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ON FIXED POINT ALGORITHMS FOR SOLVING SPLIT INVERSE PROBLEMS WITH APPLICATIONS

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Abstract. We study a certain class of split inverse problems which includes many other split-type problems. We propose a new inertial Mann-type Tseng's extragradient method to approximate the solution of this problem in real Hilbert spaces. Strong convergence of the proposed scheme to a minimum-norm solution of the problem is established when the associated single-valued operators are monotone and uniformly continuous with self-adaptive step size strategy. Moreover, we also study some classes of split inverse problems and provide some numerical implementations to illustrate our method and compare with a non-inertial version and a recently related method.
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1. Introduction

Several problems arising in many real-world applications, such as intensity-modulated radiation therapy, phase retrieval, image recovery, signal processing, data compression, and many more, can be mathematically modeled as the *Split Inverse Problem* (SIP), (see, e.g., [1, 2, 3, 4]). The SIP is defined as follows:

$$\text{Find } x^* \in H_1 \quad \text{that solves } \mathrm{IP}_1$$

such that

$$\hat{y} := Tx^* \in H_2 \quad \text{solves } \mathrm{IP}_2,$$

where H_1 and H_2 are real Hilbert spaces, IP_1 represents an inverse problem formulated in H_1 and IP_2 represents an inverse problem defined in H_2, and $T : H_1 \to H_2$ is a bounded linear operator.

The *Split Convex Feasibility Problem* (SCFP) which was first studied by Censor and Elfving [1] is the first instance of the SIP. This has been practically applied for modelling inverse problems that arise from medical image reconstruction. The SCFP finds numerous application

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in approximation theory, control theory, biomedical engineering, signal processing, geophysics, communications, and so on; see, e.g., [2, 5, 6, 7, 8, 9, 10]. The SCFP is defined as follows:

$$\text{Find } x^* \in C \text{ such that } \hat{y} = Tx^* \in Q,$$

where C and Q are nonempty, closed, and convex subsets of Hilbert spaces H_1 and H_2, respectively, and $T : H_1 \to H_2$ is a bounded linear operator.

Since the introduction of the SCFP, various SIPs that are more general than the SCFP have been introduced and studied. One of these important generalizations is the *Split Common Null Point Problem* (SCNPP), which is defined as follows:

$$\text{Find } x^* \in H_1 \text{ that solves } 0 \in F_1(x^*) \tag{1.1}$$

such that

$$\hat{y} = Tx^* \in H_2 \text{ solves } 0 \in F_2(\hat{y}), \tag{1.2}$$

where $F_1 : H_1 \to 2^{H_1}, F_2 : H_2 \to 2^{H_2}$ are two monotone operators, and $T : H_1 \to H_2$ is a bounded linear operator. The *Split Monotone Variational Inclusion Problem* (SMVIP) is another important SIP introduced by Moudafi in [11]. The SMVIP is defined as follows:

$$\text{Find } x^* \in H_1 \text{ that solves } 0 \in f_1(x^*) + F_1(x^*) \tag{1.3}$$

such that

$$\hat{y} = Tx^* \in H_2 \text{ solves } 0 \in f_2(\hat{y}) + F_2(\hat{y}), \tag{1.4}$$

where $f_1 : H_1 \to H_1, f_2 : H_2 \to H_2$ are single-valued operators, $F_1 : H_1 \to 2^{H_1}, F_2 : H_2 \to 2^{H_2}$ are multivalued operators, and $T : H_1 \to H_2$ is a bounded linear operator.

The SMVIP (1.3)-(1.4) is quite general. It includes several other optimization problems as special cases, such as the split saddle-point problems, split minimization problems, split equilibrium problems, split variational inequality problems, SCNPP (1.1)-(1.2), etc; see, e.g., [12, 13, 14, 15, 16, 17, 18].

In order to solve the SMVIP (1.3)-(1.4), Moudafi [11] introduced the following iterative method: For $x_1 \in H_1$, the sequence $\{x_n\}$ is generated as follows:

$$x_{n+1} = J_\lambda^{F_1}(I^{H_1} - \lambda f_1)(x_n + \eta T^*(J_\lambda^{F_2}(I^{H_2} - \lambda f_2) - I^{H_2})Tx_n), \quad n \geq 1, \tag{1.5}$$

where $\eta \in \left(0, \frac{2}{\|T\|}\right), I^{H_1}$ and I^{H_2} denote the identity operators on H_1 and H_2, respectively, and $J_\lambda^{F_1}$ and $J_\lambda^{F_2}$ are the resolvents of F_1 and F_2, respectively. He established the weak convergence of the sequence of iterates generated by Algorithm (1.5) under the conditions that the solution set of the SMVIP (1.3)-(1.4) is nonempty, F_1, F_2 are maximal monotone, f_1, f_2 are L_1-, L_2-co-coercive (also known as inverse strongly monotone), respectively and $\lambda \in (0, 2\alpha)$, where $\alpha := \min\{L_1, L_2\}$.

Recently, researchers proposed efficient iterative methods for approximating the solutions of the SMVIP (1.3)-(1.4) (see, e.g., [19] and the references therein). However, like the result of Moudafi [11], they require that the associated single-valued operators f_1 and f_2 are co-coercive, which stringent condition limits the scope of applications of these results (see Remark 2.1).

To remedy the above drawback, Izuchukwu *et al.* [20] recently introduced two new iterative methods for approximating the solution of the SMVIP (1.3)-(1.4) in the framework of Hilbert spaces. Their proposed methods only require that the single-valued operators f_1 and f_2 be Lipschitz continuous, and employ the inertial and relaxation techniques with the proximal contraction method. The first algorithm proposed by the authors requires the knowledge of the Lipschitz constant of the single-valued operator for its implementation while the second algorithm employs self-adaptive step size techniques, so that its execution does not depend on the Lipschitz constant of the single-valued operators f_1 and f_2 or the operator norm $\|T\|$. However, the convergence results in Izuchukwu *et al.* [20] were only weakly convergent, just like the result of Moudafi [11]. For optimization problems, strong convergence results are preferable because they are more applicable than the weak convergence results. Motivated by this, Wang *et al.* [21] made an attempt to improve on the results of Izuchukwu *et al.* [20] by proposing two new projection and contraction methods for finding the solutions of the SMVIP (1.3)-(1.4) in the framework of Hilbert spaces. Their proposed methods employ the inertial techniques with self adaptive step sizes. Under certain conditions on the control parameters, they obtained strong convergence results for the proposed algorithms.

Remark 1.1. All the results above on SMVIP (1.3)-(1.4) are not applicable when the associated single-valued operators f_1 and f_2 are non-Lipschitz. To the best of our knowledge, there are no existing results in the literature for solving the SMVIP (1.3)-(1.4) when the associated single-valued operators f_1 and f_2 are non-Lipschitz.

Recently, Reich and Tuyen [22] introduced and investigated a new class of split inverse problem named the *Split Common Null Point Problem with Multiple Output Sets* (SCNPPMOS). This problem is formulated as follows: Find a point $x^\dagger \in H$ such that

$$x^\dagger \in \Psi_1 := F^{-1}(0) \cap (\cap_{i=1}^N T_i^{-1}(F_i^{-1}(0))) \neq \emptyset. \tag{1.6}$$

where $T_i : H \to H_i,\ i = 1,2,...,N$, are bounded linear operators, and $F : H \to H,\ F_i : H_i \to 2^{H_i},\ i = 1,2,...,N$ are maximal monotone operators, and $H, H_i, i = 1,2,\ldots,N$ are Hilbert spaces. Moreover, they proposed two self-adaptive algorithms for approximating the solution of the SCNPPMOS (1.6) in the framework of Hilbert spaces. Under certain conditions on the control sequences, they proved that the sequences generated by the proposed algorithms converge strongly to the solution of the SCNPPMOS (1.6).

In addition, Alakoya and Mewomo [23] studied the *Split Variational Inequality Problem with Multiple Output Sets* (SVIPMOS). Let $H, H_i, i = 1,2,...,N$, be real Hilbert spaces, and let C, C_i be nonempty, closed, and convex subsets of real Hilbert spaces H and $H_i, i = 1,2,...,N$, respectively. Let $T_i : H \to H_i, i = 1,2,...,N$, be bounded and linear operators, and let $A : H \to H, A_i : H_i \to H_i, i = 1,2,...,N$, be mappings. The SVIPMOS is formulated as finding a point $x^* \in C$ such that

$$x^* \in \Psi_2 := VI(C,A) \cap (\cap_{i=1}^N T_i^{-1} VI(C_i, A_i)) \neq \emptyset. \tag{1.7}$$

They proposed a relaxed inertial Tseng's extragradient method for solving the SVIPMOS (1.7). Under some mild conditions, they proved that the sequence generated by the proposed algorithm converges strongly to a minimum-norm solution of the problem. Uzor et al. [24] recently introduced and studied the concept of *Split Monotone Variational Inclusion Problem with Multiple Output Sets* (SMVIPMOS). Let $H, H_i, i = 1,2,...,N$, be real Hilbert spaces, and let $T_i : H \to H_i, i = 1,2,...,N$, be bounded and linear operators. Let $F : H \to 2^H, F_i : H_i \to 2^{H_i}, i = 1,2,...,N$, be multivalued operators, and $f : H \to H, f_i : H_i \to H_i, i = 1,2,...,N$, be single-valued operators. The SMVIPMOS is formulated as finding a point $x^* \in H$ such that

$$x^* \in \Psi_3 := (f+F)^{-1}(0) \cap (\cap_{i=1}^N T_i^{-1}(f_i + F_i)^{-1}(0)) \neq \emptyset. \tag{1.8}$$

Observe that the SMVIPMOS (1.8) is quite general. It includes as special cases all the above optimization problems discussed so far in this paper. In particular, if we set $f = 0^H, f_i = 0^{H_i}, i = 1,2,...,N$, we obtain the SCNPPMOS (1.6), where 0^H and 0^{H_i} are the zero mappings on Hilbert spaces H and $H_i, i = 1,2,...,N$, respectively. Moreover, Uzor et al. [24] proposed the following inertial viscosity method for approximating the solution of SMVIPMOS (1.8) with the fixed point constraint of nonexpansive maps in the framework of Hilbert spaces:

Algorithm 1

1: Select initial data $x_0, x_1 \in H$, let $H_0 = H,\ T_0 = I^H,\ F_0 = F, f_0 = f$. Set $n := 0$.

2: Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \le \theta_n \le \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta,\ \frac{\varepsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

3: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

4: Compute

$$u_n = \sum_{i=0}^{N} \delta_{n,i}[w_n - \tau_{n,i} T_i^*(I^{H_i} - K_{\lambda_{n,i}})T_i w_n],$$

where

$$\tau_{n,i} = \frac{\psi_{n,i}||(I^{H_i} - K_{\lambda_{n,i}})T_i w_n||^2}{||T_i^*(I^{H_i} - K_{\lambda_{n,i}})T_i w_n||^2 + \Theta_{n,i}}.$$

5: Compute

$$\begin{cases} y_n = \xi_n w_n + (1 - \xi_n) S u_n; \\ x_{n+1} = \alpha_n \gamma g(w_n) + (I - \alpha_n D) y_n. \end{cases}$$

6: Set $n \leftarrow n+1$, and **go to 2**.

Here, $K_{\lambda_{n,i}} = J_{\lambda_{n,i}}^{F_i}(I^{H_i} - \lambda_{n,i} f_i),\ 0 < k_1 \le \lambda_{n,i} \le k_2 < 2\varphi,\ \varphi := \min\{\sigma, \sigma_i : i = 1,2\dots,N\}$ and I^{H_i} is the identity map on $H_i, F : H \to 2^H, F_i : H_i \to 2^{H_i}, i = 1,2,...,N$, are maximal monotone operators, and $f : H \to H, f_i : H_i \to H_i, i = 1,2,...,N$, are σ-inverse strongly monotone operator and σ_i-inverse strongly monotone operators, respectively, $S : H \to H$ is a nonexpansive mapping, $D : H \to H$ is a strongly positively bounded linear operator and $g : H \to H$ is a contraction. Under some conditions on the control parameters, the authors proved that the sequence generated by Algorithm 1 converges strongly to the common solution of the SMVIPMOS (1.8) and the fixed point problem of nonexpansive mappings.

Remark 1.2. We point out that one of the limitations of the proposed Algorithm 1 by Uzor et al. [24] is that the method is not applicable to the SMVIPMOS (1.8) when the associated single-valued operators $f_i,\ i = 0,1,2,\dots,N$ are monotone and/or non-Lipschitz. Moreover, the step sizes $\lambda_{n,i}$ depend on the inverse strongly monotonicity (co-coercive) constants of the operators $f_i,\ i = 0,1,2,\dots,N$, which could deteriorate the speed of the proposed algorithm.

Polyak in [25] first introduced an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. Recently, many authors constructed efficient iterative algorithms by using inertial the extrapolation technique; see, e.g., [26, 27, 28, 29, 30, 31, 32].

In this study, we propose a new inertial Mann-type Tseng's extragradient method with self-adaptive step sizes for approximating the solution of the SMVIPMOS (1.8) in Hilbert spaces. Our proposed method does not require the stringent co-coercive and/or Lipschitz continuity conditions assumed in the result of Uzor et al. [24] and often assumed by authors when solving monotone inclusion problems. Instead, our method only requires the associated single-valued operators to be monotone and uniformly continuous, which are more relaxed conditions than the co-coercive and Lipschitz continuity assumptions. Moreover, the proposed algorithm does not require knowledge of the operators' norm for its execution. Under mild conditions on the control parameters, we prove that the sequence generated by our proposed algorithm converges to a minimum-norm solution of the SMVIPMOS (1.8). Finally, we apply our strong convergence theorem to some classes of split inverse problems and we present several numerical experiments to demonstrate the usefulness of our result.

The remaining sections of this paper are organized as follows. We present in Section 2 some definitions and lemmas required in analyzing the convergence of the proposed algorithm and the proposed algorithm is presented in Section 3. In Section 4, we analyze the convergence of the proposed method while in Section 5 we apply our result to certain classes of split inverse problems. We present in Section 6 several numerical experiments with graphical illustrations. Finally, we give a final remark in Section 7 summarizing the results of the paper.

2. PRELIMINARIES

Some known and useful results and definitions are reviewed in this section. For the rest of the paper, we denote $H, H_i, i = 1, 2, ..., N$, as real Hilbert spaces with inner product $\langle ., . \rangle$ and induced norm $\|.\|$. For any sequence $\{x_n\} \subset H$, $x_n \rightharpoonup x$ and $x_n \to x$ denote weak and strong convergence of $\{x_n\}$ to a point $x \in H$ respectively, and $w_\omega(x_n) = \{x \in H : \exists \{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Definition 2.1. An operator $A : H \to H$ is *L-Lipschitz continuous* if there exists a constant $L > 0$ such that $||Ax - Ay|| \le L||x - y||$ for all $x, y \in H$. Here L is called the Lipschitz constant.

Definition 2.2. An operator $A : H \to H$ is *α-strongly monotone* if there exists $\alpha > 0$ such that $\langle x - y, Ax - Ay \rangle \ge \alpha \|x - y\|^2$ for all $x, y \in H$.

Definition 2.3. An operator $A : H \to H$ is said to be *α-inverse strongly monotone* (also known as *α-co-coercive*) if there exists a positive real number α such that $\langle x - y, Ax - Ay \rangle \ge \alpha||Ax - Ay||^2$ for all $x, y \in H$.

Definition 2.4. An operator $A : H \to H$ is *monotone* if the following inequality is satisfied $\langle x - y, Ax - Ay \rangle \ge 0$ for all $x, y \in H$.

Definition 2.5. An operator $A : H \to H$ is *uniformly continuous* if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|Ax - Ay\| < \varepsilon$ whenever $\|x - y\| < \delta$ for all $x, y \in H$.

Remark 2.1. The following relationships should be noted:

(1) if A is α-strongly monotone and L-Lipschitz continuous, then A is $\frac{\alpha}{L^2}$-inverse strongly monotone.

(2) if A is α-inverse strongly monotone operator, then it is $\frac{1}{\alpha}$-Lipschitz continuous and monotone but the converse statement is false.

(3) uniform continuity is a weaker notion than Lipschitz continuity.

The notion of almost Lipschitz continuity is presented next and the proof is given for completeness. For more details, one refers to [33].

Lemma 2.1. *Let $D \subset H$ be convex. An operator $A : D \to H$ is uniformly continuous if and only if, for every $\varepsilon > 0$, there exists $L^* < +\infty$ such that $||Ax - Ay|| \leq L^*||x - y|| + \varepsilon$ for all $x, y \in D$.An operator $A : D \to H$ satisfying the inequality above for every $\varepsilon > 0$ is said to be almost Lipschitz continuous.*

Proof. Assume that A is almost Lipschitz continuous on a convex domain D in the sense defined above. Let $\varepsilon > 0$ be fixed. Choose L^* such that $||Ax - Ay|| \leq L^*||x - y|| + \frac{\varepsilon}{2}$ for all $x, y \in D$. If we take $\delta = \frac{\varepsilon}{2L^*}$, then

$$\|Ax - Ay\| < \varepsilon \quad \text{whenever} \quad \|x - y\| < \delta, \quad \forall x, y \in D.$$

Hence, A is uniformly continuous. We next assume that A is uniformly continuous on a convex domain D, which implies that, for a fix $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|Ax - Ay\| < \varepsilon \quad \text{whenever} \quad \|x - y\| < \delta, \quad \forall x, y \in D.$$

Let $x, y \in D$ and $\sigma = \frac{\delta}{2}$. Define

$$r_n = n\sigma \frac{(x - y)}{||x - y||} + y, \quad n = 0, 1, 2, \cdots, \left\lfloor \frac{||x - y||}{\sigma} \right\rfloor.$$

Clearly $r_0 = y$ and $r_n \in D$ for $n = 0, 1, 2, \cdots, \left\lfloor \frac{||x-y||}{\sigma} \right\rfloor$ since D is convex. Furthermore, we obtain the following estimate

$$\begin{aligned} ||r_n - r_{n-1}|| &= \left\| \left(n\sigma \frac{(x - y)}{||x - y||} + y \right) - \left((n - 1)\sigma \frac{(x - y)}{||x - y||} + y \right) \right\| \\ &= \left\| \sigma \frac{(x - y)}{||x - y||} \right\| = \sigma. \end{aligned} \tag{2.1}$$

Similarly, we have

$$\|x - r_N\| \leq \sigma, \tag{2.2}$$

where

$$N = \left\lfloor \frac{||x - y||}{\sigma} \right\rfloor.$$

So, we obtain the following estimate by using the fact that $Ar_0 = Ay$

$$\begin{aligned} \|Ax - Ay\| &\leq \|(Ar_1 - Ar_0) + (Ar_2 - Ar_1) + (Ar_3 - Ar_2) + \cdots + (Ar_N - Ar_{N-1})\| \\ &\quad + \|Ax - Ar_N\| \\ &\leq \sum_{k=1}^{N} \|Ar_i - Ar_{i-1}\| + \|Ax - Ar_N\|. \end{aligned} \tag{2.3}$$

Using the uniform continuity of A together with (2.1) and (2.2), we have from (2.3) that

$$\|Ax - Ay\| \leq \sum_{k=1}^{N} \varepsilon + \varepsilon < N\varepsilon + \varepsilon \leq \frac{||x - y||}{\sigma}\varepsilon + \varepsilon = L^*||x - y|| + \varepsilon,$$

where $L^* = \frac{2\varepsilon}{\delta}$. Hence, A is almost Lipschitz continuous. $\square$

Definition 2.6. A function $c : H \to \mathbb{R} \cup \{+\infty\}$ is said to be weakly lower semi-continuous (w-lsc) at $x \in H$ if $c(x) \leq \liminf_{n\to\infty} c(x_n)$ holds for every sequence $\{x_n\}$ in H satisfying $x_n \rightharpoonup x$.

Definition 2.7. A convex map $c : H \to \mathbb{R}$ is said to be subdifferentiable at a point $x \in H$ if the set $\partial c(x) = \{\zeta \in H \mid c(y) \geq c(x) + \langle \zeta, y - x \rangle, \ \forall y \in H\}$ is nonempty. Each element in $\partial c(x)$ is called a subgradient of c at x. If c is subdifferentiable at each $x \in H$, then c is said to be subdifferentiable on H.

Definition 2.8. Let $\lambda > 0$ and $B : H \to 2^H$ be a multivalued operator. The effective domain of B denoted by $\text{dom}(B)$ is given by $\text{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. The operator $B : H \to 2^H$ is called

- *monotone* if $\langle u - v, x - y \rangle \geq 0$ for all $u \in B(x), v \in B(y)$.
- *maximal monotone* if the graph $\text{Gr}(B)$ of B, $\text{Gr}(B) := \{(x,u) \in H \times H | u \in B(x)\}$, is not properly contained in the graph of any other monotone mapping. In other words, B is maximal if and only if for $x \in \text{dom}(B)$ and $u \in Bx$ such that $\langle u - v, x - y \rangle \geq 0$ implies $(y, v) \in \text{Gr}(B)$.

The *resolvent* of B with parameter $\lambda > 0$ denoted by J_λ^B is given by $J_\lambda^B := (I^H + \lambda B)^{-1}$, where I^H is the identity operator of Hilbert space H. It is known that if B is maximal monotone, then J_r^B is single-valued, firmly nonexpansive and $\text{dom}(J_r^B) = H$. A fundamental example of a maximal monotone mapping is the subdifferential of a convex proper lower semicontinuous function.

Lemma 2.2. [34] *Let $F : H \to 2^H$ be a maximal monotone mapping, and let $f : H \to H$ be a hemicontinuous, monotone and bounded operator. Then the mapping $f + F$ is a maximal monotone mapping.*

Lemma 2.3. [35] *Suppose that $\{a_n\}$ is a sequence of nonnegative real numbers, $\{\alpha_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = +\infty$, and $\{z_n\}$ is a sequence of real numbers. Assume that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n z_n$ for all $n \geq 1$. If $\limsup_{k\to\infty} z_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.4. [36] *Suppose that $\{\lambda_n\}$ and $\{\theta_n\}$ are two nonnegative real sequences such that $\lambda_{n+1} \leq \lambda_n + \varphi_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} \varphi_n < +\infty$, then $\lim_{n\to\infty} \lambda_n$ exists.*

The last lemma is trivial.

Lemma 2.5. *Let H be a real Hilbert space. Then the following results hold for all $x, y \in H$ and $\delta \in (0,1)$:*

(i) $||x + y||^2 = ||x||^2 + 2\langle x, y \rangle + ||y||^2$;
(ii) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle$;
(iii) $||\delta x + (1 - \delta)y||^2 = \delta||x||^2 + (1 - \delta)||y||^2 - \delta(1 - \delta)||x - y||^2$.

3. Proposed Method

In this section, we introduce a new Tseng's extragradient algorithm involving Mann-Type iteration with inertial technique and self-adaptive stepsize for solving the SMVIPMOS (1.8). For proving the algorithm's strong convergence, we assume the following.

Let $H, H_i, i = 1, 2, ..., N$, be real Hilbert spaces.

Assumption 3.1. (a) The feasible set Ω is nonempty subset of H.
(b) $T_i : H \to H_i, i = 1, 2, ..., N$, are bounded linear operators with adjoints T_i^*.

(c) $F : H \to 2^H$, $F_i : H_i \to 2^{H_i}$, $i = 1,2,...,N$, be maximal monotone operators.
(d) $f : H \to H$, $f_i : H_i \to H_i$, $i = 1,2,...,N$, be uniformly continuous monotone operators.

In addition, we require that the control sequences satisfy the following conditions:

Assumption 3.2.

(a) $\{\alpha_n\} \subset (0,1), \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = +\infty, \lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0, \{\beta_n\} \subset [a,b] \subset (0,1-\alpha_n), \theta > 0;$

(b) $0 < c_i < c_i' < 1, 0 < \varphi_i < \varphi_i' < 1, \lim_{n\to\infty} c_{n,i} = \lim_{n\to\infty} \varphi_{n,i} = 0, \lambda_{1,i} > 0, \ \forall\, i = 0,1,2,\dots,N;$

(c) $\{\rho_{n,i}\} \subset \mathbb{R}_+, \sum_{n=1}^{\infty} \rho_{n,i} < +\infty, 0 < a_i \le \delta_{n,i} \le b_i < 1, \sum_{i=0}^{N} \delta_{n,i} = 1$ for each $n \ge 1$.

Subsequently, we give the convergence analysis of the sequence $\{x_n\}$ generated by our proposed method, Algorithm 2. The proposed algorithm is stated below:

Algorithm 2

1: Select initial data $x_0, x_1 \in H$. Let $H_0 = H$, $T_0 = I^H$, $F_0 = F$, and $f_0 = f$. Set $n := 0$.

2: Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \le \theta_n \le \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \ \frac{\varepsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \ne x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

3: Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$.

4: Compute $y_{n,i} = J^{F_i}_{\lambda_{n,i}}(T_i w_n - \lambda_{n,i} f_i T_i w_n)$.

5: Compute $u_{n,i} = y_{n,i} - \lambda_{n,i}(f_i y_{n,i} - f_i T_i w_n)$, where

$$\lambda_{n+1,i} = \begin{cases} \min\left\{\frac{(c_{n,i}+c_i)\|T_i w_n - y_{n,i}\|}{\|f_i T_i w_n - f_i y_{n,i}\|}, \ \lambda_{n,i} + \rho_{n,i}\right\}, & \text{if} \quad f_i T_i w_n - f_i y_{n,i} \ne 0, \\ \lambda_{n,i} + \rho_{n,i}, & \text{otherwise.} \end{cases}$$

6: Compute $z_n = \sum_{i=0}^{N} \delta_{n,i}\big(w_n + \eta_{n,i} T_i^*(u_{n,i} - T_i w_n)\big)$, where

$$\eta_{n,i} = \begin{cases} \frac{(\varphi_{n,i}+\varphi_i)\|T_i w_n - u_{n,i}\|^2}{\|T_i^*(T_i w_n - u_{n,i})\|^2}, & \text{if} \quad \|T_i^*(T_i w_n - u_{n,i})\| \ne 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

7: Compute

$$x_{n+1} = (1 - \alpha_n - \beta_n) w_n + \beta_n z_n.$$

8: Set $n \leftarrow n+1$, and **go to 2**.

Remark 3.1. From Assumption 3.2 (a) and (b), it follows that

$$\lim_{n\to\infty} \theta_n ||x_n - x_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| = 0.$$

Remark 3.2. When the single-valued operators $f_i, i = 0,1,2,\dots,N$ are non-Lipschitz, our method does not require any linesearch procedure, which could be computationally too expensive to implement. Instead, we employ self-adaptive step size techniques that only require simple computations with known parameters per iteration. Moreover, some of the parameters are relaxed to accommodate larger range of step sizes.

4. Convergence Results

In this section, we establish the strong convergence result of our proposed Algorithm 2 under the Assumptions 3.1 and 3.2. We begin by proving some Lemmas needed for the main result.

Lemma 4.1. *Under Assumptions 3.1 and 3.2, the sequence $\{\lambda_{n,i}\}$ generated by Algorithm 2 is well defined for each $i = 0,1,2,\dots,N$. In addition,*

$$\lim_{n\to\infty} \lambda_{n,i} = \lambda_{1,i} \in \left[\min\left\{\frac{c_i}{P_i}, \lambda_{1,i}\right\}, \lambda_{1,i} + \Phi_i\right],$$

where

$$\Phi_i = \sum_{n=1}^{\infty} \rho_{n,i}.$$

Proof. We have from Assumption 3.1(d) that f_i is uniformly continuous for each $i = 0,1,2,\dots,N$. Given $\varepsilon_i > 0$, there exists $L_i^* < +\infty$ such that

$$\|f_iT_iw_n - f_iy_{n,i}\| \le L_i^*\|T_iw_n - y_{n,i}\| + \varepsilon_i.$$

Let $f_iT_iw_n - f_iy_{n,i} \neq 0$ for all $n \ge 1$. Then

$$\begin{aligned}
\frac{(c_{n,i}+c_i)\|T_iw_n - y_{n,i}\|}{\|f_iT_iw_n - f_iy_{n,i}\|} &\ge \frac{(c_{n,i}+c_i)\|T_iw_n - y_{n,i}\|}{L_i^*\|T_iw_n - y_{n,i}\| + \varepsilon_i} \\
&= \frac{(c_{n,i}+c_i)\|T_iw_n - y_{n,i}\|}{(L_i^* + \xi_i)\|T_iw_n - y_{n,i}\|} \\
&= \frac{(c_{n,i}+c_i)}{P_i} \ge \frac{c_i}{P_i},
\end{aligned}$$

where $\varepsilon_i = \xi_i\|T_iw_n - y_{n,i}\|$ for some $\xi_i \in (0,1)$ and $P_i = L_i^* + \xi_i$. Thus, by the definition of $\lambda_{n+1,i}$, sequence $\{\lambda_{n,i}\}$ has lower bound $\min\{\frac{c_i}{P_i}, \lambda_{1,i}\}$ and has upper bound $\lambda_{1,i} + \Phi_i$. By Lemma 2.4, the limit $\lim_{n\to\infty} \lambda_{n,i}$ exists and we denote by $\lambda_i = \lim_{n\to\infty} \lambda_{n,i}$. It is obvious that $\lambda_i \in \left[\min\{\frac{c_i}{P_i}, \lambda_{1,i}\}, \lambda_{1,i} + \Phi_i\right]$ for each $i = 0,1,2\dots,N$. □

Lemma 4.2. *Let Assumptions 3.1 and 3.2 of Algorithm 2 hold. Then, there exists a positive integer N such that*

$$\varphi_i + \varphi_{n,i} \in (0,1), \quad \text{and} \quad \frac{\lambda_{n,i}(c_{n,i}+c_i)}{\lambda_{n+1,i}} \in (0,1),$$

for all $n \ge N$.

Proof. For each $i = 0,1,2,\dots,N$,, we know that $\lim_{n\to\infty} \varphi_{n,i} = 0$ and $0 < \varphi_i < \varphi_i' < 1$. So, there exists a positive integer $N_{1,i}$ such that $0 < \varphi_i + \varphi_{n,i} \le \varphi_i' < 1$ for all $n \ge N_{1,i}$. We also know that for each $i = 0,1,2,\dots,N$, $\lim_{n\to\infty} \lambda_{n,i} = \lambda_i$ and $0 < c_i < c_i' < 1$. From these, we have

$$\lim_{n\to\infty}\left(1 - \frac{\lambda_{n,i}(c_{n,i}+c_i)}{\lambda_{n+1,i}}\right) = 1 - c_i > 1 - c_i' > 0.$$

Hence, there exists a positive integer $N_{2,i}$, $i = 0,1,2,\dots,N$, such that

$$1 - \frac{\lambda_{n,i}(c_{n,i}+c_i)}{\lambda_{n+1,i}} > 0, \ \forall n \ge N_{2,i}.$$

If we set $N = \max\{N_{1,i},\ N_{2,i} : i = 0,1,2,\dots,N\}$, then the required result follows. □

Lemma 4.3. *If $\|T_i^*(T_iw_n - u_{n,i})\| \neq 0$, then the sequence $\{\eta_{n,i}\}$ defined by (3.1) is bounded below by a positive real number for each $i = 0,1,2,\dots,N$.*

Proof. If $\|T_i^*(T_iw_n - u_{n,i})\| \neq 0$, then for each $i = 0,1,2,\dots,N$, then

$$\eta_{n,i} = \frac{(\varphi_{n,i}+\varphi_i)\|T_iw_n - u_{n,i}\|^2}{\|T_i^*(T_iw_n - u_{n,i})\|^2}.$$

We use the boundedness of T_i and the fact that $\lim_{n\to\infty}\varphi_{n,i} = 0$ for each $i = 0,1,2,\dots,N$ to obtain the estimate

$$\frac{(\varphi_{n,i}+\varphi_i)\|T_iw_n - u_{n,i}\|^2}{\|T_i^*(T_iw_n - u_{n,i})\|^2} \quad \geq \quad \frac{(\varphi_{n,i}+\varphi_i)\|T_iw_n - u_{n,i}\|^2}{\|T_i\|^2\|T_iw_n - u_{n,i}\|\|^2} \geq \frac{\varphi_i}{\|T_i\|^2}.$$

Hence, we obtain a lower bound $\frac{\varphi_i}{\|T_i\|^2}$ for $\{\eta_{n,i}\}$ for each $i = 0,1,2,\dots,N$. □

Lemma 4.4. *Suppose that $\{x_n\}$ is a sequence generated by Algorithm 2 such that Assumptions 3.1 and 3.2 hold. Then the following inequality holds for all $p \in \Omega$:*

$$\|u_{n,i} - T_ip\|^2 \leq \|T_iw_n - T_ip\|^2 - \Big(1 - \frac{\lambda_{n,i}^2}{\lambda_{n+1,i}^2}(c_{n,i}+c_i)^2\Big)\|T_iw_n - y_{n,i}\|^2.$$

Proof. From the definition of $\lambda_{n+1,i}$, we have

$$\|f_iT_iw_n - f_iy_{n,i}\| \leq \frac{(c_{n,i}+c_i)}{\lambda_{n+1,i}}\|T_iw_n - y_{n,i}\|, \quad \forall n \in \mathbb{N},\ i = 0,1,\dots,N. \tag{4.1}$$

Clearly for the cases where $f_iT_iw_n - f_iy_{n,i} = 0$ and $f_iT_iw_n - f_iy_{n,i} \neq 0$, inequality (4.1) holds. Let $p \in \Omega$. It follows that $T_ip \in \Omega,\ i = 0,1,2,\dots,N$. By the definition of $u_{n,i}$ and the Lemma 2.5 together with (4.1), we obtain

$$\begin{aligned}
&\|u_{n,i} - T_ip\|^2 \\
&= \|y_{n,i} - T_ip\|^2 + \lambda_{n,i}^2\|f_iy_{n,i} - f_iT_iw_n\|^2 - 2\lambda_{n,i}\langle y_{n,i} - T_ip, f_iy_{n,i} - f_iT_iw_n\rangle \\
&= \|T_iw_n - T_ip\|^2 + \|y_{n,i} - T_iw_n\|^2 + 2\langle y_{n,i} - T_iw_n, T_iw_n - T_ip\rangle + \lambda_{n,i}^2\|f_iy_{n,i} - f_iT_iw_n\|^2 \\
&\quad - 2\lambda_{n,i}\langle y_{n,i} - T_ip, f_iy_{n,i} - f_iT_iw_n\rangle \\
&= \|T_iw_n - T_ip\|^2 - \|y_{n,i} - T_iw_n\|^2 + 2\langle y_{n,i} - T_iw_n, y_{n,i} - T_ip\rangle + \lambda_{n,i}^2\|f_iy_{n,i} - f_iT_iw_n\|^2 \\
&\quad - 2\lambda_{n,i}\langle y_{n,i} - T_ip, f_iy_{n,i} - f_iT_iw_n\rangle \\
&= \|T_iw_n - T_ip\|^2 - \|y_{n,i} - T_iw_n\|^2 - 2\langle T_iw_n - y_{n,i} - \lambda_{n,i}(f_iT_iw_n - f_iy_{n,i}), y_{n,i} - T_ip\rangle \\
&\quad + \lambda_{n,i}^2\|f_iy_{n,i} - f_iT_iw_n\|^2 \\
&\leq \|T_iw_n - T_ip\|^2 - \Big(1 - \frac{\lambda_{n,i}^2}{\lambda_{n+1,i}^2}(c_{n,i}+c_i)^2\Big)\|T_iw_n - y_{n,i}\|^2 \\
&\quad - 2\langle T_iw_n - y_{n,i} - \lambda_{n,i}(f_iT_iw_n - f_iy_{n,i}), y_{n,i} - T_ip\rangle.
\end{aligned} \tag{4.2}$$

Next, we show that

$$\langle T_iw_n - y_{n,i} - \lambda_{n,i}(f_iT_iw_n - f_iy_{n,i}), y_{n,i} - T_ip\rangle \geq 0, \quad i = 0,1,2,\dots,N. \tag{4.3}$$

From $y_{n,i}=J^{F_i}_{\lambda_{n,i}}(T_iw_n-\lambda_{n,i}f_iT_iw_n)$, we have $(I^{H_i}-\lambda_{n,i}f_i)T_iw_n\in(I^{H_i}+\lambda_{n,i}F_i)y_{n,i}$. By the maximal monotonicity of F_i , $i=0,1,2,\dots,N$, there exists $t_{n,i}\in F_iy_{n,i}$ such that $(I^{H_i}-\lambda_{n,i}f_i)T_iw_n=y_{n,i}+\lambda_{n,i}t_{n,i}$, which implies that

$$t_{n,i}=\frac{1}{\lambda_{n,i}}(T_iw_n-y_{n,i}-\lambda_{n,i}f_iT_iw_n). \tag{4.4}$$

Moreover, observe that $0\in(f_i+F_i)T_ip$ and $f_iy_{n,i}+t_{n,i}\in(f_i+F_i)y_{n,i}$. Since f_i+F_i is maximal monotone for each $i=0,1,2,\dots,N$, we have

$$\langle f_iy_{n,i}+t_{n,i},y_{n,i}-T_ip\rangle\geq 0. \tag{4.5}$$

Substituting (4.4) into (4.5), we obtain

$$\frac{1}{\lambda_{n,i}}\langle T_iw_n-y_{n,i}-\lambda_{n,i}f_iT_iw_n+\lambda_{n,i}f_iy_{n,i},y_{n,i}-T_ip\rangle\geq 0.$$

Hence, $\langle T_iw_n-y_{n,i}-\lambda_{n,i}(f_iT_iw_n-f_iy_{n,i}),y_{n,i}-T_ip\rangle\geq 0,\ i=0,1,2,\dots,N$. Consequently, by applying (4.3) in (4.2) we obtain

$$\|u_{n,i}-T_ip\|^2\leq\|T_iw_n-T_ip\|^2-\left(1-\frac{\lambda^2_{n,i}}{\lambda^2_{n+1,i}}(c_{n,i}+c_i)^2\right)\|T_iw_n-y_{n,i}\|^2, \tag{4.6}$$

which is the required result. □

Lemma 4.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 2 such that Assumptions 3.1 and 3.2 hold. Then $\{x_n\}$ is bounded.*

Proof. Let $p\in\Omega$. Using the definition of w_n and applying the triangular inequality, we have

$$\|w_n-p\|\leq\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|=\|x_n-p\|+\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|. \tag{4.7}$$

By Remark (3.1), there exists $M_1>0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\leq M_1,\ \forall\, n\geq 1.$$

Then, it follows from (4.7) that $\|w_n-p\|\leq\|x_n-p\|+\alpha_nM_1$ for all $n\geq 1$. By Lemma 4.2, there exists a positive integer N such that

$$1-\frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i}+c_i)>0,\ \forall n\geq N,\ i=0,1,2,\dots,N.$$

Therefore, it follows from (4.6) that, for all $n\geq N$ and $i=0,1,2,\dots,N$,

$$\|u_{n,i}-T_ip\|^2\leq\|T_iw_n-T_ip\|^2. \tag{4.8}$$

We have from Lemma 4.2 that there exists a positive integer N such that $0<\varphi_{n,i}+\varphi_i<1,\ i=0,1,2,\dots,N$ for all $n\geq N$. So, using Lemma 2.5 together with (4.8), we have from (4.11) that

$$\begin{aligned}
&\|w_n+\eta_{n,i}T_i^*(u_{n,i}-T_iw_n)-p\|^2\\
&=\|w_n-p\|^2+\eta^2_{n,i}\|T_i^*(u_{n,i}-T_iw_n)\|^2+2\eta_{n,i}\langle T_iw_n-T_ip,u_{n,i}-T_iw_n\rangle\\
&=\|w_n-p\|^2+\eta^2_{n,i}\|T_i^*(u_{n,i}-T_iw_n)\|^2\\
&+\eta_{n,i}[\|u_{n,i}-T_ip\|^2-\|T_iw_n-T_ip\|^2-\|u_{n,i}-T_iw_n\|^2]\\
&\leq\|w_n-p\|^2-\eta_{n,i}\left[\|u_{n,i}-T_iw_n\|^2-\eta_{n,i}\|T_i^*(u_{n,i}-T_iw_n)\|^2\right].
\end{aligned} \tag{4.9}$$

In the case where $\|T_i^*(u_{n,i}-T_iw_n)\| \neq 0$, we have from the definition of $\eta_{n,i}$ that

$$\|u_{n,i}-T_iw_n\|^2-\eta_{n,i}\|T_i^*(u_{n,i}-T_iw_n)\|^2=[1-(\varphi_{n,i}+\varphi_i)]\|T_iw_n-u_{n,i}\|^2 \geq 0. \tag{4.10}$$

Using convexity of norm square ($\|\cdot\|^2$) and (4.10), we have

$$\begin{aligned}
\|z_n-p\|^2 &= \left\|\sum_{i=0}^{N}\delta_{n,i}\big(w_n+\eta_{n,i}T_i^*(u_{n,i}-T_iw_n)\big)-p\right\|^2\\
&\leq \sum_{i=0}^{N}\delta_{n,i}\|w_n+\eta_{n,i}T_i^*(u_{n,i}-T_iw_n)-p\|^2\\
&\leq \sum_{i=0}^{N}\delta_{n,i}\left[\|w_n-p\|^2-\eta_{n,i}[1-(\varphi_{n,i}+\varphi_i)]\|T_iw_n-u_{n,i}\|^2\right]\\
&= \|w_n-p\|^2-\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}\left[1-(\varphi_{n,i}+\varphi_i)\right]\|T_iw_n-u_{n,i}\|^2\big]\\
&= \|w_n-p\|^2. \qquad (4.11)
\end{aligned}$$

Clearly from (4.9), inequality (4.11) still holds when $\|T_i^*(u_{n,i}-T_iw_n)\|=0$. Using Lemma 2.5(ii) and the inequality obtain in (4.11), we have the following estimate

$$\begin{aligned}
&\|(1-\alpha_n-\beta_n)(w_n-p)+\beta_n(z_n-p)\|^2\\
&\leq (1-\alpha_n-\beta_n)^2\|w_n-p\|^2+2(1-\alpha_n-\beta_n)\beta_n\|w_n-p\|\|z_n-p\|+\beta_n^2\|z_n-p\|^2\\
&\leq (1-\alpha_n-\beta_n)^2\|w_n-p\|^2+(1-\alpha_n-\beta_n)\beta_n\left[\|w_n-p\|^2+\|z_n-p\|^2\right]+\beta_n^2\|z_n-p\|^2\\
&\leq (1-\alpha_n-\beta_n)(1-\alpha_n)\|w_n-p\|^2+\beta_n(1-\alpha_n)\|w_n-p\|^2\\
&= (1-\alpha_n)^2\|w_n-p\|^2.
\end{aligned}$$

Hence, we have that

$$\|(1-\alpha_n-\beta_n)(w_n-p)+\beta_n(z_n-p)\| \leq (1-\alpha_n)\|w_n-p\|. \tag{4.12}$$

Using (4.12), we therefore have, for all $n \geq N$, that

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \|(1-\alpha_n-\beta_n)(w_n-p)+\beta_n(z_n-p)\|+\alpha_n\|p\|\\
&\leq (1-\alpha_n)\|w_n-p\|+\alpha_n\|p\|\\
&\leq (1-\alpha_n)\left[\|x_n-p\|+\alpha_nM_1\right]+\alpha_n\|p\|\\
&\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\left[\|p\|+M_1\right]\\
&\leq \max\left\{\|x_n-p\|,\|p\|+M_1\right\}\\
&\ \ \vdots\\
&\leq \max\left\{\|x_N-p\|,\|p\|+M_1\right\}.
\end{aligned}$$

This establishes the boundedness of $\{x_n\}$. Consequently, $\{w_n\},\{y_{n,i}\},\{u_{n,i}\}$, and $\{z_n\}$ are all bounded. □

Lemma 4.6. *Let $\{w_{n_k}\}$ and $\{z_{n_k}\}$ be subsequences of $\{w_n\}$ and $\{z_n\}$ generated by the proposed Algorithm 2 respectively. Suppose these subsequences satisfies $\lim_{k\to\infty}\|w_{n_k}-z_{n_k}\|=0$. If $w_{n_k} \rightharpoonup z \in H$, then $z \in \Omega$.*

Proof. Using the estimates in (4.11) for the subsequences $\{w_{n_k}\}$ and $\{z_{n_k}\}$, we obtain

$$\begin{aligned}\sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}[1-(\varphi_{n_k,i}+\varphi_i)]\|T_iw_{n_k}-u_{n_k,i}\|^2 &\leq \|w_{n_k}-p\|^2-\|z_{n_k}-p\|^2\\ &\leq \|w_{n_k}-z_{n_k}\|^2+2\|w_{n_k}-z_{n_k}\|\|z_{n_k}-p\|\end{aligned}$$

Hence, using the fact that $\lim\limits_{k\to\infty}\|w_{n_k}-z_{n_k}\|=0$, we have

$$\sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}[1-(\varphi_{n_k,i}+\varphi_i)]\|T_iw_{n_k}-u_{n_k,i}\|^2\to 0,\quad k\to\infty,$$

which implies that by using the definition of $\eta_{n,i}$

$$\delta_{n_k,i}(\varphi_{n_k,i}+\varphi_i)[1-(\varphi_{n_k,i}+\varphi_i)]\frac{\|T_iw_{n_k}-u_{n_k,i}\|^4}{\|T_i^*(T_iw_{n_k}-u_{n_k,i})\|^2}\to 0,\quad k\to\infty,\ \forall i=0,1,2,\dots,N,$$

So, we have

$$\frac{\|T_iw_{n_k}-u_{n_k,i}\|^2}{\|T_i^*(T_iw_{n_k}-u_{n_k,i})\|}\to 0,\quad k\to\infty,\ \forall i=0,1,2,\dots,N.$$

Using the boundedness of $\{\|T_i^*(T_iw_{n_k}-u_{n_k,i})\|\}$, it follows that

$$\|T_iw_{n_k}-u_{n_k,i}\|\to 0,\quad k\to\infty,\ \forall i=0,1,2,\dots,N. \tag{4.13}$$

Thus

$$\|T_i^*(T_iw_{n_k}-u_{n_k,i})\| \;=\; \|T_i\|\|(T_iw_{n_k}-u_{n_k,i})\|\to 0,\quad k\to\infty,\ \forall i=0,1,2,\dots,N.$$

From (4.6), we obtain

$$\begin{aligned}\Big(1-\frac{\lambda_{n_k,i}^2}{\lambda_{n_k+1,i}^2}(c_{n_k,i}+c_i)^2\Big)\|T_iw_{n_k}-y_{n_k,i}\|^2 &\leq \|T_iw_{n_k}-T_ip\|^2-\|u_{n_k,i}-T_ip\|^2\\ &\leq \|T_iw_{n_k}-u_{n_k,i}\|\big(\|T_iw_{n_k}-T_ip\|+\|u_{n_k,i}-T_ip\|\big).\end{aligned} \tag{4.14}$$

By applying (4.13) in (4.14), it follows that

$$\left(1-\frac{\lambda_{n_k,i}^2}{\lambda_{n_k+1,i}^2}(c_{n_k,i}+c_i)^2\right)\|T_iw_{n_k}-y_{n_k,i}\|^2\to 0,\quad k\to\infty,\ i=0,1,\dots,N.$$

Therefore, we have

$$\|T_iw_{n_k}-y_{n_k,i}\|\to 0,\quad k\to\infty,\ i=0,1,\dots,N. \tag{4.15}$$

For $(u_i,v_i)\in \mathrm{Gr}(F_i+f_i)$, $i=0,1,2,\dots,N$, we have that $v_i-f_iu_i\in F_iu_i$. Since $y_{n_k,i}=J_{\lambda_{n_k,i}}^{F_i}(T_iw_{n_k}-\lambda_{n_k,i}f_iT_iw_{n_k})$, we have

$$\frac{1}{\lambda_{n_k,i}}(T_iw_{n_k}-\lambda_{n_k,i}f_iT_iw_{n_k}-y_{n_k,i})\in F_iy_{n_k,i}.$$

Consequently, by the maximal monotonicity of F_i, we obtain

$$\left\langle v_i-f_iu_i-\frac{1}{\lambda_{n_k,i}}(T_iw_{n_k}-\lambda_{n_k,i}f_iT_iw_{n_k}-y_{n_k,i}),u_i-y_{n_k,i}\right\rangle\geq 0,\quad \forall i=0,1,2,\dots,N. \tag{4.16}$$

By applying the monotonicity of f_i, we have from (4.16) that

$$\begin{aligned}&\langle v_i, u_i - y_{n_k,i}\rangle \\ &\geq \langle f_i u_i + \frac{1}{\lambda_{n_k,i}}(T_i w_{n_k} - \lambda_{n_k,i} f_i T_i w_{n_k} - y_{n_k,i}), u_i - y_{n_k,i}\rangle \\ &= \frac{1}{\lambda_{n_k,i}}\langle T_i w_{n_k} - y_{n_k,i}, u_i - y_{n_k,i}\rangle + \langle f_i u_i - f_i y_{n_k,i}, u_i - y_{n_k,i}\rangle + \langle f_i y_{n_k,i} - f_i T_i w_{n_k}, u_i - y_{n_k,i}\rangle \\ &\geq \frac{1}{\lambda_{n_k,i}}\langle T_i w_{n_k} - y_{n_k,i}, u_i - y_{n_k,i}\rangle + \langle f_i y_{n_k,i} - f_i T_i w_{n_k}, u_i - y_{n_k,i}\rangle.\end{aligned} \tag{4.17}$$

By the continuity of f_i, from (4.15) we obtain

$$\|f_i T_i w_{n_k} - f_i y_{n_k,i}\| \to 0, \quad k \to \infty, \ \forall i = 0,1,2,\ldots,N. \tag{4.18}$$

Since $w_{n_k} \rightharpoonup z$ and T_i is a bounded linear operator for each $i = 0,1,2,\ldots,N$, we have $T_i w_{n_k} \rightharpoonup T_i z$ for all $i = 0,1,2,\ldots,N$. Thus, from (4.15), we obtain $y_{n_k,i} \rightharpoonup T_i z$ for all $i = 0,1,2,\ldots,N$. Therefore, by letting $k \to \infty$ and applying (4.15) and (4.18), we obtain from (4.17) that

$$\langle v_i, u_i - T_i z\rangle \geq 0, \quad \forall (u_i, v_i) \in \mathrm{Gr}(f_i + F_i), \ \forall i = 0,1,2,\ldots,N. \tag{4.19}$$

Since $f_i + F_i$ is maximal monotone for each $i = 0,1,2,\ldots,N$, it follows from Lemma 2.2 and (4.19) that $T_i z \in (f_i + F_i)^{-1}(0)$ for all $i = 0,1,2,\ldots,N$, which implies that $z \in T_i^{-1}\Big((f_i + F_i)^{-1}(0)\Big)$ for all $i = 0,1,2,\ldots,N$. Thus, we have $z \in \bigcap_{i=0}^{N} T_i^{-1}\Big((f_i + F_i)^{-1}(0)\Big)$. Hence, $z \in \Omega$ as required. □

Lemma 4.7. *Let $\{x_n\}$ be a sequence generated by Algorithm 2 under Assumptions 3.1 and 3.2. Then, the following inequality holds, for all $p \in \Omega$,*

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\Big[3M_2(1-\alpha_n)^2\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\langle p, p - x_{n+1}\rangle\Big] \\ - \beta_n(1-\alpha_n)\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1 - (\varphi_{n,i} + \varphi_i)]\|T_i w_n - u_{n,i}\|^2.\end{aligned}$$

Proof. Let $p \in \Omega$. Using the Cauchy-Schwarz inequality and Lemma 2.5, we have the following estimate

$$\begin{aligned}\|w_n - p\|^2 &= \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle \\ &\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|(\theta_n\|x_n - x_{n-1}\| + 2\|x_n - p\|) \\ &\leq \|x_n - p\|^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|,\end{aligned} \tag{4.20}$$

where $M_2 := \sup_{n\in\mathbb{N}}\{\|x_n - p\|, \theta_n\|x_n - x_{n-1}\|\} > 0$. In view of Lemma 2.5 and (4.20), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \|(1-\alpha_n-\beta_n)(w_n-p)+\beta_n(z_n-p)\|^2 - 2\alpha_n\langle p, x_{n+1}-p\rangle \\
&\le (1-\alpha_n-\beta_n)^2\|w_n-p\|^2+\beta_n^2\|z_n-p\|^2+2\beta_n(1-\alpha_n-\beta_n)\|w_n-p\|\|z_n-p\| \\
&+2\alpha_n\langle p, p-x_{n+1}\rangle \\
&\le (1-\alpha_n-\beta_n)(1-\alpha_n)\|w_n-p\|^2+\beta_n(1-\alpha_n)\|z_n-p\|^2+2\alpha_n\langle p, p-x_{n+1}\rangle \\
&\le (1-\alpha_n)^2\|w_n-p\|^2-\beta_n(1-\alpha_n)\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1-(\varphi_{n,i}+\varphi_i)]\|T_iw_n-u_{n,i}\|^2 \\
&+2\alpha_n\langle p, p-x_{n+1}\rangle \\
&\le (1-\alpha_n)^2||x_n-p||^2+3M_2\alpha_n(1-\alpha_n)^2\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\alpha_n\langle p, p-x_{n+1}\rangle \\
&-\beta_n(1-\alpha_n)\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1-(\varphi_{n,i}+\varphi_i)]\|T_iw_n-u_{n,i}\|^2 \\
&\le (1-\alpha_n)||x_n-p||^2+\alpha_n\Big[3M_2(1-\alpha_n)^2\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\langle p, p-x_{n+1}\rangle\Big] \\
&-\beta_n(1-\alpha_n)\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1-(\varphi_{n,i}+\varphi_i)]\|T_iw_n-u_{n,i}\|^2.
\end{aligned}$$

□

Theorem 4.1. *Suppose that Assumptions 3.1 and 3.2 are satisfied and that the sequence $\{x_n\}$ is generated by Algorithm 2. Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $\|x^*\| = \min\{\|p\| : p \in \Omega\}$.*

Proof. Let $\|x^*\| = \min\{\|p\| : p \in \Omega\}$, that is, $x^* = P_\Omega(0)$. Then, from Lemma 4.7, we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le (1-\alpha_n)||x_n-x^*||^2+\alpha_n\Big[3M_2(1-\alpha_n)^2\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\langle x^*, x^*-x_{n+1}\rangle\Big] \\
&= (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n d_n, \qquad (4.21)
\end{aligned}$$

where

$$d_n = 3M_2(1-\alpha_n)^2\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\langle x^*, x^*-x_{n+1}\rangle.$$

Suppose that $\{\|x_{n_k}-x^*\|\}$ is a subsequence of $\{\|x_n-x^*\|\}$ satisfying

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|\right) \ge 0. \qquad (4.22)$$

Applying Lemma 4.7, we obtain

$$\begin{aligned}
&\beta_{n_k}(1-\alpha_{n_k})\sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}[1-(\varphi_{n_k,i}+\varphi_i)]\|T_iw_{n_k}-u_{n_k,i}\|^2 \\
&\le (1-\alpha_{n_k})\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2 \\
&+\alpha_{n_k}\Big[3M_2(1-\alpha_{n_k})^2\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|+2\langle x^*, x^*-x_{n_k+1}\rangle\Big].
\end{aligned}$$

Using the fact that $\lim_{k\to\infty} \alpha_{n_k} = 0$, Remark 3.1, and assumption (4.22), we obtain

$$\beta_{n_k}(1-\alpha_{n_k})\sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}[1-(\varphi_{n_k,i}+\varphi_i)]\|T_i w_{n_k}-u_{n_k,i}\|^2 \to 0, \quad k\to\infty,$$

which then gives $\lim_{k\to\infty}\|T_i w_{n_k}-u_{n_k,i}\| = 0$ for all $i = 0,1,2,\dots,N$. Moreover, we have

$$\|T_i^*(u_{n_k,i}-T_i w_{n_k})\| \le \|T_i^*\|\|u_{n_k,i}-T_i w_{n_k}\| \to 0, \quad k\to\infty\ \forall i = 0,1,2,\dots,N. \tag{4.23}$$

By the definition of z_n and by applying (4.23), we obtain

$$\|z_{n_k}-w_{n_k}\| \le \sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}\|T_i^*(u_{n_k,i}-T_i w_{n_k})\| \to 0, \quad k\to\infty. \tag{4.24}$$

By the definition of w_n and applying Remark 3.1, we see that $\|w_{n_k}-x_{n_k}\| = \theta_{n_k}\|x_{n_k}-x_{n_k-1}\| \to 0$ as $k\to\infty$. Next, we obtain from (4.24) that $\|x_{n_k}-z_{n_k}\| \le \|x_{n_k}-w_{n_k}\|+\|w_{n_k}-z_{n_k}\| \to 0$ as $k\to\infty$. Now, from the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, we obtain

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\| &= \|(1-\alpha_{n_k}-\beta_{n_k})(w_{n_k}-x_{n_k})+\beta_{n_k}(z_{n_k}-x_{n_k})-\alpha_n x_{n_k}\|\\ &\le (1-\alpha_{n_k}-\beta_{n_k})\|w_{n_k}-x_{n_k}\|+\beta_{n_k}\|z_{n_k}-x_{n_k}\|+\alpha_{n_k}\|x_{n_k}\| \to 0, \quad k\to\infty.\end{aligned} \tag{4.25}$$

Let $x^* \in w_\omega(x_n)$ be selected arbitrarily. Using the boundedness of sequence $\{x_n\}$ and the fact that $w_\omega(x_n) \neq \emptyset$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. Hence, Lemma 4.6 and (4.24) give $x^* \in \Omega$. Also, $w_\omega(x_n) \subset \Omega$ using the fact that $x^* \in w_\omega(x_n)$ was chosen arbitrarily. Moreover, we have that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup q$ since $\{x_{n_k}\}$ is bounded. In addition, $\limsup_{k\to\infty}\langle x^*, x^*-x_{n_k}\rangle = \lim_{j\to\infty}\langle x^*, x^*-x_{n_{k_j}}\rangle$. It then clearly follows, by using $x^* = P_\Omega(0)$, that

$$\limsup_{k\to\infty}\langle x^*, x^*-x_{n_k}\rangle = \lim_{j\to\infty}\langle x^*, x^*-x_{n_{k_j}}\rangle = \langle x^*, x^*-q\rangle \le 0. \tag{4.26}$$

Combining (4.25) and (4.26), we obtain that $\limsup_{k\to\infty}\langle x^*, x^*-x_{n_{k+1}}\rangle \le 0$. Hence, we have that $\limsup_{k\to\infty} d_{n_k} \le 0$ by Remark 3.1. Lemma 2.3 and (4.21) then imply that $\{\|x_n-x^*\|\}$ converges to zero which completes the proof. □

5. Applications

This section is focused on some applications of the proposed algorithm to many classes of split inverse problems.

5.1. **Split variational inequality problem with multiple output sets.** Let H be a Hilbert space and $A : C \to H$ be a nonlinear mapping, where C is a nonempty, closed, and convex subset of H. The variational inequality problem is formulated as follows: Find $x^* \in C$ such that $\langle y-x^*, Ax^*\rangle \ge 0$ for all $y \in C$. We denote its solution set by $VI(C,A)$, see [37]. Now, we recall the indicator function of C defined by

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C,\\ \infty, & \text{if } x \notin C.\end{cases}$$

It is known that i_C is a proper, lower semicontinuous, and convex function, and its subdifferential ∂i_C is maximal monotone (see [38]). Moreover, it is known that $\partial i_C(v) = N_C(v) = \{u \in H :$

$\langle y-v,u\rangle \leq 0, \forall y \in C\}$, where N_C is the normal cone of C at a point v. Hence, the resolvent of ∂i_C can be defined for $\lambda > 0$ by $J_\lambda^{\partial i_C}(x) = (I+\lambda \partial i_C)^{-1}x$ for all $x \in H$. It was demonstrated in [39] that, for any $x \in H$ and $z \in C$, $z = J_\lambda^{\partial i_C}(x) \iff z = P_C(x)$, where P_C is the metric projection map from H onto C.

The following lemma is required to establish our next result.

Lemma 5.1. [40] *Let C be a nonempty, closed, and convex subset of a Banach space E. Suppose that $A : C \to E^*$ is a monotone and hemicontinuous operator and $F : E \to 2^{E^*}$ is an operator defined by*

$$F(v) = \begin{cases} Av + N_C(v), & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then F is maximal monotone and $F^{-1}0 = VI(C,A)$.

Here, we are interested in applying our result to the SVIPMOS (1.7). By setting $F_i = \partial_{i_{C_i}}$ and $f_i = A_i,\ i = 0,1,2,\dots,N$ in Theorem 4.1, we have the following strong convergence theorem for finding the solution of the SVIPMOS (1.7) in the framework of Hilbert spaces.

Theorem 5.1. *Let $H, H_i, i = 1,2,...,N$, be real Hilbert spaces, and let C, C_i be nonempty, closed, and convex subsets of real Hilbert spaces H and $H_i, i = 1,2,...,N$, respectively. Let $T_i : H \to H_i, i = 1,2,...,N$, be bounded and linear operators, and let $A : H \to H, A_i : H_i \to H_i, i = 1,2,...,N$, be uniformly continuous monotone operators. Suppose that the solution set $\Psi_2 \neq \emptyset$ and that Assumption A of Theorem 4.1 holds. Then, the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to $x^* \in \Psi_2$, where $\|x^*\| = \min\{\|p\| : p \in \Psi_2\}$.*

Algorithm 3

1: *Select initial points $x_0, x_1 \in H$. Let $C_0 = C,\ T_0 = I^H,\ A_0 = A$. Set $n := 0$.*

2: *Given the $(n-1)$th and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by*

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta,\ \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

3: *Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$.*

4: *Compute $y_{n,i} = P_{C_i}(T_i w_n - \lambda_{n,i} A_i T_i w_n)$.*

5: *Compute $u_{n,i} = y_{n,i} - \lambda_{n,i}(A_i y_{n,i} - A_i T_i w_n)$, where*

$$\lambda_{n+1,i} = \begin{cases} \min\left\{\frac{(c_{n,i}+c_i)\|T_i w_n - y_{n,i}\|}{\|A_i T_i w_n - A_i y_{n,i}\|},\ \lambda_{n,i} + \rho_{n,i}\right\}, & \text{if } \quad A_i T_i w_n - A_i y_{n,i} \neq 0, \\ \lambda_{n,i} + \rho_{n,i}, & \text{otherwise.} \end{cases}$$

6: *Compute $z_n = \sum_{i=0}^N \delta_{n,i}\big(w_n + \eta_{n,i} T_i^*(u_{n,i} - T_i w_n)\big)$, where*

$$\eta_{n,i} = \begin{cases} \frac{(\varphi_{n,i}+\varphi_i)\|T_i w_n - u_{n,i}\|^2}{\|T_i^*(T_i w_n - u_{n,i})\|^2}, & \text{if } \quad \|T_i^*(T_i w_n - u_{n,i})\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

7: *Compute $x_{n+1} = (1 - \alpha_n - \beta_n)w_n + \beta_n z_n$.*

8: *Set $n \leftarrow n+1$, and* **go to 2**.

5.2. **Split convex minimization problem with multiple output sets.** Suppose that $g : H \to \mathbb{R}$ is a convex and differentiable function, and $G : H \to (-\infty, +\infty]$ is a proper convex and lower semi-continuous function. It is known that if ∇g is $\frac{1}{\mu}$-Lipschitz continuous, then it is μ-inverse strongly monotone (and hence monotone), where ∇g is the gradient of g. Moreover, the subdifferential ∂G of G is maximal monotone (see [40]). In addition,

$$g(x^*) + G(x^*) = \min_{x\in H}\{g(x) + G(x)\} \iff 0 \in \nabla g(x^*) + \partial G(x^*).$$

Let $H, H_i, i = 1,2,\dots,N$ be real Hilbert spaces, and let $T_i : H \to H_i$ be bounded linear operators. Let $g : H \to \mathbb{R}, g_i : H_i \to \mathbb{R}$ be convex and differentiable functions, and let $G : H \to (-\infty, +\infty], G_i : H_i \to (-\infty, +\infty]$ be proper convex and lower semi-continuous functions. In this subsection, we are interested in applying our result to *split convex minimization problem with multiple output sets* (SCMPMOS), which is formulated as follows: Find $x^* \in H$ such that

$$x^* \in \Gamma_2 := \arg\min_H \{g(x) + G(x)\} \cap \Big(\cap_{i=1}^N T_i^{-1}\Big(\arg\min_{H_i}\{g_i(x) + G_i(x)\}\Big)\Big) \neq \emptyset, \qquad (5.1)$$

For each $i = 1,2,\dots,N$, if we set $F = \partial G, F_i = \partial G_i, f = \nabla g, f_i = \nabla g_i$ in Theorem 4.1, then we obtain the following result for approximating the solution of SCMPMOS (5.1).

Theorem 5.2. *Let $H, H_i, i = 1,2,...,N$, be real Hilbert spaces and $T_i : H \to H_i, i = 1,2,...,N$, be bounded linear operators with adjoints T_i^*. Let G, G_i, g, g_i be as defined in (5.1) above and such that $\nabla g, \nabla g_i$ are $\frac{1}{\mu}$-Lipschitz continuous and $\frac{1}{\mu_i}$-Lipschitz continuous, respectively. Suppose that Assumption A of Theorem 4.1 holds and the solution set $\Gamma_2 \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm 4 below converges strongly to $x^* \in \Gamma$, where $x^* = \min\{\|p\| : p \in \Gamma_2\}$.*

Algorithm 4

1: *Select initial data $x_0, x_1 \in H$. Let $H_0 = H$, $T_0 = I^H$, $\partial G_0 = \partial G$, and $\nabla g_0 = \nabla g$. Set $n := 0$.*
2: *Given the $(n-1)$th and nth iterates, choose θ_n such that $0 \le \theta_n \le \hat{\theta}_n$ with $\hat{\theta}_n$ defined by*

$$\hat{\theta}_n = \begin{cases} \min\Big\{\theta,\ \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\Big\}, & \text{if } x_n \neq x_{n-1},\\ \theta, & \text{otherwise.}\end{cases}$$

3: *Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$.*
4: *Compute $y_{n,i} = J^{\partial G_i}_{\lambda_{n,i}}(T_i w_n - \lambda_{n,i}\nabla g_i T_i w_n)$.*
5: *Compute $u_{n,i} = y_{n,i} - \lambda_{n,i}(\nabla g_i y_{n,i} - \nabla g_i T_i w_n)$, where*

$$\lambda_{n+1,i} = \begin{cases} \min\Big\{\frac{(c_{n,i}+c_i)\|T_i w_n - y_{n,i}\|}{\|\nabla g_i T_i w_n - \nabla g_i y_{n,i}\|},\ \lambda_{n,i} + \rho_{n,i}\Big\}, & \text{if } \nabla g_i T_i w_n - \nabla g_i y_{n,i} \neq 0,\\ \lambda_{n,i} + \rho_{n,i}, & \text{otherwise.}\end{cases}$$

6: *Compute $z_n = \sum_{i=0}^N \delta_{n,i}\big(w_n + \eta_{n,i} T_i^*(u_{n,i} - T_i w_n)\big)$, where*

$$\eta_{n,i} = \begin{cases} \frac{(\varphi_{n,i}+\varphi_i)\|T_i w_n - u_{n,i}\|^2}{\|T_i^*(T_i w_n - u_{n,i})\|^2}, & \text{if } \|T_i^*(T_i w_n - u_{n,i})\| \neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

7: *Compute*

$$x_{n+1} = (1 - \alpha_n - \beta_n)w_n + \beta_n z_n.$$

8: *Set $n \leftarrow n+1$, and* **go to 2.**

5.3. **Generalized split monotone variational inclusion problem.** In this subsection, we apply our result to generalized split monotone variational inclusion problem. Let $H_i, i = 1,2,...,N$, be real Hilbert spaces and let $S_i : H_i \to H_{i+1}, i = 1,2,...,N-1$, be bounded linear operators, such that $S_i \neq 0$. Let $G_i : H_i \to 2^{H_i}, i = 1,2,...,N$, be multivalued operators, and $g_i : H_i \to H_i, i = 1,2,...,N$, be single-valued operators. The *generalized split monotone variational inclusion problem* (GSMVIP) is formulated as follows: Find an element $x^* \in H_1$ such that

$$x^* \in \Gamma_3 := (g_1+G_1)^{-1}(0) \cap S_1^{-1}((g_2+G_2)^{-1}(0)) \cap \ldots S_1^{-1}(S_2^{-1} \ldots (S_{N-1}^{-1}((g_N+G_N)^{-1}(0)))) \neq \emptyset. \tag{5.2}$$

That is, $x^* \in H_1$ such that

$$0 \in (g_1+G_1)(x^*), 0 \in (f_2+F_2)(S_1x^*), \ldots, 0 \in (g_N+G_N)(S_{N-1}(S_{N-2} \ldots S_1x^*)).$$

We note that if we set $H = H_1, f = g_1, F = G_1, f_i = g_{i+1}, F_i = G_{i+1}, 1 \leq i \leq N-1, T_1 = S_1, T_2 = S_2S_1, \ldots,$ and $T_{N-1} = S_{N-1}S_{N-2} \ldots S_1$, then the SMVIPMOS (1.8) becomes the GSMVIP (5.2). Therefore, we have the following strong convergence theorem for finding the solution of GSMVIP (5.2).

Theorem 5.3. *Let $H_i, i = 1,2,...,N$, be real Hilbert spaces and let $S_i : H_i \to H_{i+1}, i = 1,2,...,N-1$, be bounded and linear operators with adjoints S_i^* such that $S_i \neq 0$. Let g_i, G_i, $1,2,...,N$ be as defined above in (5.2), and suppose Assumption A of Theorem 4.1 holds and the solution set $\Gamma_3 \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm 4 below converges strongly to $x^* \in \Gamma_3$, where $x^* = \min\{\|p\| : p \in \Gamma_3\}$.*

Algorithm 5

1: *Select initial data $x_0, x_1 \in H_1$. Let $S_0 = I^{H_1}$, $\hat{S}_{N-1} = S_{N-1}S_{N-2} \ldots S_0, \hat{S}_{N-1}^* = S_0^*S_1^* \ldots S_{N-1}^*$, $i = 1,2,\ldots,N$. Set $n := 0$.*

2: *Given the $(n-1)$th and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by*

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

3: *Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$.*

4: *Compute $y_{n,i} = J_{\lambda_{n,i}}^{G_i}(\hat{S}_{i-1}w_n - \lambda_{n,i}g_i\hat{S}_{i-1}w_n)$.*

5: *Compute $u_{n,i} = y_{n,i} - \lambda_{n,i}(g_iy_{n,i} - g_i\hat{S}_{i-1}w_n)$, where*

$$\lambda_{n+1,i} = \begin{cases} \min\left\{\frac{(c_{n,i}+c_i)\|\hat{S}_{i-1}w_n - y_{n,i}\|}{\|g_i\hat{S}_{i-1}w_n - g_iy_{n,i}\|}, \lambda_{n,i} + \rho_{n,i}\right\}, & \text{if } \quad g_i\hat{S}_{i-1}w_n - g_iy_{n,i} \neq 0, \\ \lambda_{n,i} + \rho_{n,i}, & \text{otherwise.} \end{cases}$$

6: *Compute $z_n = \sum_{i=1}^N \delta_{n,i}\big(w_n + \eta_{n,i}\hat{S}_{i-1}^*(u_{n,i} - \hat{S}_{i-1}w_n)\big)$, where*

$$\eta_{n,i} = \begin{cases} \frac{(\varphi_{n,i}+\varphi_i)\|\hat{S}_{i-1}w_n - u_{n,i}\|^2}{\|\hat{S}_{i-1}^*(\hat{S}_{i-1}w_n - u_{n,i})\|^2}, & \text{if } \quad \|\hat{S}_{i-1}^*(\hat{S}_{i-1}w_n - u_{n,i})\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

7: *Compute $x_{n+1} = (1 - \alpha_n - \beta_n)w_n + \beta_nz_n$.*

8: *Set $n \leftarrow n+1$, and* **go to 2**.

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6. Numerical Illustrations

In this section, we provide some numerical experiments and demonstrate the efficiency and accuracy of our proposed algorithm to solve the SMVIPMOS (1.8). We compare the performance of our proposed method, Algorithm 2 (Proposed Alg.) with the non-inertial version (Non-inertial Alg.) and Algorithm 1 proposed by Uzor *et al.* (Uzor *et al.* Alg.). All the numerical computations and codes were carried out using Matlab version R2021(b).

In all the experiments, we use $\|E_n\| = \|x_{n+1} - x_n\| < \varepsilon$, where $\varepsilon = 10^{-6}$ as the stopping criterion. Parameters used in all the experiments for all the algorithms involved are presented in Table 1.

Table 1. Methods Parameters for Examples 6.1 and 6.2

Proposed Alg. 2	$\lambda_{1,i} = i + 1.25$	$\theta = 0.99$	$\alpha_n = \frac{1}{2n+3}$	$\beta_n = 0.999(1-\alpha_n)$
	$c_i = 0.89$	$c_{n,i} = \frac{1}{(1+n)^2}$	$\varepsilon_n = \frac{7}{(2n+3)^3}$	$\rho_{n,i} = \frac{100}{n^2}$
	$\delta_{n,i} = \frac{1}{n+1}$	$\phi_i = 0.02$	$\phi_{n,i} = \frac{1}{(5+n)^3}$	
Non-inertial Alg.	$\lambda_{1,i} = i + 1.25$	$\theta = 0$	$\alpha_n = \frac{1}{2n+3}$	$\beta_n = 0.999(1-\alpha_n)$
	$c_i = 0.89$	$c_{n,i} = \frac{1}{(1+n)^2}$	$\varepsilon_n = \frac{7}{(2n+3)^3}$	$\rho_{n,i} = \frac{100}{n^2}$
	$\delta_{n,i} = \frac{1}{n+1}$	$\phi_i = 0.02$	$\phi_{n,i} = \frac{1}{(5+n)^3}$	
Uzor *et al.* Alg.	$\lambda_{1,i} = i + 1.25$	$\lambda_{n,i} = 0.12$	$\theta = 0.99$	$\alpha_n = \frac{1}{2n+3}$
	$\varepsilon_n = \frac{7}{(2n+3)^3}$	$\xi_n = 0.9$	$\Theta_{n,i} = 1.5$	$\gamma = \frac{2}{5}$
	$\delta_{n,i} = \frac{1}{n+1}$	$\psi_{n,i} = 0.65$	$D(x) = \frac{x}{5}$	$S(x) = g(x) = \frac{x}{3}$

Example 6.1. Let $H_i = \mathbb{R}^k, i = 0,1,\ldots,5$ and for $i = 0,1,\ldots,5$. Define the mappings $T_i, f_i, F_i : \mathbb{R}^k \to \mathbb{R}^k$ by

$$T_i(x) = \frac{4}{i+4}x, \quad f_i(x) = (i+1)(x+\sin x), \quad F_i(x) = 7(i+1)x, \ \forall x \in \mathbb{R}^k.$$

Then, $T_i^*(y) = \frac{4}{i+4}y, \ \forall y \in \mathbb{R}^k$. For a fixed $N = 5$, we generate starting points x_0, x_1 of different length k populated with random real entries. The numerical results are reported in Figures 1-4 and Table 2.

TABLE 2. Numerical Results for Example 6.1

	Proposed Alg. 2		Non-inertial Alg.		Uzor *et al. Alg.*	
k	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
5	56	0.0060	64	0.0069	219	0.0062
50	68	0.0016	73	0.0015	246	0.0034
100	71	0.0017	75	0.0019	250	0.0045
500	75	0.0074	79	0.0086	259	0.0134

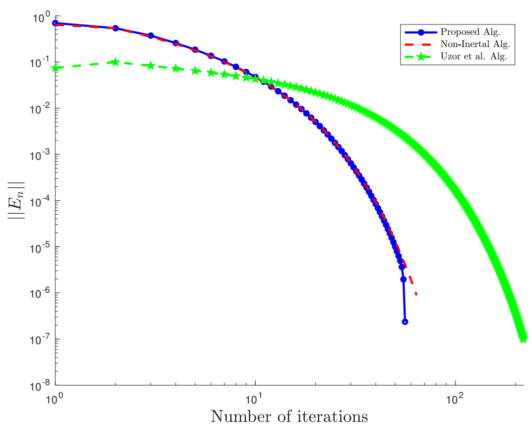

FIGURE 1. Example 6.1 with k = 5

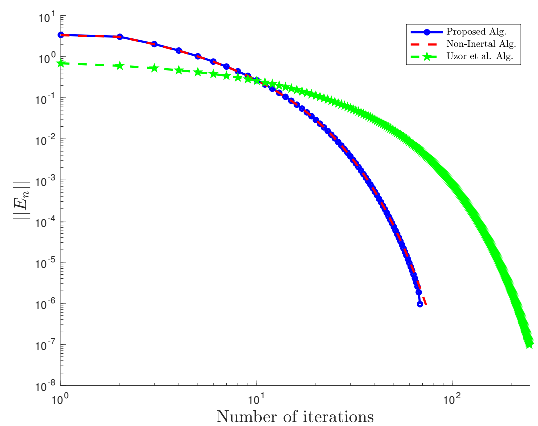

FIGURE 2. Example 6.1 with k = 50

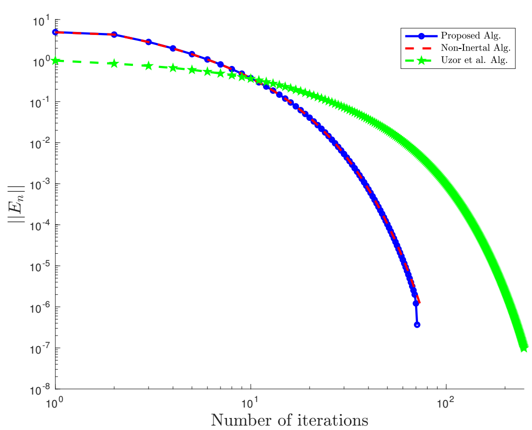

FIGURE 3. Example 6.1 with k=100

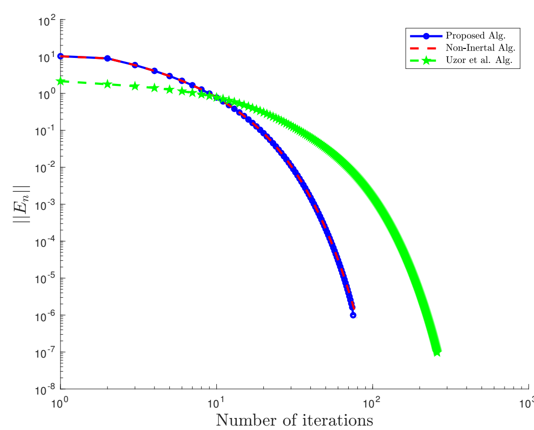

FIGURE 4. Example 6.1 with k=500

Example 6.2. Let $H_i = (\ell_2(\mathbb{R}), \|\cdot\|_2),\ i = 0, 1, \ldots, 5$, where

$$\ell_2(\mathbb{R}) := \left\{ x = (x_1, x_2, \ldots, x_j, \ldots), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^2 < +\infty \right\},\ ||x||_2 = \left(\sum_{j=1}^{\infty} |x_j|^2 \right)^{\frac{1}{2}}$$

for all $x \in \ell_2(\mathbb{R})$. For each $i = 0, 1, \ldots, 5$, we define the mappings $T_i, f_i, F_i : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ by

$$T_i(x) = \frac{3}{i+3}x,\ \forall x \in \ell_2(\mathbb{R}),$$

$$f_i(x) = 5(i+1)x,\ \forall x \in \ell_2(\mathbb{R}),$$

and

$$F_i(x) = 3(i+1)x,\ \forall x \in \ell_2(\mathbb{R}).$$

Then,

$$T_i^*(y) = \frac{3}{i+3}y,\ \forall y \in \ell_2(\mathbb{R}).$$

We choose different starting points as follows for a fixed $N = 6$:

Case I: $x_0 = (4, 1, \frac{1}{4}, \cdots)$; $x_1 = (-2, 1, -\frac{1}{2}, \cdots)$;

Case II: $x_0 = (5, 1, \frac{1}{5}, \cdots)$; $x_1 = (-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \cdots)$.

Case III: $x_0 = (4, 1, \frac{1}{4}, \cdots)$; $x_1 = (0.1, -0.01, 0.001, \cdots)$;

Case IV: $x_0 = (6, 1, \frac{1}{6}, \cdots)$; $x_1 = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots)$.

The numerical results are reported in Figures 5-8 and Table 3.

TABLE 3. Numerical Results for Example 6.1

	Proposed Alg. 2		Non-inertial Alg.		Uzor *et al. Alg.*	
Case	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
I	76	0.6539	80	0.6764	198	1.0915
II	65	0.5286	72	0.5957	175	0.9057
III	36	0.3001	61	0.5611	103	0.5513
IV	58	0.5082	69	0.6538	159	0.9558

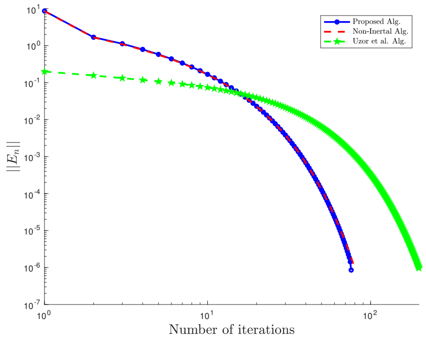

FIGURE 5. Example 6.2: Case I

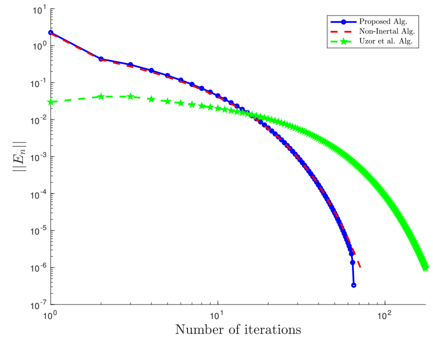

FIGURE 6. Example 6.2: Case II

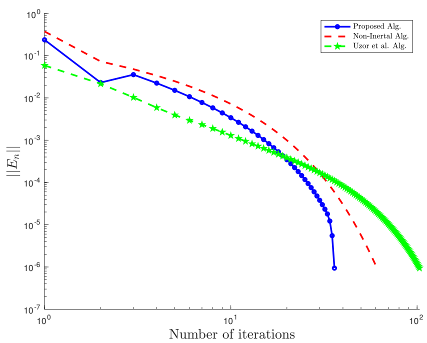

FIGURE 7. Example 6.2: Case III

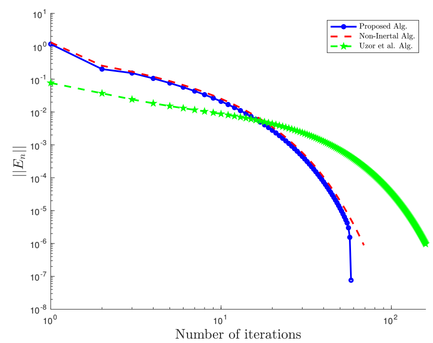

FIGURE 8. Example 6.2: Case IV

Remark 6.1.
The following observations are presented from the above numerical Examples 6.1 - 6.2 in the following remarks.

(1). From Figures 1 - 8 and Tables 2 and 3, our proposed Algorithm 2 is easy to implement, efficient, and accurate in handling applications in both finite and infinite dimensional spaces.

(2). We compared our proposed Algorithm 2 to its Non-inertial case and the state-of-the-art algorithms proposed by Uzor et al. [24]. The results in 1 - 8 and Tables 2 and 3 indicate that our proposed algorithm outperformed all methods compared for these examples with respect to the number of iterations and CPU time.

7. CONCLUDING REMARK

We studied the split monotone variational inclusion problem with multiple output sets. We proposed an inertial Mann-type Tseng's extragradient algorithm with self-adaptive step sizes for finding the solution of the problem in real Hilbert spaces. Our proposed method does not require the co-coercive condition or the Lipschitz continuity of the associated single-valued operators, which are often assumed in the literature when solving monotone inclusion problems.

Moreover, under some mild conditions on the control sequences and without prior knowledge of the operator norms, we proved that the sequence generated by our proposed method converges strongly to the minimum-norm solution of the problem. Finally, we applied our result to certain classes of split inverse problems and we carried out several numerical experiments to demonstrate the efficiency of the proposed algorithm.

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