & 0 \end{bmatrix} = \begin{bmatrix} \Delta^1 & & \\ & \Delta^2 & \\ & & \mathbf{0} \end{bmatrix},$$

where A is the adjacency matrix of G. Then $D = 2(E - I) + 2\Delta$ is a unit spherical EDM of embedding dimension 3 that satisfies (3.1). Moreover, an orthonormal representation of G is given by $p^1 = e^1$, $p^2 = -e^1$, $p^3 = e^2$, $p^4 = -e^2$ and $p^5 = e^3$, where e^i is the ith standard unit vector in $\mathbb{R}^3$.

To illustrate Kuperberg's Theorem, first, if we define $\tilde{\Delta}^2 = \begin{bmatrix} \Delta^2 & \\ & \mathbf{0} \end{bmatrix}$. Then $\mathbb{R}^3$ can be split into 2 orthogonal subspaces L_1 and L_2 where L_1 consists of the x-axis and contains points p^1 and p^2; while L_2 consists of the y–z plane and contains points p^3, p^4 and p^5. Notice that the origin is in the relative interior of the convex hull of p^1 and p^2, while the origin lies on the relative boundary of the convex hull of p^3, p^4 and p^5.

On the other hand, if we define $\tilde{\Delta}^1 = \begin{bmatrix} \Delta^1 & \\ & \mathbf{0} \end{bmatrix}$. Then, in this case, the subspace L_1 consists of the x–z plane and contains points p^1, p^2 and p^5, while L_2 consists of the y-axis and contains points p^3 and p^4.

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This note is dedicated to Henry Wolkowicz on the occasion of his 75th birthday. I learned about the beautiful subject of Euclidean Distance Matrices (EDMs) from Henry. Thus on this occasion, it is only fitting to contribute this note on the power of EDMs to simplify the proofs of few Geometric theorems.

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