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A NOTE ON A FIXED POINT THEOREM IN MODULATED LTI-SPACES

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Dedicated to Professor Simeon Reich on the occasion of his 75th Birthday

Abstract. The aim of the paper is to re-visit the 1990 Khamsi-Kozlowski-Reich Fixed Point Theorem, which initiated a flourishing field of fixed point theory in modular function spaces. Our result generalises this theorem as well as other classical fixed point theorems, including celebrated 1965 result of Kirk. As the common setting for our investigation, we choose the modulated LTI-spaces defined as modular spaces equipped with a sequential convergence structure, which allows also to use convergence types not associated with any topology (like convergence almost everywhere).
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1. Introduction

The aim of the paper is to re-visit the 1990 Khamsi-Kozlowski-Reich Fixed Point Theorem ([9, Theorem 3.5]), which initiated a flourishing field of fixed point theory in modular function spaces (see e.g. [10], the literature referenced there, and a multitude of results published since then), that is, in spaces of measurable functions, where norms are replaced by a more general construct of modulars. As observed many times, many fixed point results including the cited Khamsi-Kozlowski-Reich Theorem are analogs of classical Banach space results proven in the context of modular function spaces. The original Khamsi-Kozlowski-Reich Theorem is a good example of this phenomenon, as it can be considered a modular function space analog of the Kirk Theorem.

The reader is referred to the book [14] to learn more about the origins of the theory of modular function spaces in a more general context. For instance, in the theory introduced there the function modulars are not assumed to be convex functions, in contrast to a simpler setting of [10]. In the current paper, we do not generally assume convexity of modulars. The main improvement over the 1990 version of the Khamsi-Kozlowski-ReichTheorem is that we consider the setting of modular spaces equipped with a sequential convergence structure, that generalise both normed and modular function spaces. Hence, the version produced in this paper, Theorem 3.2 is a common generalisation of both the 1965 result by Kirk, and the 1990 result by Khamsi, Kozlowski and Reich.

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As the common setting for our investigation, we choose the modulated LTI-spaces defined as modular spaces equipped with a sequential convergence structure, and introduced by the author in the recently published paper [17], where a problem of existence and uniqueness of approximants in such spaces is being analysed. The framework of convergence spaces was originally introduced by Kisyński in [13] (see also [3]), following much earlier ideas of Fréchet [4] and Urysohn [20]. The choice of this setting allows using convergence types not associated with a topology. An important example of this case is convergence almost everywhere.

2. Modulated Convergence Spaces

Let X be a real vector space. Let us recall the definition of modular on X, [18], and associated terminology.

Definition 2.1. A functional $\rho : X \to [0,\infty]$ is called a modular if

(1) $\rho(x) = 0$ if and only if $x = 0$
(2) $\rho(-x) = \rho(x)$
(3) $\rho(\alpha x+\beta y) \leq \rho(x)+\rho(y)$ for any $x,y \in X$, and $\alpha,\beta \geq 0$ with $\alpha+\beta = 1$

and convex modular if instead of (3) the following holds

(3') $\rho(\alpha x+\beta y) \leq \alpha\rho(x)+\beta\rho(y)$ for any $x,y \in X$, and $\alpha,\beta \geq 0$ with $\alpha+\beta = 1$

The vector space $X_\rho = \{x \in X : \rho(\lambda x) \to 0,\ as\ \lambda \to 0\}$ is called a modular space.

The notions shown in Definition 2.2 below were introduced in [15] for general modular spaces. They follow the same pattern as their equivalents in modular function spaces (see e.g [14, 10]).

Definition 2.2. Let ρ be a modular defined on a vector spaces X.

(a) We say that $\{x_n\}$, a sequence of elements of X_ρ is ρ-convergent to x and write $x_n \overset{\rho}{\to} x$ if $\rho(x_n - x) \to 0$.
(b) A sequence $\{x_n\}$ where $x_n \in X_\rho$ is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n,m \to \infty$.
(c) X_ρ is called ρ-complete if every ρ-Cauchy is ρ-convergent to an $x \in X_\rho$.
(d) A set $B \subset X_\rho$ is called ρ-closed if for any sequence of $x_n \in B$, the convergence $x_n \overset{\rho}{\to} x$ implies that x belongs to B.
(e) A set $B \subset X_\rho$ is called ρ-bounded if its ρ-diameter $\delta_\rho(B) = \sup\{\rho(x-y) : x \in B, y \in B\}$ is finite.
(f) A set $K \subset X_\rho$ is called ρ-compact if for any $\{x_n\}$ in K, there exists a subsequence $\{x_{n_k}\}$ and an $x \in K$ such that $\rho(x_{n_k} - x) \to 0$.
(g) Let $x \in X_\rho$ and $C \subset X_\rho$. The ρ-distance between x and C is defined as

$$d_\rho(x,C) = inf\{\rho(x-y) : y \in C\}.$$

(h) A ρ-ball $B_\rho(x,r)$ is defined by $B_\rho(x,r) = \{y \in X_\rho : \rho(x-y) \leq r\}$.

Let us recall from [17] basic concepts related to the sequential convergence and modulated convergence spaces, remembering that the framework of convergence spaces was originally introduced in [13], see also [3], and recent papers [15, 16].

Definition 2.3. Let X be any nonempty set. A relation ζ between sequences $\{x_n\}_{n=1}^{\infty}$ of elements of X and elements x of X, denoted by $x_n \xrightarrow{\zeta} x$, is called a sequential convergence on X if

(1) if $x_n = x$ for all $n \in \mathbb{N}$ then $x_n \xrightarrow{\zeta} x$,
(2) if $x_n \xrightarrow{\zeta} x$ and $\{x_{n_k}\}$ is a proper subsequence of $\{x_n\}$, then $x_{n_k} \xrightarrow{\zeta} x$.

The pair (X, ζ) (or shortly X) is called a convergence space.

Given a sequential convergence ζ on X, we can introduce the notions of closed and sequentially compact sets.

Definition 2.4. Let (X, ζ) be a convergence space. A set $K \subset X$ is called closed if whenever $x_n \in K$ all $n \in \mathbb{N}$ and $x_n \xrightarrow{\zeta} x$, then $x \in K$. Similarly, K is called sequentially compact if from every sequence $\{x_n\}$ of elements of K we can choose a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{\zeta} x$ for an $x \in K$.

Definition 2.5. A sequential convergence ζ is called an L-convergence on X if

(3) if $x_n \xrightarrow{\zeta} x$ and $x_n \xrightarrow{\zeta} y$, then $x = y$.

The pair (X, ζ) (or shortly X) is called an L-space.

Definition 2.6. An L-convergence ζ on X is called L^*-convergence if, in addition, it satisfies the following condition

(*) if every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ contains a subsequence $\{x_{n_{k_p}}\}$ such that $x_{n_{k_p}} \xrightarrow{\zeta} x$, then $x_n \xrightarrow{\zeta} x$.

Similarly, X is called an L^*-space.

Let ζ be a sequential convergence on X. Let us denote by $T(\zeta)$ the class of all subsets U of X such that from $x \in U$ and $x_n \xrightarrow{\zeta} x$ it follows that there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for $n \geq n_0$. We will call them open sets in the sense of $T(\zeta)$. It is easy to see that these open sets form a topology, as it was already observed by Birkhoff in 1936, [1]. Note also that the same topology $T(\zeta)$ can be as well determined by closed sets, where $F \subset X$ is called a closed set if $x \in F$, whenever $x_n \in F$ and $x_n \xrightarrow{\zeta} x$. Note that $T(\zeta)$ does not need to be Hausdorff even when ζ is an L^*-convergence, see [3].

Let now τ be a topology on X. We say that a sequence $\{x_n\}$ of elements of X converges to an $x \in X$ (and write $x_n \xrightarrow{\tau} x$) if from $x \in U \in \tau$ it follows that $x_n \in U$ for $n \in \mathbb{N}$ greater than some $n_0 \in \mathbb{N}$. It is easy to see that such convergence, denoted by $C(\tau)$, is a sequential convergence which, in addition, satisfies $(*)$. If τ is Hausdorff then the limit is unique and hence $C(\tau)$ is an L^*-convergence.

The next, fundamental result was attributed by Kisyński to Urysohn [20] and can be formulated as follows.

Proposition 2.1. *Let ζ be an L-convergence on X, $x_n, x \in X$. Then $x_n \xrightarrow{C(T(\zeta))} x$ if and only if from every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ we can choose a subsequence $\{x_{n_{k_p}}\}$ such that $x_{n_{k_p}} \xrightarrow{\zeta} x$.*

An interesting and important example is when ζ is an almost everywhere convergence of equivalence classes of Lebesgue-measurable functions on $[0,1]$ (it is easy to check that this is an L-convergence). We know that the m -almost everywhere convergence implies but is not equivalent to the convergence in measure m. As a matter of fact, $f_n \to f$ in measure if and only if from every subsequence $\{f_{n_k}\}$ we can choose a subsequence $\{f_{n_{k_p}}\}$ such that $f_{n_{k_p}} \to f$ m -almost everywhere. From Proposition 2.1 we conclude immediately that the convergence almost everywhere cannot be generated from any topology.

Let us define LTI-convergence, LTI-spaces and modulated LTI-spaces.

Definition 2.7. Let X be a real vector space and let ζ be an L-convergence on X. We say that ζ is an LTI-convergence (translation invariant convergence) if $x_n \xrightarrow{\zeta} x$ implies that $x_n - y \xrightarrow{\zeta} x - y$ for any $y \in X$. In this case, the pair (X, ζ) is called an LTI-space.

Definition 2.8. Let ρ be a modular defined on X and let ζ be an L-convergence on X_ρ. The triplet (X_ρ, ρ, ζ) is called a modulated LTI-space if (X_ρ, ζ) is an LTI-space and the following two conditions are satisfied

(i) $x_n \xrightarrow{\zeta} x \Rightarrow \rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$,

(ii) if $x_n \xrightarrow{\rho} x$ then there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\zeta} x$, where x, $x_n \in X$.

The assertions of the following Proposition are easy consequences of the appropriate definitions.

Proposition 2.2. *Let (X_ρ, ρ, ζ) be a modulated LTI-space. Then the following assertions are true.*

(i) Every ζ-closed set is also ρ-closed.
(ii) Every ρ-compact set is also sequentially ζ-compact.
(iii) Every ρ-ball $B_\rho(x,r)$ is ζ-closed (and hence also ρ-closed).
(iv) Every sequentially ζ-compact set is ζ-closed.
(v) Every ζ-closed subset of a sequentially ζ-compact set is sequentially ζ-compact.
(vi) ρ-convergence is an L^*-convergence.

Remark 2.1. Banach spaces with ρ being a norm and ζ standing for the convergence in weak topology, and modular function spaces (with the Fatou property), with the $\rho - a.e$ convergence (including Lebesgue spaces, Orlicz spaces, variable Lebesgue spaces with ζ being convergence $a.e.$ w.r.t. to measure) are typical examples of ρ-complete modulated LTI- spaces. We refer the reader to [17] for a more comprehensive list of examples.

3. Results

For the last 30 years, the evolution of fixed point theory has demonstrated the great usefulness of modular space techniques. Therefore, some fundamental fixed point existence theorems serve as an excellent example of application of the theory of modulated convergence spaces introduced in the previous section. Because of the important role played by normal structure (since the 1965 paper by Kirk [11]) in fixed point theory, we have chosen this property as an illustration of the power of the theory introduced in the current note.

Let us recall basic definitions. As in the previous section, X is a vector space and ρ is a modular defined on X.

Definition 3.1. [15, Def. 3.1]
Let C be a ρ-bounded subset of X_ρ.

(1) A mapping $T : C \to C$ is called ρ-nonexpansive if $\rho(T(x) - T(y)) \leq \rho(x - y)$ for any $x, y \in C$.
(2) The quantity $r_\rho(x,C) = \sup\ \{\rho(x - y) :\ y \in C\}$ will be called the ρ-*Chebyshev radius of* C *with respect to* x.
(3) The ρ-*Chebyshev radius of* C is defined by $R_\rho(C) = \inf\ \{r_\rho(x,C) :\ x \in C\}$.

Note that $R_\rho(C) \leq r_\rho(x,C) \leq \delta_\rho(C)$, for any $x \in C$ and any ρ-bounded nonempty subset C of X_ρ.

Let C be a ρ-bounded subset of X_ρ such that $\delta_\rho(C) > 0$. and let $\mathscr{A}$ be a class of subsets of C. With this in mind, let us introduce the following two definitions.

Definition 3.2. A class $\mathscr{A}$ is said to be ρ-normal if, for each $A \in \mathscr{A}$, not reduced to a single point, we have $R_\rho(A) < \delta_\rho(A)$. Alternatively, we say in this situation that the set C has a ρ-normal structure.

Definition 3.3. We say that $\mathscr{A}$ is countably compact if any decreasing sequence $\{A_n\}_{n\geq 1}$ of nonempty elements of $\mathscr{A}$, has a nonempty intersection.

The following technical result is a general modular version of a result for function modulars in the cited paper [9] by Khamsi, Kozlowski and Reich, and relates to Kirk's lemma [12], being in its turn an abstraction of a result obtained by Gillespie and Williams [6]. We provide the proof for the sake of completeness. Please note that, in contrast to previous results, our result is proved for general modular spaces, where no convexity or any other additional structure is assumed.

Lemma 3.1. *Let X_ρ be any modular space, and let $K \subset X_\rho$ be ρ-bounded. Let $\mathscr{K}$ be a class of subsets of K which is stable under arbitrary intersections and contains all sets of the form $K \cap B_\rho(x, p)$, where $x \in K$ and $p > 0$. Suppose that $T : K \to K$ is ρ-nonexpansive. Then, for each $\varepsilon > 0$, there exists $K_\varepsilon \in \mathscr{K}$ such that $T(K_\varepsilon) \subset K_\varepsilon$ and for which*

$$\delta_\rho(K_\varepsilon) \leq R_\rho(K) + \varepsilon\delta_\rho(K). \tag{3.1}$$

Proof. Let us denote $r = R_\rho(K) + \varepsilon\delta_\rho(K)$. If $\delta_\rho(K) = 0$, then (3.1) is trivially satisfied with $K_\varepsilon = K$. Hence we can assume that $\delta_\rho(K) > 0$. Denote $K_* = \{z \in K : K \subset B_\rho(z,r)\}$, which is not empty in view of the definition of r. Define a class $\mathscr{D} = \{D \in \mathscr{K} : K_* \subset D \subset K,\ T(D) \subset D\}$ and note that $\mathscr{D} \neq \emptyset$ since $K \in \mathscr{D}$. Set $F = \bigcap \mathscr{D}$. It is easy to see that $F \in \mathscr{K}$ and that $K_* \subset F$. It follows from the definition of $\mathscr{D}$ that $T(F) \subset F$. Denoting $A = K_* \cup T(F)$, we conclude then that $A \subset F$. Because of this and the fact that $F \in \mathscr{K}$ we conclude that $cov(A) = \bigcap\{D \in \mathscr{K} : A \subset D\} \subset F$, which implies that

$$T(cov(A)) \subset T(F) \subset A \subset cov(A), \tag{3.2}$$

proving that $cov(A) \in \mathscr{D}$. Because of this and the fact that $cov(A) \subset F$ we have $cov(A) = F$ (by the definition of F). Define $K_\varepsilon = \bigcap_{u\in F} B_\rho(u,r) \cap F$ and observe that $K_\varepsilon \in \mathscr{K}$ because $\mathscr{K}$ is stable under intersections and contains intersections of ρ-balls with K (remember that $F \subset K$).

It follows from the definition of K_* and the fact that $K_* \subset F$ that $K_* \subset K_\varepsilon$. Since $K_* \neq \emptyset$ it follows that $K_\varepsilon \neq \emptyset$. Let us prove now that $T(K_\varepsilon) \subset K_\varepsilon$. Indeed, let $x \in K_\varepsilon$, then $x \in F$, which implies that $T(x) \in F$. Let $u \in F$ then $\rho(T(x) - T(u)) \leq \rho(x-u) \leq r$ because T is ρ-nonexpansive, $x \in K_\varepsilon$ and $u \in F$. Hence $T(F) \subset B_\rho(T(x), r)$. Let $z \in K_*$, then $\rho(T(x) - z) \leq r$ because $K \subset B_\rho(z,r)$, hence $K_* \subset B_\rho(T(x), r)$. We conclude the that $A = K_* \cup T(F) \subset B_\rho(T(x), r)$, which in turn implies that $F = cov(A) \subset B_\rho(T(x), r)$. This means that for every $u \in F$, $u \in B_\rho(T(x), r)$ and hence $T(x) \in B_\rho(u, r)$, which implies that $T(x) \in C_\varepsilon$ (we already know that $T(x) \in F$). Our assertion that $T(C_\varepsilon) \subset C_\varepsilon$ is therefore proved. Finally, let $x, y \in C_\varepsilon$, then $x \in F$ and $y \in B_\rho(x, r)$, which implies that $\rho(x-y) \leq r$ and consequently that $\delta_\rho(K_\varepsilon) \leq r$. The proof of the Lemma is therefore complete. □

We are now ready to prove a fixed point result that generalises (among many other results) the classical Kirk Theorem [11] for Banach spaces and, at the same time, the Khamsi-Kozlowski-Reich Theorem for modular function spaces [9, Theorem 3.5]. We quote both these fundamental results below. The commentary about the relations between these two results and Theorem 3.3 of this paper will be provided after the proof of the latter result.

Theorem 3.1 (Kirk Theorem)**.** *Let C be a nonempty, bounded, closed, and convex subset of a reflexive Banach space, and suppose that C has normal structure. If T is a nonexpansive mapping of C into itself, then T has a fixed point.*

Theorem 3.2 (Khamsi-Kozlowski-Reich Theorem)**.** *Let ρ have the Fatou property. Suppose that a ρ-bounded, ρ-a.e. compact C subset of a modular function space L_ρ has ρ-normal structure. If $T : C \to C$ is ρ-nonexpansive, then it has a fixed point.*

Theorem 3.3. *Let (X_ρ, ρ, ζ) be a modulated LTI-space. Let $C \subset X_\rho$ be nonempty ρ-bounded, sequentially ζ-compact. Define $\mathscr{A}$ as a class of all nonempty, ζ-closed subset of C and assume that $\mathscr{A}$ is ρ-normal. Let $T : C \to C$ be a ρ-nonexpansive mapping. Then T has a fixed point in C.*

Proof. First, we will demonstrate that $\mathscr{A}$ is countably compact. Indeed, let $\{A_n\}$ be a decreasing sequence of nonempty sets from $\mathscr{A}$. We need to show that the interception of $\{A_n\}$ is nonempty. Let $x_n \in A_n \subset C$ for every $n \in \mathbb{N}$. Since C is sequentially ζ-compact there is a subsequence $\{x_{n_k}\}$ and $x \in C$ such that $x_{n_k} \xrightarrow{\zeta} x$. We claim that $x \in \bigcap_{k=1}^{\infty} A_{n_k}$. To this end, suppose to the contrary that $x \notin A_{n_k}$ for $k \geq k_0$ for some $k_0 \in \mathbb{N}$. It follows from Proposition 2.2 (v) that the set $A_{n_{k_0}}$ is sequentially ζ-compact and therefore $x \in A_{n_{k_0}}$ (being the ζ-limit of $\{x_{n_k}\}$), which contradicts the indirect assumption, and proves that $x \in \bigcap_{k=1}^{\infty} A_{n_k}$ and hence that $\mathscr{A}$ is countably compact.

Let $\mathscr{D} = \{D \in \mathscr{A} : D \neq \emptyset,\ T : D \to D\}$ and note that $\mathscr{D} \neq \emptyset$ because $C \in \mathscr{D}$. Let us define $\tilde{\delta}_\rho : \mathscr{D} \to [0, \infty)$ by

$$\tilde{\delta}_\rho(D) = \inf\{\delta_\rho(B) : B \in \mathscr{D},\ B \subset D\}. \tag{3.3}$$

Set $D_1 = C$. By definition of $\tilde{\delta}(D_1)$, there exists $D_2 \in \mathscr{D}$ such that $D_2 \subset D_1$ and $\delta_\rho(D_2) < \tilde{\delta}(D_1) + 1$. Using the same argument we can inductively construct a sequence $\{D_n\}$ such that $D_{n+1} \in \mathscr{D}$, $D_{n+1} \subset D_n$ and

$$\delta_\rho(D_{n+1}) < \tilde{\delta}(D_n) + \frac{1}{n}. \tag{3.4}$$

Since $\mathscr{A}$ is countably compact, then $D_\infty = \bigcap_{n\geq 1} D_n$ is not empty. Clearly $D_\infty \in \mathscr{D}$ and $T : D_\infty \to D_\infty$ is ρ-nonexpansive. It remains to be proved that D_∞ is reduced to a single point. Let us define a class $\mathscr{F} = \{D \in \mathscr{D} : D \subset D_\infty\}$. Fix temporarily an arbitrary $\varepsilon > 0$. Observe that we can apply Lemma 3.1 taking D_∞ instead of K and $\mathscr{F}$ instead of $\mathscr{K}$. Hence, there exists $D_\varepsilon \in \mathscr{D}$ such that $D_\varepsilon \subset D_\infty$, $T : D_\varepsilon \to D_\varepsilon$ and

$$\delta_\rho(D_\varepsilon) \leq R_\rho(D_\infty) + \varepsilon\delta_\rho(D_\infty). \tag{3.5}$$

Let us observe that for every n, $\tilde{\delta}(D_n) \leq \delta_\rho(D_\varepsilon)$ because of the definition of $\tilde{\delta}(D_n)$ and the fact that $D_\varepsilon \in \mathscr{D}$, $D_\varepsilon \subset D_\infty \subset D_n$ and $T(D_\varepsilon) \subset D_\varepsilon$. Using this fact and combining it with (3.4) and (3.5) we have

$$\delta_\rho(D_\infty) - \frac{1}{n} \leq \delta_\rho(D_{n+1}) - \frac{1}{n} \leq \tilde{\delta}_\rho(D_n) \leq \delta_\rho(D_\varepsilon) \leq R_\rho(D_\infty) + \varepsilon\delta_\rho(D_\infty). \tag{3.6}$$

Passing with n to infinity we get $\delta_\rho(D_\infty) \leq R_\rho(D_\infty) + \varepsilon\delta_\rho(D_\infty)$, which by arbitrariness of $\varepsilon > 0$ gives us $\delta_\rho(D_\infty) \leq R_\rho(D_\infty)$. In view of the assumption that $\mathscr{A}$ is normal, this is possible only if D_∞ is reduced to a single point which is then a fixed point for T. □

In [9], the Khamsi-Kozlowski-Reich Theorem (Theorem 3.2) was presented as an analog of Kirk's fundamental result in Banach spaces (Theorem 3.3) in modular function spaces. It was not really a generalisation of the Kirk Theorem because, obviously, not every Banach space is a modular function space. As intimidated in Introduction to this paper, the similarity between these two results called for a structure allowing to generalise both these fundamental results. As observed in Remark 2.1, Banach spaces and modular function spaces with the Fatou property are examples of modulated LTI-spaces, and hence our Theorem 3.3 generalises both these classical results, as announced earlier in this paper.

Brailey Sims observed in [19] that Banach spaces which are uniformly convex in every direction ($UCED$) have weak normal structure (i.e., every weak compact convex set has normal structure), an important result with origins in work by Garkavi [5] (similar results were obtained also in hyperbolic spaces, see e.g. the books by Goebel and Reich [7] and by Khamsi and Kirk [8]). Using then Kirks' theorem, we can conclude that $UCED$ Banach spaces enjoy the weak fixed point property. As it turns out, we have an analogous result in modulated LTI-spaces. To see this, we introduced a relevant notion of $UCED$, in which we follow a similar notion introduced for modular function spaces in [9], see also a more recent application of $UCED$ for modulated topological vector spaces in [16] and modular function spaces in [2].

Definition 3.4. [15] [Def. 3.6] Let ρ be a convex modular. For any nonzero $u \in X_\rho$ and $r > 0$, we define the r-modulus of uniform convexity of ρ in the direction of u as

$$\delta(r,u) = \inf\left\{1 - \frac{1}{r}\rho\left(y + \frac{1}{2}u\right)\right\},$$

where the infimum is taken over all $y \in X_\rho$ such that $\rho(y) \leq r$ and $\rho(y+u) \leq r$.
We say that X_ρ is ρ uniformly convex in every direction ($\rho - UCED$) if $\delta(r,u) > 0$ for every nonzero $u \in X_\rho$ and all $r > 0$.

We will need the following result proved in [15], and inspired by an analogous result for modular function spaces [9][Proposition 3.10]. For the completeness sake, we reproduce its short proof.

Proposition 3.1. [15] *[Prop.3.7] Let a modular space X_ρ be $\rho - UCED$, and let $C \subset X_\rho$ be convex, ρ-bounded and not a singleton. Then C has a ρ-nondiametral point.*

Proof. Take any $x \neq y$ elements of C. Fix temporarily any $h \in C$ and set $u = x - y$, $w = y - h$ and $r = \delta_\rho(C)$. Then $\rho(w) = \rho(y - u) \leq r$ and $\rho(w + u) = \rho(x - h) \leq r$. By the definition of $\delta(r, u)$ we have then

$$\rho\left(w + \frac{1}{2}u\right) \leq r(1 - \delta(r, u)),$$

which by a straightforward calculation gives us

$$\rho\left(\frac{x + y}{2} - h\right) \leq r(1 - \delta(r, u)).$$

Hence,

$$\sup_{h \in C} \rho\left(\frac{x + y}{2} - h\right) \leq \delta_\rho(C)(1 - \delta(r, u)) < \delta_\rho(C),$$

because $\delta(r, u) > 0$. Consequently, $\frac{x+y}{2}$ is not a ρ-diametral point in C. □

By combining Theorem 3.3 with Proposition 3.1, we obtain the following result being an extension of the Browder fixed point theorem to the case of modulated LTI-spaces.

Theorem 3.4. *Let (X_ρ, ρ, ζ) be a $\rho - UCED$ modulated LTI-space. Let $C \subset X_\rho$ be convex, ρ-bounded and ζ-sequentially compact. If $T : C \to C$ is ρ-nonexpansive, then T has a fixed point.*

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