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SPLIT MODELING APPROACH TO NON-COOPERATIVE STRATEGIC GAMES

SHIPRA SINGH[1], AVIV GIBALI[2,3,*]

[1]*Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim, Norway*
[2]*Department of Mathematics, Braude College, 2161002 Karmiel, Israel*
[3]*Center for Mathematics and Scientific Computation, University of Haifa, Mount Carmel, 3498838 Haifa, Israel*

Abstract. In this paper, we examine how can one achieve equilibrium when two related non-cooperative strategic games are being played. We propose a split generalized Nash equilibrium problem for two non-cooperative strategic games and also define an equivalent split quasi-variational inequality problem. Further, by using the techniques of proving existence of quasi-variational inequality problems, we establish the existence of equilibria. Moreover, as an application, we investigate our split generalized Nash equilibrium problem in the terms of river basin pollution problem.

Keywords. Nash equilibrium problem; Split quasi-variational inequality problem; River basin pollution problem.
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1. Introduction

The study of equilibrium problems was started by Cournot [1], who studied an oligopolistic economy. However, it was Nash [2, 3] who introduced this concept formally. Subsequently, Arrow and Debreu [4] extended it to the generalized Nash equilibrium problem (GNEP), which is useful in mathematical modeling, for instance, routing problems in communication networks [5], and in engineering applications [6]. For an overview of the methods for solving GNEPs, we refer the reader to the survey paper [7] and the references therein. We emphasize here that GNEPs are connected to quasi-variational inequality problems, a fact which was first observed by Bensoussan [8]. Thereafter, Harker [9] investigated these problems in Euclidean spaces. Quasi-variational inequality problems have been proven an efficient tool to study the GNEPs. Very recently, Bueno and Cotrina [10] studied the projected solutions of GNEP with the help of quasi-variational inequality problems. For more recent relevant works, we refer to [11, 12] and the references therein. Moreover, we would also like to mention the very recent interesting relevant articles [13, 14] which studied the GNEP, its reformulation in terms of variational inequality problems, and applications to the COVID-19 pandemic.

On the other hand, a split inverse problem concerns a model in which two vector spaces, connected by a bounded linear operator, are given. In addition, two inverse problems are also involved. The first instance of a split inverse problem is the split convex feasibility problem

*Corresponding author.
E-mail address: avivg@braude.ac.il (A. Gibali).
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(SCFP), introduced by Censor and Elfving [15], where each inverse problem is a convex feasibility problem in its corresponding space. Censor et al. [16] used this reformulation successfully in the field of intensity-modulated radiation therapy (IMRT) treatment planning. Since then, SCFP was generalized in various ways to split minimization, split common fixed points, split variational inequality problems and many more. In addition, extensions to infinite dimensional Hilbert and Banach spaces continue to attract a lot of interest. With regards to applications, the split modelling approach is very flexible and allows different choices of inverse problems. It also prevents the need to "translate" everything into one of the two spaces for further processing of the solution. Treatment planning is one of the most known applications of split inverse problem, but many other real-world problems can also successfully be solved by a split reformulation.

In some situations, it is possible to shift the economic constraints from one space to another. Unfortunately, the resulting constraints often turn out to be more complicated than the original ones. Therefore, it is more advantageous to leave the constraints in their respective spaces and build a framework for merging them. At this point, we would like to mention that Li [17, 18] proposed such a mathematical framework for the extended Bertrant duopoly model of price competition in terms of the split Nash equilibrium problem. Some more works on split equilibrium problems are well documented in [19, 20, 21] and the references therein. It is evident that this area needs deeper and more detailed studies. We emphasize that the split generalized Nash equilibrium problem has not been investigated so far, neither from the perspective of forging appropriate variational inequality tools for solving it, nor from the perspective of constructing a unified mathematical model for tackling several economic world related problems, such as pollution control, traffic equilibrium, and oligopolistic market equilibrium problems. Therefore, to pursue further explorations and present novel results for split modeling approaches to equilibrium problems, particularly in non-cooperative strategic games, we introduce a split generalized Nash equilibrium problem and also propose a split quasi-variational inequality problem. We build an equivalent relationship between both problems. Then, we establish the existence result of equilibria as well. To provide an application of the formulated split generalized Nash equilibrium problems in pollution control problems, we interpret a river basin pollution model in terms of such an equilibrium problem.

The outline of our paper is as follows. Preliminaries and formulations of the problem are presented in Section 2. The equivalence of the split generalized Nash equilibrium problem with the split quasi-variational inequality problem is established in Section 3. The existence of equilibria is obtained in Section 4. A motivational example of the river basin pollution problem is given in Section 5. Finally, Section 6 concludes our paper.

2. Preliminaries and Problem Formulations

In game theory and microeconomics theory, it is vital to study the behavior of two related strategic games when these are played for achieving equilibrium in the both games. In these two related games, players of one game choose the strategies that come from the linear transformation of the strategy of other game. In order to study the behavior of this kind of game, we intend to formulate a split generalized Nash equilibrium problem (SGNEP), which comprises two related non-cooperative strategic games, denoted by $K^1(N)$ and $K^2(M)$, with N and M players, respectively. Let $x^\mu \in \mathbb{R}^{n_\mu}$ and $y^\nu \in \mathbb{R}^{m_\nu}$ be the vectors of strategies of the players

$\mu = 1,2,\dots,N$ of the game $K^1(N)$ and players $\nu = 1,2,\dots,M$ of the game $K^2(M)$, respectively. Let $x^{-\mu} \in \mathbb{R}^{n-n_\mu}$ and $y^{-\nu} \in \mathbb{R}^{m-m_\nu}$ be the vectors of strategies of all the players' decision variables except those of the players μ and ν, respectively, and let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ be the vectors of strategies of all the players in their respective games. Here $n = \sum_{\mu=1}^{N} n_\mu$ and $m = \sum_{\nu=1}^{M} m_\nu$. We think of $x^{-\mu}$ and $y^{-\nu}$ as the strategies of the rival players to x^μ and y^ν, respectively. In order to emphasize the strategy vectors of the players μ and ν, we rewrite the strategy vectors $x = (x^1, x^2, \dots, x^{\mu-1}, x^\mu, x^{\mu+1}, \dots, x^N) \in \mathbb{R}^n$ and $y = (y^1, y^2, \dots, y^{\nu-1}, y^\nu, y^{\nu+1}, \dots, y^M) \in \mathbb{R}^m$ of all the players as $x = (x^\mu, x^{-\mu}) \in \mathbb{R}^n$ and $y = (y^\nu, y^{-\nu}) \in \mathbb{R}^m$. For the given vectors $x^{-\mu}$ and $y^{-\nu}$ of the rival players of their respective games, the nonempty, closed, and convex feasible sets (strategy sets) of the players μ and ν are denoted by $K_\mu(x^{-\mu}) \subset \mathbb{R}^{n_\mu}$ and $L_\nu(y^{-\nu}) \subset \mathbb{R}^{m_\nu}$, respectively. Each player $\mu = 1,2,\dots,N$ and $\nu = 1,2,\dots,M$ has an objective function $f^\mu : \mathbb{R}^n \to \mathbb{R}$ and $g^\nu : \mathbb{R}^m \to \mathbb{R}$, respectively. These functions are continuously differentiable and known as the cost/loss functions. We assume that both of the objective functions are defined on the full vector of strategies, which includes the strategies of the rival players too. We consider the nonempty, closed and convex subsets $K \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^m$ and we are also given a bounded linear operator $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(K) \subset L$. This operator allows us to say that the both games $K^1(N)$ and $K^2(M)$ are related. Now, we define the split generalized Nash equilibrium problem as follows:

(SGNEP) to find a strategy vector $x \in K$ such that, for all $\mu = 1,2,\dots,N$, we have $x^\mu \in K_\mu(x^{-\mu})$ and

$$f^\mu(x^\mu, x^{-\mu}) \le f^\mu(p^\mu, x^{-\mu}),\ \forall\ p^\mu \in K_\mu(x^{-\mu}),$$

while the strategy vector $y = Tx \in L$ is such that, for all $\nu = 1,2,\dots,M$, we have $y^\nu \in L_\nu(y^{-\nu})$ and

$$g^\nu(y^\nu, y^{-\nu}) \le g^\nu(q^\nu, y^{-\nu}),\ \forall\ q^\nu \in L_\nu(y^{-\nu}).$$

To formulate the split quasi-variational inequality problem, we define the two set-valued maps $\Gamma_1 : K \to 2^K$ and $\Gamma_2 : L \to 2^L$ as

$$\Gamma_1(x) = \prod_{\mu=1}^{N} K_\mu(x^{-\mu})\ \forall x \in K \text{ and } \Gamma_2(y) = \prod_{\nu=1}^{M} L_\nu(y^{-\nu})\ \forall y \in L, \text{ respectively.}$$

We also consider two functions $F\colon K \to \mathbb{R}^n$ and $G\colon L \to \mathbb{R}^m$. Now, split quasi-variational inequality problem is defined as follow:

(SQVIP) to find a vector $x \in K$ such that $x \in \Gamma_1(x)$ and

$$\langle F(x), z - x\rangle \ge 0,\ \forall z \in \Gamma_1(x),$$

while the vector $y = Tx \in L$ is such that $y \in \Gamma_2(y)$, and

$$\langle G(y), w - y\rangle \ge 0,\ \forall w \in \Gamma_2(y).$$

Definition 2.1. A subset D of K is said to be compactly open (respectively, compactly closed) in K if, for any nonempty compact subset L of K, the intersection $D \cap L$ is open (respectively, closed) in L.

Remark 2.1. (a) It is evident from the definition above that every open (respectively, closed) set is compactly open (respectively, compactly closed).

(b) The union or intersection of a finite number of compactly open (respectively, compactly closed) sets is compactly open (respectively, compactly closed).
(c) If $A \subset K_1$ and $B \subset K_2$ are compactly open (respectively, compactly closed) in K_1 and K_2, respectively, then $A \times B \subset K_1 \times K_2$ is compactly open (respectively, compactly closed) in $K_1 \times K_2$.

Definition 2.2. A family $\{g^\mu\}_{\mu=1}^N$ of maps $g^\mu : K \to \mathbb{R}^n$ is called hemicontinuous if, for all $x, y \in K$ and $\lambda \in [0,1]$, the mapping $\lambda \to \sum_{\mu=1}^{N} \langle g^\mu(x+\lambda z), z^\mu \rangle$ with $z^\mu = y^\mu - x^\mu$ is continuous, where z^μ is the μ^{th} component of z.

Theorem 2.1. [22] *Assume that $S, T : K \to 2^K$ are set-valued maps and that the following hypotheses are satisfied:*

(1) $\forall\, x \in K,\ S(x) \subset T(x)$,
(2) $\forall\, x \in K,\ S(x) \neq \emptyset$,
(3) $\forall\, x \in K,\ T(x)$ *is convex,*
(4) $\forall\, y \in K,\ S^{-1}(\{y\}) = \{x \in K : y \in S(x)\}$ *is compactly open,*
(5) there exists a nonempty, closed and compact subset D of K and $\bar{y} \in D$ such that $K \setminus D \subset S^{-1}(\{\bar{y}\})$.

Then there exists $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x})$.

3. Equivalence of Split Generalized Nash Equilibrium Problem

Theorem 3.1. *Assume that $F(x) = (\nabla f^\mu(x))_{\mu=1}^N$, $G(y) = (\nabla g^\nu(y))_{\nu=1}^M$, for each $x \in K,\ y \in L$, and that for each $\mu \in \{1,2,\ldots,N\}$, $\nu \in \{1,2,\ldots,M\}$, $x^{-\mu}$, $y^{-\nu}$ the functions f^μ and g^ν are convex on K and L in the arguments x^μ and y^ν, respectively. Then $x \in K$ is a split generalized Nash equilibrium if and only if it is the solution of (SQVIP).*

Proof. Let $x \in K$ be a split generalized Nash equilibrium. It follows that, for all $\mu = 1,2,\ldots,N$, we have $x^\mu \in K_\mu(x^{-\mu})$ and

$$f^\mu(x^\mu, x^{-\mu}) \leq f^\mu(p^\mu, x^{-\mu}),\ \forall\, p^\mu \in K_\mu(x^{-\mu}),$$

and $y = Tx \in L$ is such that, for all $\nu = 1,2,\ldots,M$, $y^\nu \in L_\nu(y^{-\nu})$ and

$$g^\nu(y^\nu, y^{-\nu}) \leq g^\nu(q^\nu, y^{-\nu}),\ \forall\, q^\nu \in L_\nu(y^{-\nu}).$$

Since $K_\mu(x^{-\mu})$ and $L_\nu(y^{-\nu})$ are convex sets, $\lambda p^\mu + (1-\lambda)x^\mu \in K_\mu(x^{-\mu})$ and $\lambda q^\nu + (1-\lambda)y^\nu \in L_\nu(y^{-\nu})$ for all $p^\mu \in K_\mu(x^{-\mu})$ and $q^\nu \in L_\nu(y^{-\nu})$, and $\lambda \in [0,1]$. Then the above inequality can be rewritten as

$$f^\mu(x^\mu + \lambda(p^\mu - x^\mu), x^{-\mu}) - f^\mu(x^\mu, x^{-\mu}) \geq 0,\ \forall\, p^\mu \in K_\mu(x^{-\mu}),$$

and $y = Tx \in L$ is such that, for all $\nu = 1,2,\ldots,M$, $y^\nu \in L_\nu(y^{-\nu})$ and

$$g^\nu(y^\nu + \lambda(q^\nu - y^\nu), y^{-\nu}) - g^\nu(y^\nu, y^{-\nu}) \geq 0,\ \forall\, q^\nu \in L_\nu(y^{-\nu}).$$

Dividing the above inequalities by λ, taking the limit $\lambda \to 0$, and using Taylor's series, we obtain

$$\langle \nabla f^\mu(x^\mu, x^{-\mu}), p^\mu - x^\mu \rangle \geq 0,\ \forall\, p^\mu \in K_\mu(x^{-\mu})$$

and $y = Tx \in L$ is such that, for all $\nu = 1,2,\dots,M$, $y^\nu \in L_\nu(y^{-\nu})$ and

$$\langle \nabla g^\nu(y^\nu, y^{-\nu}), q^\nu - y^\nu \rangle \geq 0,\ \forall\, q^\nu \in L_\nu(y^{-\nu}).$$

By the hypotheses, $F(x) = (\nabla f^\mu(x))_{\mu=1}^N$ and $G(y) = (\nabla g^\nu(y))_{\nu=1}^M$, we have

$$\langle F(x), p - x\rangle = \sum_{\mu=1}^{N} \langle \nabla f^\mu(x^\mu, x^{-\mu}), p^\mu - x^\mu \rangle \geq 0,\ \forall\, p \in \Gamma_1(x),$$

and for $y = Tx \in L$, we have $y \in \Gamma_2(y)$ and

$$\langle G(y), q - y\rangle = \sum_{\nu=1}^{M} \langle \nabla g^\nu(y^\nu, y^{-\nu}), q^\nu - y^\nu \rangle \geq 0,\ \forall\, q \in \Gamma_2(y).$$

Since we already have $x \in \Gamma_1(x)$, it follows that x is the solution of (SQVIP).

Conversely, let $x \in K$ be the solution to (SQVIP). First, we prove that, for each $\mu = 1,2,\dots,N$, $x^\mu \in K_\mu(x^{-\mu})$ satisfies the following

$$\begin{aligned} &\langle \nabla f^\mu(x), p^\mu - x^\mu \rangle \geq 0,\ \forall\, p^\mu \in K_\mu(x^{-\mu}) \\ \text{and } y = Tx \in L \text{ is such that for all } &\nu = 1,2,\dots,M, \text{we have } y^\nu \in L_\nu(y^{-\nu}) \text{ and} \\ &\langle \nabla g^\nu(y), q^\nu - y^\nu \rangle \geq 0,\ \forall\, q^\nu \in L_\nu(y^{-\nu}). \end{aligned} \tag{3.1}$$

To this end, suppose to the contrary. Then, we have following three cases:

(1) there exist $\overline{\mu} \in \{1,2,\dots,N\}$, and a strategy vector $\overline{p}^{\overline{\mu}} \in K_{\overline{\mu}}(x^{-\overline{\mu}})$ such that for $x^{\overline{\mu}} \in K_{\overline{\mu}}(x^{-\overline{\mu}})$ the following hold

$$\begin{aligned} &\langle \nabla f^{\overline{\mu}}(x), \overline{p}^{\overline{\mu}} - x^{\overline{\mu}} \rangle < 0, \\ \text{and } y = Tx \in L \text{ is such that there exists } &\overline{\nu} \in \{1,2,\dots,M\}, \text{ and a strategy vector} \\ \overline{q}^{\overline{\nu}} \in L_{\overline{\nu}}(y^{-\overline{\nu}}) \text{ such that, for } &y^{\overline{\nu}} \in L_{\overline{\nu}}(y^{-\overline{\nu}}), \langle \nabla g^{\overline{\nu}}(y), \overline{q}^{\overline{\nu}} - y^{\overline{\nu}} \rangle < 0. \end{aligned} \tag{3.2}$$

(2) there exist $\overline{\mu} \in \{1,2,\dots,N\}$, and a strategy vector $\overline{p}^{\overline{\mu}} \in K_{\overline{\mu}}(x^{-\overline{\mu}})$ such that, for $x^{\overline{\mu}} \in K_{\overline{\mu}}(x^{-\overline{\mu}})$, $\langle \nabla f^{\overline{\mu}}(x), \overline{p}^{\overline{\mu}} - x^{\overline{\mu}} \rangle < 0$ and $y = Tx \in L$ is such that there exists $\overline{\nu} \in \{1,2,\dots,M\}$, and a strategy vector $\overline{q}^{\overline{\nu}} \in L_{\overline{\nu}}(y^{-\overline{\nu}})$ such that for $y^{\overline{\nu}} \in L_{\overline{\nu}}(y^{-\overline{\nu}})$, $\langle \nabla g^{\overline{\nu}}(y), \overline{q}^{\overline{\nu}} - y^{\overline{\nu}} \rangle \geq 0$.

(3) there exist $\overline{\mu} \in \{1,2,\dots,N\}$, and a strategy vector $\overline{p}^{\overline{\mu}} \in K_{\overline{\mu}}(x^{-\overline{\mu}})$ such that, for $x^{\overline{\mu}} \in K_{\overline{\mu}}(x^{-\overline{\mu}})$, $\langle \nabla f^{\overline{\mu}}(x), \overline{p}^{\overline{\mu}} - x^{\overline{\mu}} \rangle \geq 0$ and $y = Tx \in L$ is such that there exists $\overline{\nu} \in \{1,2,\dots,M\}$, and a strategy vector $\overline{q}^{\overline{\nu}} \in L_{\overline{\nu}}(y^{-\overline{\nu}})$ such that, for $y^{\overline{\nu}} \in L_{\overline{\nu}}(y^{-\overline{\nu}})$, $\langle \nabla g^{\overline{\nu}}(y), \overline{q}^{\overline{\nu}} - y^{\overline{\nu}} \rangle < 0$.

For Case 1, we consider the strategy vectors $h \in \mathbb{R}^n$ and $\overline{h} \in \mathbb{R}^m$ as

$$h := \begin{cases} h^\mu = \overline{p}^{\overline{\mu}},\ \overline{\mu} = \mu, \\ h^\mu = x^\mu,\ \overline{\mu} \neq \mu, \end{cases} \quad \text{and } \overline{h} := \begin{cases} \overline{h}^\nu = \overline{q}^{\overline{\nu}},\ \overline{\nu} = \nu, \\ \overline{h}^\nu = y^\nu,\ \overline{\nu} \neq \nu. \end{cases}$$

Then $h \in \Gamma_1(x)$ and $\overline{h} \in \Gamma_2(y)$. Now, we have

$$\begin{aligned} \langle F(x), h - x\rangle = \sum_{\mu=1}^{N} \langle \nabla f^\mu(x), h^\mu - x^\mu \rangle = \langle \nabla f^{\overline{\mu}}(x), \overline{p}^{\overline{\mu}} - x^{\overline{\mu}} \rangle \\ \text{and } y = Tx \in L \text{ such that} \\ \langle G(y), \overline{h} - y\rangle = \sum_{\nu=1}^{M} \langle \nabla g^\nu(y), \overline{h}^\nu - y^\nu \rangle = \langle \nabla g^{\overline{\nu}}(y), \overline{q}^{\overline{\nu}} - y^{\overline{\nu}} \rangle. \end{aligned} \tag{3.3}$$

The corresponding inequalities of (3.2) and (3.3) imply

$$\langle F(x), h-x\rangle < 0 \text{ and } y = Tx \in L \text{ satisfies } \langle G(y), \overline{h}-y\rangle < 0. \tag{3.4}$$

By using same techniques, we can find the following inequalities for the cases 2 and 3, respectively,

$$\langle F(x), h-x\rangle < 0 \text{ and } y = Tx \in L \text{ satisfies } \langle G(y), \overline{h}-y\rangle \geq 0, \tag{3.5}$$

$$\langle F(x), h-x\rangle \geq 0 \text{ and } y = Tx \in L \text{ satisfies } \langle G(y), \overline{h}-y\rangle < 0. \tag{3.6}$$

Inequalities (3.4), (3.5), and (3.6) contradict the fact that $x \in K$ is the solution to (SQVIP). Thus inequality (3.1) is validated. Further, for each μ and ν, the convexities of the functions f^μ and g^ν in the arguments x^μ and y^ν, respectively give the following

$$f^\mu(p^\mu, x^{-\mu}) - f^\mu(x^\mu, x^{-\mu}) \geq \langle \nabla f^\mu(x), p^\mu - x^\mu\rangle \;\forall\, p^\mu \in K_\mu(x^{-\mu}), \tag{3.7}$$

$$g^\nu(q^\nu, y^{-\nu}) - g^\nu(y^\nu, y^{-\nu}) \geq \langle \nabla g^\nu(y), q^\nu - y^\nu\rangle \;\forall\, q^\nu \in L_\nu(y^{-\nu}). \tag{3.8}$$

By combining (3.1), (3.7), and (3.8), we obtain that, for each $\mu = 1,2,\ldots,N$, $x^\mu \in K_\mu(x^{-\mu})$ satisfies $f^\mu(p^\mu, x^{-\mu}) - f^\mu(x^\mu, x^{-\mu}) \geq 0$ for all $p^\mu \in K_\mu(x^{-\mu})$ and $y = Tx \in L$ is such that, for all $\nu = 1,2,\ldots,M$, $y^\nu \in L_\nu(y^{-\nu})$ and $g^\nu(q^\nu, y^{-\nu}) - g^\nu(y^\nu, y^{-\nu}) \geq 0$ for all $q^\nu \in L_\nu(y^{-\nu})$, which implies that x is the split generalized Nash equilibrium. □

4. Existence of Equilibrium

In this section, we prove the existence of (SGNEP) by using the techniques of [22, 23] for proving existence results of quasi-variational inequality problems. Throughout this section, for better understanding of the strategy vectors x^μ for each player $\mu = 1,2,\ldots,N$ and y^ν for each player $\nu = 1,2,\ldots,M$, the subsets $K \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^m$ are, respectively, given as $K = \prod_{\mu=1}^N X_\mu$, $X_{-\mu} = \prod_{\hat{\mu}=1,\,(\hat{\mu}\neq\mu)}^N X_{\hat{\mu}}$, and $L = \prod_{\nu=1}^M Y_\nu$, $Y_{-\nu} = \prod_{\hat{\nu}=1,\,(\hat{\nu}\neq\nu)}^M Y_{\hat{\nu}}$, where $\{X_\mu\}_{\mu=1}^N$ and $\{Y_\nu\}_{\nu=1}^M$ are a family of nonempty, closed, and convex subsets with each $X_\mu \subset \mathbb{R}^{n_\mu}$ and $Y_\nu \subset \mathbb{R}^{m_\nu}$. With this notation, the strategy vectors of all N and M players $x \in K$ and $y \in L$ can be written as $x = (x^\mu, x^{-\mu}) \in X_\mu \times X_{-\mu}$ and $y = (y^\nu, y^{-\nu}) \in Y_\nu \times Y_{-\nu}$, respectively. Moreover, for all $x^{-\mu} \in X_{-\mu}$ and $y^{-\nu} \in Y_{-\nu}$, $K_\mu(x^{-\mu}) \subset X_\mu$, and $L_\nu(y^{-\nu}) \subset Y_\nu$, and for each $\hat{x}^\mu \in X_\mu$ and $\hat{y}^\nu \in Y_\nu$, $K_\mu^{-1}(\{\hat{x}^\mu\}) \subset X_{-\mu}$, and $L_\nu^{-1}(\{\hat{y}^\nu\}) \subset Y_{-\nu}$. By using these mathematical framework, it is easy to see that

$$\Gamma_1^{-1}(\{\hat{x}\}) = \bigcap_{\mu=1}^N [X_\mu \times K_\mu^{-1}(\{\hat{x}^\mu\})] \;\forall\, \hat{x} \in K \text{ and } \Gamma_2^{-1}(\{\hat{y}\}) = \bigcap_{\nu=1}^M [Y_\nu \times L_\nu^{-1}(\{\hat{y}^\nu\})] \;\forall\, \hat{y} \in L.$$

We consider that for each $\mu = 1,2,\ldots,N$, $\nu = 1,2,\ldots,N$, X_μ, Y_ν are compactly open and for all $\hat{x}^\mu \in X_\mu$, $\hat{y}^\nu \in Y_\mu$, the sets $K_\mu^{-1}(\{\hat{x}^\mu\})$ and $L_\nu^{-1}(\{\hat{y}^\nu\})$ are compactly open in $X_{-\mu}$ and $Y_{-\nu}$. Therefore, Remark (2.1) (b) and (c) imply that $\Gamma_1^{-1}(\{\hat{x}\})$ and $\Gamma_2^{-1}(\{\hat{y}\})$ are compactly open for all $\hat{x} \in K$ and $\hat{y} \in L$. Further, we also assume that the sets $A = \{x \in K : x \in \Gamma_1(x)\}$ and $\hat{A} = \{y \in L : y \in \Gamma_2(y)\}$ are compactly closed.

Theorem 4.1. *Let $x \in K$ and $y \in L$ be arbitrary strategy vectors, $F(x) = (\nabla f^\mu(x))_{\mu=1}^N$, $G(y) = (\nabla g^\nu(y))_{\nu=1}^M$, and, for each $\mu \in \{1,2,\ldots,N\}$, $\nu \in \{1,2,\ldots,M\}$, and given $x^{-\mu}$ and $y^{-\nu}$, the functions f^μ and g^ν be convex on the set K and L in the arguments x^μ and y^ν, respectively. Assume that there exists a nonempty, closed, and compact subset $D \subset K$ and $\hat{x} \in D$ such that $\langle \nabla f^\mu(\hat{x}), \hat{x} - z\rangle < 0 \;\forall\, z \in K \setminus D$ with $\hat{x} \in \Gamma_1(\hat{x})$, and there exists a nonempty, closed, and compact* *subset $\hat{D} \subset L$ and $\hat{y} \in \hat{D}$ such that $\langle \nabla g^{\nu}(\hat{y}), \hat{y} - \hat{z} \rangle < 0$ for all $\hat{z} \in L \setminus \hat{D}$ with $\hat{y} \in \Gamma_2(\hat{y})$. Then (SQVIP) has a solution.*

Proof. First, we define two set-valued maps $\Upsilon_1, \Upsilon_2 : K \to 2^K$ as follows: for each $x \in K$,

$$\Upsilon_1(x) = \{\hat{x} \in K : \sum_{\mu=1}^{N} \langle \nabla f^{\mu}(\hat{x}), \hat{x}^{\mu} - x^{\mu} \rangle < 0\}$$

and

$$\Upsilon_2(x) = \{\hat{x} \in K : \sum_{\mu=1}^{N} \langle \nabla f^{\mu}(x), \hat{x}^{\mu} - x^{\mu} \rangle < 0\}.$$

We also define two set-valued maps $\Omega_1, \Omega_2 : L \to 2^L$ as follows: for each $y \in L$,

$$\Omega_1(y) = \{\hat{y} \in L : \sum_{\nu=1}^{M} \langle \nabla g^{\nu}(\hat{y}), \hat{y}^{\nu} - y^{\nu} \rangle < 0\}$$

and

$$\Omega_2(y) = \{\hat{y} \in L : \sum_{\nu=1}^{M} \langle \nabla g^{\nu}(y), \hat{y}^{\nu} - y^{\nu} \rangle < 0\}.$$

By using the convexity of each function f^{μ} on the set K in the argument of X_{μ}, we obtain, for all x_1 and $x_2 \in K$,

$$f^{\mu}(x_1) - f^{\mu}(x_2) \geq \langle \nabla f^{\mu}(x_2), x_1^{\mu} - x_2^{\mu} \rangle. \tag{4.1}$$

By changing the variables x_1 and x_2 in the inequality (4.1), we have

$$f^{\mu}(x_2) - f^{\mu}(x_1) \geq \langle \nabla f^{\mu}(x_1), x_2^{\mu} - x_1^{\mu} \rangle. \tag{4.2}$$

By adding inequalities (4.1) and (4.2), we see that

$$\langle \nabla f^{\mu}(x_1), x_2^{\mu} - x_1^{\mu} \rangle \leq \langle \nabla f^{\mu}(x_2), x_2^{\mu} - x_1^{\mu} \rangle,$$

which also yields the following

$$\sum_{\mu=1}^{N} \langle \nabla f^{\mu}(x_1), x_2^{\mu} - x_1^{\mu} \rangle \leq \sum_{\mu=1}^{N} \langle \nabla f^{\mu}(x_2), x_2^{\mu} - x_1^{\mu} \rangle. \tag{4.3}$$

Similarly, by using the convexity of each function g^{ν} on the set L in the argument of Y_{ν}, we obtain, for all $y_1 = Tx_1 \in L$ and $y_2 = Tx_2 \in L$,

$$\sum_{\nu=1}^{M} \langle \nabla g^{\nu}(y_1), y_2^{\nu} - y_1^{\nu} \rangle \leq \sum_{\nu=1}^{M} \langle \nabla g^{\nu}(y_2), y_2^{\nu} - y_1^{\nu} \rangle. \tag{4.4}$$

Inequalities (4.3) and (4.4) imply that $\Upsilon_1(x) \subset \Upsilon_2(x)$ for all $x \in K$ and $\Omega_1(y) \subset \Omega_2(y)$ for all $y \in L$. Further, we define four more set-valued maps $S_1, T_1 : K \to 2^K$ and $S_2, T_2 : L \to 2^L$ as follows:

$$S_1(x) := \begin{cases} \Gamma_1(x) \cap \Upsilon_1(x), & \text{if } x \in A, \\ \Gamma_1(x), & \text{if } x \in K \setminus A, \end{cases} \quad T_1(x) := \begin{cases} \Gamma_1(x) \cap \Upsilon_2(x), & \text{if } x \in A, \\ \Gamma_1(x), & \text{if } x \in K \setminus A, \end{cases}$$

and

$$S_2(y) := \begin{cases} \Gamma_2(y) \cap \Omega_1(y), & \text{if } y \in \hat{A}, \\ \Gamma_2(y), & \text{if } y \in L \setminus \hat{A}, \end{cases} \quad T_2(y) := \begin{cases} \Gamma_2(y) \cap \Omega_2(y), & \text{if } y \in \hat{A}, \\ \Gamma_2(y), & \text{if } y \in L \setminus \hat{A}. \end{cases} \quad .$$

Evidently, the point images of the set-valued maps Γ_1 and Υ_2, that is, $\Gamma_1(x)$ and $\Upsilon_2(x)$ are convex for all $x \in K$, and the point images of the set-valued maps Γ_2 and Ω_2, that is, $\Gamma_2(y)$ and $\Omega_2(y)$ are convex for all $y \in L$. Therefore, the point images of the set-valued map T_1, that is, $T_1(x)$, and T_2, that is, $T_2(y)$ are also convex for all $x \in K$ and $y \in L$. Moreover, $S_1(x) \subset T_1(x)$ for all $x \in K$, and $S_2(y) \subset T_2(y)$ for all $y \in L$. Now, for all $\hat{x} \in K$, we have

$$\begin{aligned}
S_1^{-1}(\{\hat{x}\}) &= \{x \in K : \hat{x} \in S_1(x)\} \\
&= \{x \in A : \hat{x} \in \Gamma_1(x) \cap \Upsilon_1(x)\} \cup \{x \in K \setminus A : \hat{x} \in \Gamma_1(x)\} \\
&= [A \cap (\Gamma_1^{-1}(\{\hat{x}\}) \cap \Upsilon_1^{-1}(\{\hat{x}\}))] \cup [K \setminus A \cap \Gamma_1^{-1}(\{\hat{x}\})] \\
&= [(A \cap (\Gamma_1^{-1}(\{\hat{x}\}) \cap \Upsilon_1^{-1}(\{\hat{x}\}))) \cup K \setminus A] \\
&\cap [(A \cap (\Gamma_1^{-1}(\{\hat{x}\}) \cap \Upsilon_1^{-1}(\{\hat{x}\}))) \cup \Gamma_1^{-1}(\{\hat{x}\})] \\
&= [K \cap ((\Gamma_1^{-1}(\{\hat{x}\}) \cap \Upsilon_1^{-1}(\{\hat{x}\})) \cup K \setminus A)] \\
&\cap [(A \cup \Gamma_1^{-1}(\{\hat{x}\})) \cap (\Gamma_1^{-1}(\{\hat{x}\}) \cap \Upsilon_1^{-1}(\{\hat{x}\}))] \\
&= [(\Gamma_1^{-1}(\{\hat{x}\}) \cap \Upsilon_1^{-1}(\{\hat{x}\})) \cup K \setminus A] \cap \Gamma - 1^{-1}(\{\hat{x}\}) \\
&= (\Gamma_1^{-1}(\{\hat{x}\}) \cap \Upsilon_1^{-1}(\{\hat{x}\})) \cup (K \setminus A \cap \Gamma_1^{-1}(\{\hat{x}\})).
\end{aligned}$$

In the same way, we can prove that

$$S_2^{-1}(\{\hat{y}\}) = (\Gamma_2^{-1}(\{\hat{y}\}) \cap \Omega_1^{-1}(\{\hat{y}\})) \cup (L \setminus \hat{A} \cap \Gamma_2^{-1}(\{\hat{y}\})), \ \forall\, \hat{y} \in L.$$

Further, for each $\hat{x} \in K$ and $\hat{y} \in L$, the complement of $\Upsilon_1^{-1}(\{\hat{x}\})$ in K and $\Omega_1^{-1}(\{\hat{y}\})$ in L can be written as in following respective ways:

$$[\Upsilon_1^{-1}(\{\hat{x}\})]^c = \{x \in K : \sum_{\mu=1}^{N} \langle \nabla f^{\mu}(\hat{x}), \hat{x}^{\mu} - x^{\mu} \rangle \geq 0\}$$

and

$$[\Omega_1^{-1}(\{\hat{y}\})]^c = \{y \in L : \sum_{\nu=1}^{M} \langle \nabla g^{\nu}(\hat{y}), \hat{y}^{\nu} - y^{\nu} \rangle \geq 0\}.$$

It is evident that $[\Upsilon_1^{-1}(\{\hat{x}\})]^c$ is closed in K, and $[\Omega_1^{-1}(\{\hat{y}\})]^c$ is closed in L. Consequently, set $\Upsilon_1^{-1}(\{\hat{x}\})$ is open in K, and set $\Omega_1^{-1}(\{\hat{y}\})$ is open in L. Remark 2.1(a) implies that $\Upsilon_1^{-1}(\{\hat{x}\})$ is compactly open for all $\hat{x} \in K$, and $\Omega_1^{-1}(\{\hat{y}\})$ is compactly open for all $\hat{y} \in L$. We also note that, for all $\hat{x} \in K$, the sets $\Gamma_1^{-1}(\{\hat{x}\})$ and $K \setminus A$ are compactly open, and for all $\hat{y} \in L$, sets $\Gamma_2^{-1}(\{\hat{y}\})$ and $L \setminus \hat{A}$ are compactly open. Hence the sets $S_1^{-1}(\{\hat{x}\})$ and $S_2^{-1}(\{\hat{y}\})$ are now seen to also be compactly open for each $\hat{x} \in K$ and for each $\hat{y} \in L$, respectively.

Now, we claim that there exists a point $x^* \in A$ such that $\Gamma_1(x^*) \cap \Upsilon_1(x^*) = \phi$, and, for the point $y^* = Tx^* \in \hat{A}$, $\Gamma_2(y^*) \cap \Omega_1(y^*) = \phi$. To this end, we assume to the contrary. Thus we have following three cases

(1) for all $x^* \in A$, $\Gamma_1(x^*) \cap \Upsilon_1(x^*) \neq \phi$ and for all $y^* = Tx^* \in \hat{A}$, $\Gamma_2(y^*) \cap \Omega_1(y^*) \neq \phi$;
(2) for all $x^* \in A$, $\Gamma_1(x^*) \cap \Upsilon_1(x^*) \neq \phi$ and for all $y^* = Tx^* \in \hat{A}$, $\Gamma_2(y^*) \cap \Omega_1(y^*) = \phi$;
(3) for all $x^* \in A$, $\Gamma_1(x^*) \cap \Upsilon_1(x^*) = \phi$ and for all $y^* = Tx^* \in \hat{A}$, $\Gamma_2(y^*) \cap \Omega_1(y^*) \neq \phi$.

For the Case 1, the sets $\Gamma_1(x)$ and $\Gamma_2(y)$ are, respectively, nonempty and convex for all $x \in K$ and $y \in L$, $S_1(x) \neq \phi$ for each $x \in K$ and $S_2(y) \neq \phi$ for each $y \in L$. The hypotheses imply that there exist a nonempty, closed, and compact subset $D \subset K$ and a point $\hat{x} \subset D$ such that

$K \setminus D \subset S_1^{-1}(\{\hat{x}\})$, and there exist a nonempty, closed, and compact subset $\hat{D} \subset L$ and a point $\hat{y} \subset \hat{D}$ such that $L \setminus \hat{D} \subset S_2^{-1}(\{\hat{y}\})$. Thus, all the conditions of Theorem (2.1) are satisfied, so we conclude that there exists a point $z \in K$ such that $z \in T_1(z)$, and there exists a point $\hat{z} \in L$ such that $\hat{z} \in T_2(\hat{z})$. The definitions of the sets A and $\hat{A}$, and the set-valued maps T_1 and T_2 yield that $\{x \in K : x \in T_1(x)\} \subset A$ and $\{y \in L : y \in T_2(y)\} \subset \hat{A}$. Hence $z \in \Gamma_1(z) \cap \Upsilon_2(z)$ and $z \in A$, and $\hat{z} \in \Gamma_2(\hat{z}) \cap \Omega_2(\hat{z})$ and $\hat{z} \in \hat{A}$. Consequently, $z \in \Upsilon_2(z)$ and $\hat{z} \in \Omega_2(\hat{z})$ give the following inequalities

$$\sum_{\mu=1}^{N} \langle \nabla f^{\mu}(z), z^{\mu} - z^{\mu} \rangle < 0 \text{ and } \sum_{\nu=1}^{M} \langle \nabla g^{\nu}(\hat{z}), \hat{z}^{\nu} - \hat{z}^{\nu} \rangle < 0, \text{ respectively.}$$

which are impossible. By using the same techniques, we can find the contradiction $\sum_{\mu=1}^{N} \langle \nabla f^{\mu}(z), z^{\mu} - z^{\mu} \rangle < 0$ for Case 2, and $\sum_{\nu=1}^{M} \langle \nabla g^{\nu}(\hat{z}), \hat{z}^{\nu} - \hat{z}^{\nu} \rangle < 0$ for Case 3. The contradiction proves our claim, i.e., there exists $x^* \in A$ such that

$$\sum_{\mu=1}^{N} \langle \nabla f^{\mu}(x), x^{\mu} - x^{*\mu} \rangle \geq 0, \forall x \in \Gamma_1(x^*),$$

and for $y^* = Tx^* \in \hat{A}$,

$$\sum_{\nu=1}^{M} \langle \nabla g^{\nu}(y), y^{\nu} - y^{*\nu} \rangle \geq 0, \ \forall\, y \in \Gamma_2(y^*).$$

By using the convexity of $K_{\mu}(x^{-*\mu})$, $\Gamma_1(x^*)$, and $L_{\nu}(y^{-*\nu})$, $\Gamma_2(y^*)$, we can rewrite the above inequality as follows:

$$\sum_{\mu=1}^{N} \langle \nabla f^{\mu}(x^* + \lambda(x - x^*)), x^{\mu} - x^{*\mu} \rangle \geq 0, \ \forall\, x \in \Gamma_1(x^*) \text{ and } \lambda \in [0,1]$$

and for $y^* = Tx^* \in \hat{A}$,

$$\sum_{\nu=1}^{M} \langle \nabla g^{\nu}(y^* + \lambda(y - y^*), y^{\nu} - y^{*\nu} \rangle \geq 0, \ \forall\, y \in \Gamma_2(y^*) \text{ and } \lambda \in [0,1].$$

Since $\nabla f^{\mu}(.)$ and $\nabla g^{\nu}(.)$ are hemicontinuous, by taking the limit $\lambda \to 0^+$ in the above inequality, we obtain

$$\sum_{\mu=1}^{N} \langle \nabla f^{\mu}(x^*), x^{\mu} - x^{*\mu} \rangle \geq 0, \ \forall\, x \in \Gamma_1(x^*)$$

and for $y^* = Tx^* \in \hat{A}$,

$$\sum_{\nu=1}^{M} \langle \nabla g^{\nu}(y^*), y^{\nu} - y^{*\nu} \rangle \geq 0, \ \forall\, y \in \Gamma_2(y^*).$$

By using the hypotheses, the above inequality can be rewritten as

$$x^* \in A \text{ such that } x^* \in \Gamma_1(x^*) \text{ and } \langle F(x^*), x - x^* \rangle \geq 0, \ \forall\, x \in \Gamma_1(x^*),$$

$$\text{and } y^* = Tx^* \in \Gamma_2(y^*) \text{ satisfies } \langle g(y^*), y - y^* \rangle \geq 0, \ \forall\, y \in \Gamma_2(y^*).$$

Therefore, x^* is a solution to (SQVIP). □

5. A Motivational Example

In this section, we demonstrate that how (SGNEP) can be applied to a river basin pollution problem [24]. For the investigations of the river basin pollution problem, we refer to [25, 26, 27]. We assume that a city is situated along the two sides of a river, and N and M industrial factories (paper mills, chemical factories, pharmaceutical manufacturing companies, and so on) are located along each side of the river, respectively. In the sequel, we call them industrial agents. Nowadays, it is a very common scenario that industrial factories often dump waste garbage, such as dirty water, used chemicals and oils, sewage, and cafeteria waste, directly into a community water source (river, lake or stream). Waste dumped contains several contaminants which mix and create pollution concentration along the river. We assume that m basin authorities (monitoring stations) are located along the river. Each basin authority is empowered to set a maximum pollutant concentration level which we denote by $\chi_s \in \mathbb{R}_+$, where $s = 1,2,\ldots,m$. Further, let $e^\mu, \hat{e}^\nu \in \mathbb{R}_+$ be the pollutant emission coefficient for the industrial agent μ and ν, and $x^\mu, y^\nu \in \mathbb{R}_+$ be the chosen emitted pollutant concentration level by the industrial agent $\mu = 1,2,\ldots,N$ and $\nu = 1,2,\ldots,M$, respectively. Let $x^{-\mu} \in \mathbb{R}_+^{N-1}$ be the chosen emitted pollutant concentration level by the all industrial agents of one side of river except the agent μ, and $y^{-\nu} \in \mathbb{R}_+^{M-1}$ be the chosen emitted pollutant concentration level by the all industrial agents of another side of river except the agent ν. Let $x = (x^\mu, x^{-\mu}) \in \mathbb{R}_+^N$ be the chosen emitted pollutant concentration level by the all industrial agents of one side and $y = (y^\nu, y^{-\nu}) \in \mathbb{R}_+^M$ be the chosen emitted pollutant concentration level by the all industrial agents of another side. Waste materials, dumped by the industrial agents into the river, spread, decay and then finally reach the basin authorities. Thus, the amount of pollution received by the basin authority $s = 1,2,\ldots,m$ by the industrial agents of one side of river is $\sum_{\mu=1}^{N} \delta_s^\mu e^\mu x^\mu$, where δ_s^μ is the decay-and-transportation coefficient from the agent μ to the monitoring station s and by the industrial agents of another side of river is $\sum_{\nu=1}^{M} \delta_s^\nu \hat{e}^\nu y^\nu$, where δ_s^ν is the decay-and-transportation coefficient from the agent ν to the monitoring station s. The basin authorities impose constraints on the pollution, so that industrial agents control their pollutant emission into the river. Thus, the pollution constraint imposed by the authority s on the industrial agents of both sides of river are given by,

$$\sum_{\mu=1}^{N} \delta_s^\mu e^\mu x^\mu \leq \chi_s \text{ and } \sum_{\nu=1}^{M} \delta_s^\nu \hat{e}^\nu y^\nu \leq \chi_s, \text{ for } s = 1,2,\ldots,m, \text{ respectively.}$$

The nonempty set of entire feasible pollution concentration levels of industrial agents of both sides of river are given by

$$K = \{x \in \mathbb{R}_+^N \colon \sum_{\mu=1}^{N} \delta_s^\mu e^\mu x^\mu \leq \chi_s \text{ for } s = 1,2,\ldots,m\} \text{ and}$$

$$L = \{y \in \mathbb{R}_+^M \colon \sum_{\nu=1}^{M} \delta_s^\nu \hat{e}^\nu y^\nu \leq \chi_s \text{ for } s = 1,2,\ldots,m\}, \text{ respectively.}$$

We bear in the mind that the industrial agents of both sides of river are dependent on each other, at least because of the finiteness of the amount of dumping pollutants into the river. Therefore, for any given $x^{-\mu}$ and $y^{-\nu}$, the nonempty, closed, and convex feasible pollution concentration level set of each industrial agent μ and ν are denoted by $K_\mu(x^{-\mu})$ and $L_\nu(y^{-\nu})$, respectively, and these are defined as $K_\mu(x^{-\mu}) = \{x^\mu \in \mathbb{R}_+ : (x^\mu, x^{-\mu}) \in K\}$ and $L_\nu(y^{-\nu}) = \{y^\nu \in \mathbb{R}_+ : (y^\nu, y^{-\nu}) \in L\}$. Each agent of both sides of river wishes to maximize its profit, where the profit of industrial agent μ is defined by the difference between the revenue $[p_1 - p_2 \sum_{\mu=1}^{N} x^\mu]x^\mu$ and the cost $[a^\mu + b^\mu x^\mu]x^\mu$. The profit of industrial agent ν is defined by the difference between the revenue $[q_1 - q_2 \sum_{\nu=1}^{M} y^\nu]y^\nu$ and the cost $[c^\nu + d^\nu y^\nu]y^\nu$, where p_1, p_2, q_1, and q_2 are the economic constants which follow the inverse demand law and $a^\mu, b^\mu, c^\nu, d^\nu \in \mathbb{R}_+$ are the cost coefficient functions. Now, for given $x^{-\mu}$ and $y^{-\nu}$, the aim of the industrial agents μ and ν of each side of river is to choose an emitted pollutant concentration level $x^\mu \in K_\mu(x^{-\mu})$ and $y^\nu \in L_\nu(y^{-\nu})$ such that it solves the following split optimization problem

$$\max_{x^\mu} \left[\left\{ \left(p_1 - p_2 \sum_{\mu=1}^{N} x^\mu \right) x^\mu \right\} - \{a^\mu + b^\mu x^\mu\} x^\mu \right]$$

and the vector $y = Tx \in L$ solves

$$\max_{y^\nu} \left[\left\{ \left(q_1 - q_2 \sum_{\nu=1}^{M} y^\nu \right) y^\nu \right\} - \{c^\nu + d^\nu y^\nu\} y^\nu \right].$$

An equilibrium of the above defined split optimization problem is a split generalized Nash equilibrium in the sense of our (SGNEP), where

$$f^\mu(x^\mu, x^{-\mu}) = \left[\{a^\mu + b^\mu x^\mu\} x^\mu - \left\{ \left(p_1 - p_2 \sum_{\mu=1}^{N} x^\mu \right) x^\mu \right\} \right],$$

$$\text{and } g^\nu(y^\nu, y^{-\nu}) = \left[\{c^\nu + d^\nu y^\nu\} y^\nu - \left\{ \left(q_1 - q_2 \sum_{\nu=1}^{M} y^\nu \right) y^\nu \right\} \right].$$

6. Conclusion

This paper is concerned with the areas of non-cooperative strategic games and split inverse problems. We formulated a split generalized Nash equilibrium problem and a split quasi-variational inequality problem. We proved an equivalence between these two problems as well as the existence of equilibria. Furthermore, as an application of our split generalized Nash equilibrium problem, we reformulated a river basin pollution problem in the terms of such a problem. The model developed in this paper provides a foundation for our future studies that attempt to test the numerical experiments of the above mentioned motivational example.

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