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A NEW RELAXED INERTIAL FORWARD-BACKWARD-FORWARD METHOD FOR SOLVING THE CONVEX MINIMIZATION PROBLEM WITH APPLICATIONS TO IMAGE INPAINTING

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Abstract. The purpose of this paper is to study the convex minimization problem in real Hilbert spaces. We introduce a new relaxed forward-backward-forward method by using the inertial technique and the adaptive stepsize. We then provide its convergence theorem under suitable assumptions. Finally, we analyse the proposed method to image inpainting and give comparisons of our method with other methods in the literature.

Keywords. Adaptive stepsize; Convex minimization problem; Forward-backward-forward method; Inertial method; Weak convergence.

1. Introduction

In recent years, much attention has been devoted to the convex minimization problem which can be applied to image processing, signal recovery, support machines classification, and so on; see, e.g., [1, 2, 3, 4, 5] and the references therein.

Let H be a real Hilbert space. Let $g : H \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function, and ∂g denotes its subdifferential. Let $f : H \to \mathbb{R}$ be a convex and differentiable function with the Lipschitz continuous gradient, denoted by ∇f. The convex minimization problem is formulated as follows:

$$\text{find a point } x^* \in H \text{ such that } 0 \in (\partial g + \nabla f)(x^*). \tag{1.1}$$

Recently, efficient iterative methods have been introduced and investigated for solving (1.1) in various spaces; see, e.g., [6, 7, 8, 9, 10, 11, 12] and the references therein. One of the methods is the forward-backward (FB) method, which is defined by: $x_1 \in H$ is an initial and

$$x_{n+1} = prox_{\lambda_n g}(x_n - \lambda_n \nabla f(x_n)),$$

where $prox_g$ is the proximal operator (see below) of g and λ_n is a positive stepsize chosen in $(0, 2/L)$, where L is the Lipschitz constant of ∇f.

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Alvarez and Attouch [13] introduced the following inertial proximal algorithm. Let $x_0 = x_1$ is chosen arbitrarily, define

$$x_{n+1} = prox_{\lambda_n g}(x_n + \theta_n(x_n - x_{n-1})),$$

where $\{\lambda_n\}$ is a sequence in $(0,\infty)$ and $\{\theta_n\}$ is a sequence in $[0,\infty)$. Polyak [14] proposed an inertial extrapolation as an acceleration process to solve the convex minimization problem. In 2005, Combettes and Wajs [15] introduced the following relaxed forward-backward method (RFB). Let $\varepsilon \in (0, \min\{1, \frac{1}{L}\})$ and $x_0 \in H$, and define

$$\begin{aligned} y_n &= x_n - \lambda_n \nabla f(x_n) \\ x_{n+1} &= x_n + \alpha_n(prox_{\lambda_n g} y_n - x_n), \end{aligned}$$

where $\lambda_n \in [\varepsilon, \frac{2}{L} - \varepsilon]$, $\alpha_n \in [\varepsilon, 1]$, and L is the Lipschitz constant of the gradient of ∇f.

In the spirit of Nesterov [16], Cruz and Nghia [17] proposed a fast multistep forward-backward method (MFB) with linesearch. Let $\Omega = \mathrm{dom} g$. Take $x_0 = x_1 \in \mathrm{dom} g$, $t_0 = 1$, $\sigma > 0$, $\delta \in (0, \frac{1}{2})$, and $\gamma \in (0,1)$:

$$\begin{aligned} t_{n+1} &= \frac{1 + \sqrt{1 + 4t_{n-1}^2}}{2} \\ \theta_n &= \frac{t_{n-1} - 1}{t_n} \\ y_n &= x_n + \theta_n(x_n - x_{n-1}) \\ z_n &= P_\Omega(y_n) \\ x_{n+1} &= prox_{\lambda_n g}(z_n - \lambda_n \nabla f(z_n)) \end{aligned}$$

where $\lambda_n = \sigma \gamma^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\lambda_n \|\nabla f(x_{n+1}) - \nabla f(z_n)\| \le \delta \|x_{n+1} - z_n\|.$$

Tseng [18] proposed the forward-backward-forward method (FBF), which is generated by $x_1 \in H$ and

$$\begin{aligned} y_n &= prox_{\lambda_n g}(x_n - \lambda_n \nabla f(x_n)) \\ x_{n+1} &= y_n - \lambda_n(\nabla f(y_n) - \nabla f(x_n)), \end{aligned}$$

where $\lambda_n \in (0, 1/L)$.

Motivated by the previous works, we introduce a new forward-backward-forward method. We use the adaptive stepsize and the inertial technique in our method and also obtain a convergence theorem for the proposed algorithm. Finally, we present numerical experiments to illustrate an application to image inpainting. Some comparisons to other methods are also given to demonstrate the efficiency of our method.

2. Basic Definitions and Lemmas

In this section, we recall some basic definitions and lemmas. Let H be a real Hilbert space. The symbols $\rightharpoonup$ and $\to$ are borrowed to denote weak and strong convergence, respectively. The following equality is trivial but useful in Hilbert spaces:

$$\|\beta x + (1-\beta)y\|^2 = \beta\|x\|^2 + (1-\beta)\|y\|^2 - \beta(1-\beta)\|x-y\|^2, \ \forall x, y \in H, \tag{2.1}$$

where $\beta \in (0,1)$.

Recall that a mapping $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$.

The orthogonal projection P_Ω from H onto a nonempty, closed, and convex subset $\Omega \subset H$ is defined by $P_\Omega x := \arg\min_{y\in\Omega} \|x - y\|^2$ for all $x \in H$.

Lemma 2.1. [19] *Let Ω be a nonempty, closed, and convex subset of a real Hilbert space H. Then, for any $x \in H$, the following assertions hold:*

(1) $\langle x - P_\Omega x, z - P_\Omega x\rangle \leq 0$ *for all* $z \in \Omega$*;*
(2) $\|P_\Omega x - P_\Omega y\|^2 \leq \langle P_\Omega x - P_\Omega y, x - y\rangle$ *for all* $x, y \in H$*;*
(3) $\|P_\Omega x - z\|^2 \leq \|x - z\|^2 - \|P_\Omega x - x\|^2$ *for all* $z \in \Omega$*.*

Let $g : H \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, and its domain is denoted by $\mathrm{dom} g = \{x \in H | g(x) < +\infty\}$. For any $x \in \mathrm{dom} g$, the subdifferential of g at x is defined by

$$\partial g(x) = \{v \in H | \langle v, y - x\rangle \leq g(y) - g(x),\ y \in H\}.$$

Recall that the proximal operator $\mathrm{prox}_g : \mathrm{dom}(g) \to H$ is given by

$$\mathrm{prox}_g(x) = (I + \partial g)^{-1}(z)$$

for all $z \in H$. It is known that the proximal operator is single-valued. Moreover,

$$\frac{z - \mathrm{prox}_{\lambda g}(z)}{\lambda} \in \partial g(\mathrm{prox}_{\lambda g}(z)) \quad \text{for all } z \in H,\ \lambda > 0.$$

Definition 2.1. Let S be a nonempty subset of H. A sequence $\{x_n\}$ in H is said to be quasi-Fejér convergent to S if and only if, for all $x \in S$, there exists a positive sequence $\{\varepsilon_n\}$ with $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$ and $\|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 + \varepsilon_n$ for all $n \geq 1$. If $\{\varepsilon_n\}$ is a null sequence, we say that $\{x_n\}$ is Fejér convergent to S.

Lemma 2.2. [6] *The subdifferential operator ∂g is maximal monotone. Moreover, the graph of ∂g, $\mathrm{Gph}(\partial g) = \{(x, v) \in H \times H : v \in \partial g(x)\}$ is demiclosed, i.e., if the sequence $\{(x_n, v_n)\} \subset \mathrm{Gph}(\partial g)$ satisfies conditions that $\{x_n\}$ converges weakly to x and $\{v_n\}$ converges strongly to v, then $(x, v) \in \mathrm{Gph}(\partial g)$.*

Lemma 2.3. [20] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real positive sequences such that $a_{n+1} \leq (1 + c_n)a_n + b_n$. If $\Sigma_{n=1}^{\infty} c_n < +\infty$ and $\Sigma_{n=1}^{\infty} b_n < +\infty$, then $lim_{n\to+\infty} a_n$ exists.*

Lemma 2.4. [21] *Let $\{a_n\}$ and $\{\theta_n\}$ be real positive sequences such that $a_{n+1} \leq (1 + \theta_n)a_n + \theta_n a_{n-1}$, $n \geq 1$. Then, $a_{n+1} \leq K \cdot \prod_{i=1}^{n}(1 + 2\theta_i)$, where $K = \max\{a_1, a_2\}$. Moreover, if $\sum_{n=1}^{\infty} \theta_n < +\infty$, then $\{a_n\}$ is bounded.*

Lemma 2.5. [9, 19] *If $\{x_n\}$ is quasi-Fejér convergent to S, then*

(i) $\{x_n\}$ is bounded.
(ii) If all weak accumulation points of $\{x_n\}$ is in S, then $\{x_n\}$ weakly converges to a point in S.

3. MAIN RESULTS

In this section, we introduce an algorithm using the inertial extrapolation and the adaptive stepsize. we assume that the solution set of convex minimization problem (1.1) is nonempty, i.e., $S = \operatorname{argmin}(f+g) \neq \emptyset$.

We next introduce a relaxed inertial forward-backward-forward method for solving (1.1).

Algorithm 3.1. Inertial modified relaxed forward-backward-forward method (IMRFB)
Initialization: Let $x_0 = x_1 \in H$, $\lambda_1 > 0$, $\rho_1 \in (0,1)$, $\mu \in (0,1)$, and $\theta_1 \geq 0$.
Iterative Step: Let Ω be a nonempty, closed, and convex subset of H. Given $n \geq 1$, calculate x_{n+1} as follows:
Step 1. Compute the inertial step:

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2. Compute the forward-backward-forward step:

$$\begin{aligned} y_n &= prox_{\lambda_n g}(w_n - \lambda_n \nabla f(w_n)), \\ z_n &= (1-\rho_n)w_n + \rho_n(y_n - \lambda_n(\nabla f(y_n) - \nabla f(w_n))), \end{aligned}$$

Step 3. Compute the projection step:

$$x_{n+1} = P_\Omega(z_n).$$

Step 4. Compute the stepsize step:

$$\lambda_{n+1} = \begin{cases} \min\{\lambda_n, \frac{\mu\|y_n - w_n\|}{\|\nabla f(y_n) - \nabla f(w_n)\|}\} & if\ \nabla f(y_n) - \nabla f(w_n) \neq 0; \\ \lambda_n & otherwise. \end{cases} \tag{3.1}$$

Set $n = n+1$ and return to **Step 1**.

Using the proof line as in [22], we obtain the following lemma.

Lemma 3.1. *Let $\mu \in (0,1)$ and $\lambda_1 > 0$. The sequence $\{\lambda_n\}$ generated by (3.1) is nonincreasing and*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\{\lambda_1, \frac{\mu}{L}\}.$$

Hence,

$$\|\nabla f(y_n) - \nabla f(w_n)\| \leq \frac{\mu}{\lambda_{n+1}}\|y_n - w_n\|. \tag{3.2}$$

Theorem 3.1. *Let $\{x_n\}$ be generated by Algorithm 3.1. If $\sum_{n=1}^{\infty}\theta_n < +\infty$ and $0 < \liminf_{n\to\infty}\rho_n \leq \limsup_{n\to\infty}\rho_n < 1$, then the sequence $\{x_n\}$ weakly converges to an element of S.*

Proof. Let $x^* \in S$. Then, by Lemma 2.1(3), we have

$$\|x_{n+1} - x^*\|^2 = \|P_\Omega(z_n) - x^*\|^2 \leq \|z_n - x^*\|^2 - \|P_\Omega(z_n) - z_n\|^2. \tag{3.3}$$

Using (2.1) and setting $v_n = y_n - \lambda_n(\nabla f(y_n) - \nabla f(w_n))$, we obtain

$$\begin{aligned} \|z_n - x^*\|^2 &= \|(1-\rho_n)(w_n - x^*) + \rho_n(v_n - x^*)\|^2 \\ &= (1-\rho_n)\|w_n - x^*\|^2 + \rho_n\|v_n - x^*\|^2 - \rho_n(1-\rho_n)\|v_n - w_n\|^2. \end{aligned} \tag{3.4}$$

From definition of v_n and (3.2), we have

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|w_n - y_n\|^2 + \|y_n - v_n\|^2 + \|v_n - x^*\|^2 + 2\langle w_n - y_n, y_n - v_n\rangle \\
&\quad + 2\langle y_n - v_n, v_n - x^*\rangle + 2\langle v_n - x^*, w_n - y_n\rangle \\
&= \|w_n - y_n\|^2 + \|y_n - v_n\|^2 + \|v_n - x^*\|^2 + 2\langle w_n - y_n, y_n - x^*\rangle \\
&\quad + 2\langle y_n - v_n, v_n - y_n + y_n - x^*\rangle \\
&= \|w_n - y_n\|^2 + \|y_n - v_n\|^2 + \|v_n - x^*\|^2 + 2\langle w_n - y_n, y_n - x^*\rangle \\
&\quad - 2\langle y_n - v_n, y_n - v_n\rangle + 2\langle y_n - v_n, y_n - x^*\rangle \\
&= \|w_n - y_n\|^2 - \|y_n - v_n\|^2 + \|v_n - x^*\|^2 + 2\langle w_n - v_n, y_n - x^*\rangle \\
&= \|w_n - y_n\|^2 - \lambda_n^2\|\nabla f(y_n) - \nabla f(w_n)\|^2 + \|v_n - x^*\|^2 \\
&\quad + 2\langle w_n - v_n, y_n - x^*\rangle \\
&\geq \|w_n - y_n\|^2 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2 + \|v_n - x^*\|^2 + 2\langle w_n - v_n, y_n - x^*\rangle.
\end{aligned}$$

It follows that

$$\|v_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \|w_n - y_n\|^2 + \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2 - 2\langle w_n - v_n, y_n - x^*\rangle. \tag{3.5}$$

Since $(I - \lambda_n \nabla f)(w_n) \in (I - \lambda_n \partial g)(y_n)$, we have

$$\begin{aligned}
w_n &\in y_n + \lambda_n \partial g(y_n) + \lambda_n \nabla f(w_n) \\
&= y_n - \lambda_n(\nabla f(y_n) - \nabla f(w_n)) + \lambda_n(\partial g + \nabla f)(y_n) \\
&= v_n + \lambda_n(\partial g + \nabla f)(y_n).
\end{aligned}$$

Hence,

$$\frac{1}{\lambda_n}(w_n - v_n) \in (\partial g + \nabla f)(y_n).$$

This, together with $0 \in (\partial g + \nabla f)(x^*)$ and the monotonicity of $\partial g + \nabla f$, implies

$$\langle w_n - v_n, y_n - x^*\rangle \geq 0. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \|w_n - x^*\|^2 - \|w_n - y_n\|^2 + \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\|w_n - y_n\|^2 \\
&= \|w_n - x^*\|^2 - \left(1 - \frac{\lambda_n^2\mu}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2.
\end{aligned} \tag{3.7}$$

From (3.4) and (3.7), we have

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq (1-\rho_n)\|w_n - x^*\|^2 + \rho_n\|w_n - x^*\|^2 - \rho_n\left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2 \\
&\quad - \rho_n(1-\rho_n)\|v_n - w_n\|^2 \\
&= \|w_n - x^*\|^2 - \left(1 - \frac{\lambda_n^2\mu}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2 - \rho_n(1-\rho_n)\|v_n - w_n\|^2.
\end{aligned} \tag{3.8}$$

This shows that

$$\|z_n - x^*\| \leq \|w_n - x^*\|. \tag{3.9}$$

From (3.3) and (3.9), we also have

$$\|x_{n+1} - x^*\| \quad \leq \quad \|z_n - x^*\| \leq \|w_n - x^*\|.$$

Hence,

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|w_n - x^*\| \\ &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\| \\ &\leq \|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\| \\ &\leq \|x_n - x^*\| + \theta_n(\|x_n - x^*\| + \|x_{n-1} - x^*\|). \end{aligned} \tag{3.10}$$

Therefore

$$\|x_{n+1} - x^*\| \leq (1+\theta_n)\|x_n - x^*\| + \theta_n\|x_{n-1} - x^*\|.$$

By Lemma 2.4, we conclude that

$$\|x_{n+1} - x^*\| \leq K\prod_{i=1}^{n}(1+2\theta_i)$$

where $K = \max\{\|x_1 - x^*\|, \|x_2 - x^*\|\}$. Since $\sum_{n=1}^{\infty}\theta_n < +\infty$ and $\{x_n\}$ is bounded, we have $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\| < +\infty$. By Lemma 2.3 and (3.10), we have $\lim_{n\to\infty}\|x_n - x^*\|$ exists.

Next, we consider

$$\begin{aligned} \|w_n - x^*\|^2 &= \|x_n - x^*\|^2 + 2\theta_n\langle x_n - x^*, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - x^*\|^2 + 2\theta_n\|x_n - x^*\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.11}$$

From (3.3), (3.8), and (3.11), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leq\ & \|x_n - x^*\|^2 + 2\theta_n\|x_n - x^*\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\ & -\rho_n\left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2 - \rho_n(1-\rho_n)\|v_n - w_n\|^2 - \|x_{n+1} - z_n\|^2, \end{aligned}$$

which yields that $\|w_n - y_n\| \to 0$, $\|v_n - w_n\| \to 0$, and $\|x_{n+1} - z_n\| \to 0$ as $n \to \infty$. From definition of w_n, we see that $\|x_n - w_n\| \to 0$ as $n \to \infty$. Since ∇f is uniformly continuous, we obtain

$$\lim_{n\to\infty}\|\nabla f(w_n) - \nabla f(y_n)\| = 0. \tag{3.12}$$

From definition of v_n and (3.12), we have

$$\|y_n - v_n\| = \lambda_n\|\nabla f(y_n) - \nabla f(w_n)\| \to 0. \tag{3.13}$$

On the other hand, we see that

$$\|z_n - y_n\|^2 = (1-\rho_n)\|w_n - y_n\|^2 + \rho_n\|v_n - y_n\|^2 - \rho_n(1-\rho_n)\|w_n - v_n\|^2.$$

Since $\|w_n - y_n\| \to 0$, $\|v_n - w_n\| \to 0$ and (3.13), it follows that $\|z_n - y_n\| \to 0$ as $n \to \infty$. Also, we have $\|x_n - y_n\| \leq \|x_n - w_n\| + \|w_n - y_n\| \to 0$ as $n \to \infty$, and

$$\|z_n - x_n\| \leq \|z_n - y_n\| + \|y_n - x_n\| \to 0. \tag{3.14}$$

Hence, we obtain

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - x_n\| \to 0 \tag{3.15}$$

as $n \to \infty$. Since the sequence $\{x_n\}$ is bounded, the set of its weak accumulation points is nonempty. Take any weak accumulation point $\bar{x}$ of the sequence y_n. Since $\|x_n - y_n\| \to 0$, there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ weakly converging to $\bar{x}$. From $y_{n_k} = prox_{\lambda_{n_k} g}(w_{n_k} - \lambda_{n_k} \nabla f(w_{n_k}))$, it follows that

$$\frac{w_{n_k} - \lambda_{n_k} \nabla f(w_{n_k}) - y_{n_k}}{\lambda_{n_k}} \in \partial g(y_{n_k}).$$

This implies that

$$\frac{w_{n_k} - y_{n_k}}{\lambda_{n_k}} - \nabla f(w_{n_k}) + \nabla f(y_{n_k}) \in \partial g(y_{n_k}) + \nabla f(y_{n_k}). \tag{3.16}$$

Letting $k \to \infty$ in (3.16), we obtain by Lemma 2.5 $0 \in (\partial g + \nabla f)(\bar{x})$. Thus $\bar{x} \in \operatorname{argmin}(f+g)$.

Next, we demonstrate that $\bar{x} \in \Omega$. Since P_Ω is nonexpansive, by (3.14) and (3.15), we have

$$\begin{aligned} \|P_\Omega(x_n) - x_n\| &\leq \|P_\Omega(x_n) - P_\Omega(z_n)\| + \|P_\Omega(z_n) - x_n\| \\ &\leq \|x_n - z_n\| + \|x_{n+1} - x_n\| \to 0 \end{aligned}$$

as $n \to \infty$. Hence, by the demiclosedness of P_Ω, we obtain $\bar{x} \in \Omega$. By Lemma 2.5, we conclude that sequence $\{x_n\}$ weakly converges to a point in S. This completes the proof. □

4. Numerical Experiments in Image Inpainting

In this section, we present numerical experiments that support our main result. We aim to apply our result to an image inpainting problem, which is the following minimization:

$$\min_{x \in \mathbb{R}^{M \times N}} \frac{1}{2} \|A(x - x_0)\|_F^2 + \tau \|x\|_* \tag{4.1}$$

where $x_0 \in \mathbb{R}^{M \times N} (M < N)$, A is a linear map that selects a subset of the entries of an $M \times N$ matrix by setting each unknown entry in the matrix to 0, x is matrix of known entries $A(x_0)$, and $\tau > 0$ is regularization parameter.

In particular, we aim to solve the following image inpainting problem [23, 24]:

$$\min_{x \in \mathbb{R}^{M \times N}} \frac{1}{2} \|P_\Omega(x) - P_\Omega(x_0)\|_F^2 + \tau \|x\|_* \tag{4.2}$$

where $\|\cdot\|_F$ is the Frobenius matrix norm, and $\|\cdot\|_*$ is the nuclear matrix norm. Define P_Ω as follows:

$$P_\Omega(x) = \begin{cases} x_{ij}, & (i,j) \in \Omega \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

Image inpainting problem (4.2) is problem (1.1) when $f(x) = \frac{1}{2}\|P_\Omega(x) - P_\Omega(x_0)\|_F^2$ and $g(x) = \tau \|x\|_*$. We know that $\nabla f(x) = P_\Omega(x) - P_\Omega(x_0)$ is 1-Lipschitz continuous and $prox_g$ is obtained by the singular value decomposition (SVD) [25].

To measure the quality of images, we use the signal-to-noise ratio (SNR) and the structural similarity index (SSIM) [26] which are defined by:

$$\text{SNR} = 20 \log \frac{\|x\|_F}{\|x - x_r\|_F} \tag{4.4}$$

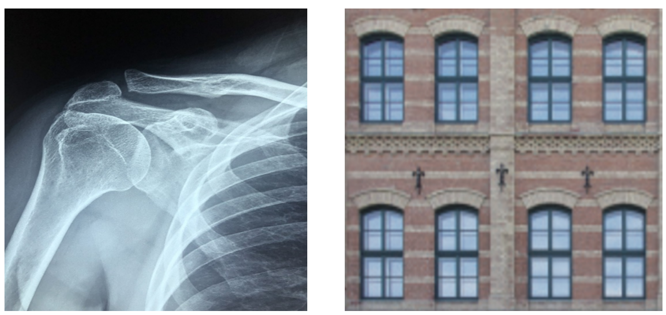
and

$$\text{SSIM} = \frac{(2a_x a_{x_r} + c_1)(2\sigma_{xx_r} + c_2)}{(a_x^2 + a_{x^r}^2 + c_1)(\sigma_x^2 + \sigma_{x_r}^2 + c_2)} \tag{4.5}$$

where x is the original image, x_r is the restored image, a_x and a_{x_r} are the mean values of the original image a and restored image x_r, respectively, σ_x^2 and $\sigma_{x_r}^2$ are the variances, $\sigma_{xx_r}^2$ is the covariance of two images, $c_1 = (0.01L)^2$ and $c_2 = (0.03L)^2$, and L is the dynamic range of pixel values. SSIM ranges from 0 to 1, and 1 means perfect recovery.

Next, we present the performance of IRFBF and its comparison to the projected version of RFB and MFB. In all tests, the starting point $x_0 = x_1 = (0,0,0,...,0) \in \mathbb{R}^N$. Set
$\lambda_n = 1/\|A\|^2, \alpha_n = 0.09$ for RFB;
$\sigma = 0.1, \delta = 0.2, \gamma = 0.5$ for MFB;
$\lambda_1 = 0.2, \mu = 0.2, \rho = 2$ for IRFBF.

We test two images. For the first one, we use x-ray image with size 700×525 (see Figure 1(a)). For the second one, we use windows image with size 466×572 (see Figure 1(b)).



(A) x-ray image (B) Windows image

FIGURE 1. The original images

The maximum number of iterations was set to be 300th. All codes are written in Matlab (version R2020b) on MacBook Pro M1 with ram 8 GB. The numerical results are presented as follows:

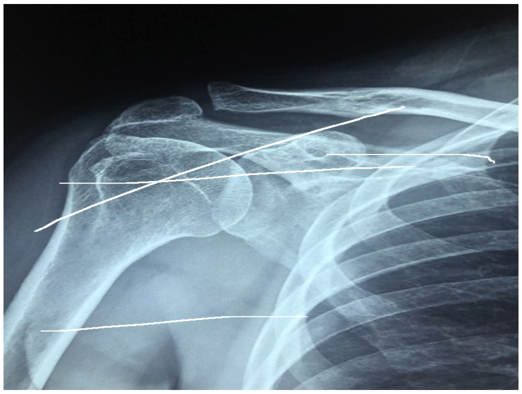
TABLE 1. The SNR and SSIM for each methods.

Methods	x-ray		windows	
	SNR	SSIM	SNR	SSIM
RFB	19.9964	0.9614	12.5741	0.9365
MFB	20.4239	0.9649	12.6261	0.9388
IRFBF	22.0834	0.9653	13.6382	0.9391

From Table 1, we can see that our algorithm **(IRFBF)** is effective and has higher SNR and SSIM than **FB** and **MFB** for both images. This means that our proposed algorithm is better than other methods.

Next, we demonstrate the figures of inpainting for each methods.

We see that the proposed method does not require the computation of Lipschitz constant of the gradient of functions. Moreover, the linesearch of iterations is not necessary in algorithms.



(A) Painted image

(B) Restored image by RFB (SNR=19.9964)

(C) Restored image by MFB (SNR=20.4239)

(D) Restored image by IRFBF (SNR=22.0834)

FIGURE 2. (a) is the painted x-ray image, (b),(c), and (d) are the restored images by RFB, MFB, and IRFBF, respectively.

5. CONCLUSION

In this paper, we introduced a relaxed inertial forward-backward-forward method with inertial effect and adaptive stepsize for solving the convex minimization problem in real Hilbert spaces. We proved its weak convergence theorem under mild conditions. We also applied to image inpainting and presented numerical results to demonstrat the efficiency of the our method, which does outperform other iterative methods.

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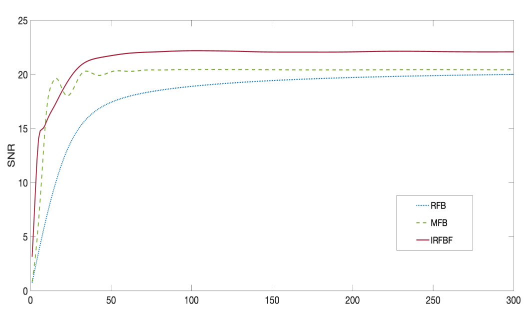
(A) Painted image

(B) Restored image by RFB (SNR=12.5741)

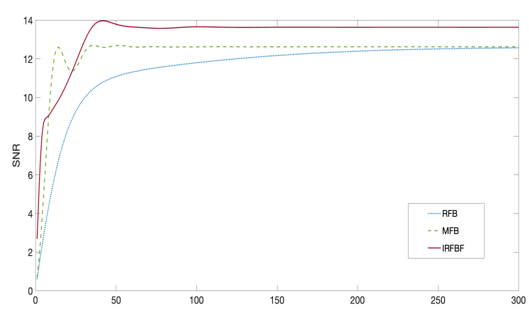
(C) Restored image by MFB (SNR=12.6261)

(D) Restored image by IRFBF (SNR=13.6382)

FIGURE 3. (a) is the painted windows image, (b),(c), and (d) are the restored images by RFB, MFB, and IRFBF, respectively.



(A) SNR plotting of x-ray image



(B) SNR plotting of windows image

FIGURE 4. Graphs of SNR values of two test images for each methods

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