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ANISOTROPIC (p,q)-EQUATIONS WITH A LOCALLY DEFINED REACTION

SERGIU AIZICOVICI[1], NIKOLAOS S. PAPAGEORGIOU[2], VASILE STAICU[3,*]

[1]*Department of Mathematics, Ohio University, Athens, OH 45701, USA*
[2]*Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece*
[3]*CIDMA - Center for Research and Development in Mathematics and Applications,*
Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

Abstract. We consider a nonlinear Dirichlet problem driven by the anisotropic (p,q)-Laplacian, and a Carathéodory reaction $f(z,x)$ ($z \in \Omega \subseteq \mathbb{R}^N$, $x \in \mathbb{R}$), which is only locally defined around zero in $x \in \mathbb{R}$. We prove a mltiplicity theorem providing sign information for all the solutions, which are also ordered. Also, under a symmetry condition on $f(z,\cdot)$, we generate a whole sequence of nodal smooth solutions, converging to zero in $C_0^1\left(\overline{\Omega}\right)$.

Keywords. Anisotropic equation; Critical groups; Constant sign and nodal solutions; Locally defined reaction; Symmetry condition.
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1. INTRODUCTION

In this paper, we study the following Dirichlet problem driven by the anisotropic (p,q)-Laplacian

$$-\triangle_{p(z)}u(z) - \triangle_{q(z)}u(z) = f(z,u(z)) \text{ in } \Omega,\ u\,|_{\partial\Omega}= 0. \tag{1.1}$$

In this problem, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2 boundary $\partial\Omega$. Consider the set

$$E_1 = \left\{ r \in C\left(\overline{\Omega}\right) : 1 < \min_{\overline{\Omega}} r \right\},$$

and let $r \in E_1$. By $\triangle_{r(z)}$, we denote the anisotropic $r-$ Laplace differential operator defined by

$$\triangle_{r(z)}u = div\left(|Du|^{r(z)-2}Du\right), \text{ for all } u \in W_0^{1,r(z)}(\Omega),$$

where $|\cdot|$ denotes the norm in $\mathbb{R}^N$. In contrast with the isotropic $r-$ Laplacian (that is, $r(z) = r > 1$ for all $z \in \overline{\Omega}$), the anisotropic operator is not homogeneous, and this is a source of difficulties in the analysis of anisotropic problems. Problem (1.1) is driven by the sum of two such operators with different exponents (double phase problem).

In the reaction (right hand side of (1.1)), the function $f(z,x)$ is a Carathéodory function (that is, $z \mapsto f(z,x)$ is measurable and $x \mapsto f(z,x)$ is continuous) which is only locally defined in x,

*Corresponding author.
E-mail address: aizicovs@ohio.edu (S. Aizicovici), npapg@math.ntua.gr (N. S. Papageorgiou), vasile@ua.pt (V. Staicu).
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that is, the restrictions on $f(z,\cdot)$ concern only its behavior near zero. There are no conditions about the behavior of $f(z,\cdot)$ as $x \to \pm\infty$.

Our aim is to prove a multiplicity theorem for problem (1.1) (three solutions theorem), providing sign information for all the solutions. Moreover, imposing a symmetry condition on $f(z,\cdot)$, we generate a whole sequence of smooth nodal solutions converging to zero in $C_0^1(\overline{\Omega})$, complementing a recent similar result of the authors [1]. Our work here extends that of Tan-Fang [2] (see Theorems 1.2, 1.3, and 1.4).

The multiplicity results of Tan-Fang [2] were obtained by imposing global conditions on $f(z,\cdot)$ (that is, both near zero and near $\pm\infty$), and the authors did not provide sign information for all the solutions. Moreover, in Theorem 1.4 of Tan-Fang [2], which produces a whole sequence of solutions under a symmetry condition, the authors did not demonstrate that the solutions are nodal and that the convergence is in $C_0^1(\overline{\Omega})$ (they show convergence to zero in $W_0^{1,p(z)}(\Omega)$). Finally we point out that in Tan-Fang [2], the equation is driven by the $p(z)$-Laplacian only.

In the past, multiplicity theorems for the problems with a locally defined reaction were proved only for parametric isotropic problems. We here mention the works of Gasinski-Papageorgiou [3] and Papageorgiou-Radulescu-Repovs [4]. Similar remarks apply to the results concerning asymptotically vanishing nodal solutions. We refer to the works of Leonardi-Papageorgiou [5], Papageorgiou-Zhang [6], and Wang [7].

2. Mathematical Background - Hypotheses

For the analysis of problem (1.1), we use Lebesgue and Sobolev spaces with variable exponents. A comprehensive presentation of the theory of those spaces can be found in the books of Cruz Uribe-Fiorenza [8] and of Diening-Harjulehto-Hasto-Ruzicka [9].

By $L^0(\Omega)$, we denote the space of all Lebesgue measurable functions $u:\Omega\to\mathbb{R}$. As usual, we identify two such functions which differ only on a Lebesgue null set. Let $r \in E_1 = \left\{ r \in C(\overline{\Omega}) : 1 < \min_{\overline{\Omega}} r \right\}$. Then the anisotropic Lebesgue space $L^{r(z)}(\Omega)$ is defined by

$$L^{r(z)}(\Omega) = \left\{ u \in L^0(\Omega) : \rho_r(u) = \int_\Omega |u|^{r(z)}\,dz < \infty \right\}.$$

On this space, we define the so called "Luxemburg norm" by

$$\|u\|_{r(z)} = \inf\left\{ \lambda > 0 : \int_\Omega \left(\frac{|u(z)|}{\lambda}\right)^{r(z)} dz \le 1 \right\}.$$

Equipped with this norm, $L^{r(z)}(\Omega)$ becomes a separable and reflexive (in fact uniformly convex) Banach space.

The Luxemburg norm $\|\cdot\|_{r(z)}$ and the modular function $\rho_r(\cdot)$ are closely related. In what follows, given $r \in E_1$, we define

$$r_- = \min_{\overline{\Omega}} r \text{ and } r_+ = \max_{\overline{\Omega}} r.$$

Proposition 2.1. *If $r \in E_1$ and $\{u_n, u\}_{n\in\mathbb{N}} \subseteq L^{r(z)}(\Omega)$, then*

(a) $\|u\|_{r(z)} = \lambda \Longleftrightarrow \rho_r\left(\frac{u}{\lambda}\right) = 1$;

(b) $\|u\|_{r(z)} < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho_r(u) < 1$ *(resp.* $= 1, > 1$*);*

(c) $\|u\|_{r(z)} \leq 1 \Rightarrow \|u\|_{r(z)}^{r_+} \leq \rho_r(u) \leq \|u\|_{r(z)}^{r_-}$ *and*

$$\|u\|_{r(z)} \geq 1 \Rightarrow \|u\|_{r(z)}^{r_-} \leq \rho_r(u) \leq \|u\|_{r(z)}^{r_+};$$

(d) $\|u_n\|_{r(z)} \to 0$ *(resp.* $\to +\infty$*)* $\Longleftrightarrow \rho_r(u_n) \to 0$ *(resp.* $\to +\infty$*);*

(e) $\|u_n - u\|_{r(z)} \to 0 \Longleftrightarrow \rho_r(u_n - u) \to 0.$

The modular function $\rho_r(\cdot)$ *is continuous and convex, hence weakly lower semicontinuous too.*

Given $r \in E_1$, let $r' \in E_1$ be defined by

$$r'(z) = \frac{r(z)}{r(z)-1} \text{ for all } z \in \overline{\Omega}$$

(that is,

$$\frac{1}{r(z)} + \frac{1}{r'(z)} = 1, \text{ for all } z \in \overline{\Omega}).$$

We have $L^{r(z)}(\Omega)^* = L^{r'(z)}(\Omega)$ and can state the following Holder-type inequality

$$\int_\Omega |uv|\, dz \leq \left[\frac{1}{r_-} + \frac{1}{r'_-}\right] \|u\|_{r(z)} \|v\|_{r'(z)}$$

$$\text{for all } u \in L^{r(z)}(\Omega), \text{ all } v \in L^{r'(z)}(\Omega).$$

We can also define the variable exponent (anisotropic) Sobolev spaces. So, for $r \in E_1$, we define

$$W^{1,r(z)}(\Omega) = \left\{u \in L^{r(z)}(\Omega) : |Du| \in L^{r(z)}(\Omega)\right\},$$

where Du is the weak gradient of u. This space is equipped with the following norm

$$\|u\|_{1,r(z)} = \|u\|_{r(z)} + \|Du\|_{r(z)} \text{ for all } u \in W^{1,r(z)}(\Omega)$$

with

$$\|Du\|_{r(z)} = \||Du|\|_{r(z)}.$$

Then $W^{1,r(z)}(\Omega)$ becomes a separable and reflexive (in fact uniformly convex) Banach space. Let

$$C^{0,1}\left(\overline{\Omega}\right) = \left\{u : \overline{\Omega} \to \mathbb{R} : \ u \text{ is Lipschitz continuous}\right\}.$$

Given $r \in E_1 \cap C^{0,1}\left(\overline{\Omega}\right)$, we define

$$W_0^{1,r(z)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,r(z)}}.$$

This is a Banach space which is separable and reflexive (in fact uniformly convex), and the Poincaré inequality holds, that is,

$$\|u\|_{r(z)} \leq \widehat{C}\|Du\|_{r(z)} \text{ for some } \widehat{C} > 0, \text{ all } u \in W_0^{1,r(z)}(\Omega).$$

Therefore, on $W_0^{1,r(z)}(\Omega)$, we can consider the equivalent norm

$$\|u\| = \|Du\|_{r(z)} \text{ for all } u \in W_0^{1,r(z)}(\Omega).$$

For these spaces, we have the following useful embeddings (anisotropic Sobolev embedding theorem):

Proposition 2.2. *If $r \in E_1 \cap C^{0,1}\left(\overline{\Omega}\right)$ with $r(z) < N$ for all $z \in \overline{\Omega}$, then*

$$W_0^{1,r(z)}(\Omega) \hookrightarrow L^{\tau(z)}(\Omega)$$

continuously (resp. compactly) for all $\tau \in C(\Omega)$ with $1 \le \tau(z) \le r^*(z)$ for all $z \in \overline{\Omega}$ (resp. $1 \le \tau(z) < r^*(z)$ for all $z \in \overline{\Omega}$), where

$$r^*(z) = \frac{Nr(z)}{N - r(z)} \text{ for all } z \in \overline{\Omega}.$$

We have

$$W_0^{1,r(z)}(\Omega)^* = W^{-1,r'(z)}(\Omega)$$

and consider the nonlinear operator $A_{r(z)} : W_0^{1,r(z)}(\Omega) \to W^{-1,r'(z)}(\Omega)$ defined by

$$\left\langle A_{r(z)}(u), h\right\rangle = \int_\Omega |Du|^{r(z)-2} (Du, Dh)_{\mathbb{R}^N} dz \tag{2.1}$$

$$\text{for all } u,\ v \in W_0^{1,r(z)}(\Omega),$$

where $(\cdot,\cdot)_{\mathbb{R}^N}$ denotes the inner product in $\mathbb{R}^N$. This operator defined by (2.1) has the following properties (see Fan-Zhang [10], Theorem 3.1).

Proposition 2.3. *The operator $A_{r(z)} : W_0^{1,r(z)}(\Omega) \to W^{-1,r'(z)}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone, too), and of type $(S)_+$, that is, "if $u_n \xrightarrow{w} u$ in $W_0^{1,r(z)}(\Omega)$ and*

$$\limsup_{n\to\infty} \left\langle A_{r(z)}(u_n), u_n - u\right\rangle \le 0,$$

then

$$u_n \to u \text{ in } W_0^{1,r(z)}(\Omega) \text{ as } n \to \infty\text{"}.$$

Here and in what follows, " $\xrightarrow{w}$ " stands for the weak convergence.

We also use the space

$$C_0^1\left(\overline{\Omega}\right) = \left\{u \in C^1\left(\overline{\Omega}\right) : u|_{\partial\Omega} = 0\right\},$$

which is an ordered Banach space with positive (order) cone

$$C_+ = \left\{u \in C_0^1\left(\overline{\Omega}\right) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior, given by

$$int\ C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}, \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\right\}$$

with

$$\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N},$$

where $n(\cdot)$ is the outward unit normal on $\partial\Omega$. Let $u : \Omega \to \mathbb{R}$ be measurable. Then we define

$$u^{\pm}(z) = \max\{\pm u(z), 0\} \text{ for all } z \in \Omega.$$

Both $u^{\pm}(\cdot)$ are measurable and

$$u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Moreover, if $u \in W_0^{1,r(z)}(\Omega)$, then $u^{\pm} \in W_0^{1,r(z)}(\Omega)$. If $u,\ v : \Omega \to \mathbb{R}$ are measurable functions with $u(z) \le v(z)$ for all $z \in \Omega$, then we define

$$[u,v] = \left\{ h \in W_0^{1,r(z)}(\Omega) : u(z) \le h(z) \le v(z) \text{ for a.a. } z \in \Omega \right\}.$$

Also by $int_{C_0^1(\overline{\Omega})}[u,v]$, we denote the interior in $C_0^1\left(\overline{\Omega}\right)$ of $[u,v] \cap C_0^1\left(\overline{\Omega}\right)$.

Suppose that $\varphi \in C^1\left(W_0^{1,r(z)}(\Omega)\right)$. The critical set of φ is defined by

$$K_{\varphi} = \left\{ u \in W_0^{1,r(z)}(\Omega) : \varphi'(u) = 0 \right\}.$$

Let $u \in K_{\varphi}$ be isolated and $k \in \mathbb{N}_0$. By $C_k(\varphi,u)$, we denote the $k^{th}-$critical group of φ at u, with coefficients in the field of reals. So, $C_k(\varphi,u)$ is a linear space and no torsion phenomena can occur (see Papageorgiou-Radulescu-Repovs [11], Sections 6.1 and 6.2).

Now we introduce our hypotheses on the data of problem (1.1).

$\mathbf{H_0}:$ $p,q \in C^{0,1}\left(\overline{\Omega}\right)$, $1 < q(z) < p(z) < N$ for all $z \in \overline{\Omega}$.

$\mathbf{H_1}:$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a. a. $z \in \Omega$ and

(i) there exist $\theta_- < 0 < \theta_+$ and $a_0 \in L^{\infty}(\Omega)$ such that

$$f(z,\theta_+) \le -C_0 < 0 < C_0 \le f(z,\theta_-) \text{ for a.a. } z \in \Omega,$$

$$|f(z,x)| \le a_0(z) \text{ for a. a. } z \in \Omega, \text{ all } |x| \le \widehat{\theta} := \max\{\theta_+, -\theta_-\};$$

(ii) there exists $\tau \in C\left(\overline{\Omega}\right)$ with $\tau_+ < q_-$ and $\delta > 0$ such that

$$C_1 |x|^{\tau(z)} \le f(z,x)x \text{ for a. a. } z \in \Omega, \text{ all } |x| \le \delta;$$

(iii) there exists $\widehat{\xi} > 0$ such that for a. a. $z \in \Omega$, the function

$$x \to f(z,x) + \widehat{\xi}\,|x|^{p(z)-2}x$$

is nondecreasing on $\left[-\widehat{\theta}, \widehat{\theta}\right]$.

Remark 2.1. We see that our hypotheses on $f(z,\cdot)$ concern the interval $\left[-\widehat{\theta}, \widehat{\theta}\right]$. The values of $f(z,\cdot)$ outside $\left[-\widehat{\theta}, \widehat{\theta}\right]$ are irrelevant and the function can be arbitrary there.

3. A Multiplicity Theorem

In this section, we prove a multiplicity theorem for problem (1.1) (three solutions theorem), and we provide sign information for all the solutions.

First, we produce two nontrivial constant sign smooth solutions.

Proposition 3.1. *If hypotheses* $\mathbf{H_0}$, $\mathbf{H_1}$ *hold, then problem* (1.1) *has at least two constant sign solutions*

$$u_0 \in int\ C_+ \text{ with } u_0(z) < \theta_+ \text{ for all } z \in \overline{\Omega},$$
$$v_0 \in -int\ C_+ \text{ with } \theta_- < v_0(z) \text{ for all } z \in \overline{\Omega}.$$

Proof. We introduce the Carathéodory function $g_+(z,x)$ defined by

$$g_+(z,x) = \begin{cases} f(z,x^+) & \text{if} \quad x \le \theta_+ \\ f(z,\theta_+) & \text{if} \quad \theta_+ < x. \end{cases} \tag{3.1}$$

Let

$$G_+(z,x) = \int_0^x g_+(z,s)\,ds$$

and consider the C^1- functional $\psi_+ : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\,dz - \int_\Omega G(z,u)\,dz$$
$$\text{for all } u \in W_0^{1,p(z)}(\Omega).$$

Using (3.1), we see that

$$\psi_+(u) \ge \frac{1}{p_+}\left[\rho_p(Du) + \rho_q(Du)\right] - C_2 \text{ for some } C_2 > 0.$$

Hence, $\psi_+(\cdot)$ is coercive (see Proposition 2.1). Also, using the anisotropic Sobolev embedding theorem (see Proposition 2.2), we infer that $\psi_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$\psi_+(u_0) = \inf\left\{\psi_+(u) : u_0 \in W_0^{1,p(z)}(\Omega)\right\}. \tag{3.2}$$

Let $u \in int\ C_+$ and choose $t \in (0,1)$ small, such that $0 \le tu(z) \le \delta$ for all $z \in \overline{\Omega}$, with $\delta > 0$ as in hypothesis $\mathbf{H_1}(ii)$. Clearly, we can always assume that $\delta < \min\{\theta_+, -\theta_-\}$. Using (3.1) and hypothesis $\mathbf{H_1}(ii)$, we have

$$\psi_+(tu) \le \frac{t^{q_-}}{q_-}\left[\rho_p(Du) + \rho_q(Du)\right] - \frac{t^{\tau_+}}{\tau_+} C_1 \rho_\tau(u)$$
$$\le C_3 t^{q_-} - C_4 t^{\tau_+} \text{ for some } C_3,\ C_4 > 0.$$

Since $\tau_+ < q_-$ (see hypothesis $\mathbf{H_1}(ii)$), choosing $t \in (0,1)$ even smaller if necessary, we have $\psi_+(tu) < 0$. Hence

$$\psi_+(u_0) < 0 = \psi_+(0) \text{ (see (3.2))}.$$

Thus $u_0 \ne 0$. From (3.2), we have

$$\left\langle \psi_+'(u_0), h\right\rangle = 0 \text{ for all } h \in W_0^{1,p(z)}(\Omega).$$

Hence

$$\langle V(u_0), h\rangle = \int_\Omega g_+(z,u_0)\,h dz \text{ for all } h \in W_0^{1,p(z)}(\Omega). \tag{3.3}$$

Here $V : W_0^{1,p(z)}(\Omega) \to W^{-1,p(z)}(\Omega)$ is defined by

$$V(u) = A_{p(z)}(u) + A_{q(z)}(u) \text{ for all } u \in W_0^{1,p(z)}(\Omega).$$

On account of Proposition 2.3, $V(\cdot)$ is continuous, strictly monotone (thus maximal monotone, too) and of type $(S)_+$. In (3.3), we first choose the test function $h=-u_0^- \in W_0^{1,p(z)}(\Omega)$. We obtain $\rho_q\left(Du_0^-\right) \le 0$. Hence,

$$u_0 \ge 0,\ u_0 \ne 0$$

(see Proposition 2.1). Next, in (3.3), we use test function $h=[u_0-\theta_+]^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}\left\langle V(u_0),[u_0-\theta_+]^+\right\rangle &= \int_\Omega f(z,\theta_+)[u_0-\theta_+]^+\,dz \text{ (see (3.1))}\\ &\le 0=\left\langle V(\theta_+),[u_0-\theta_+]^+\right\rangle.\end{aligned}$$

Here we consider $V(\cdot)$ defined on $W^{1,p(z)}(\Omega)$ with values in $W^{1,p(z)}(\Omega)^*$ and as such, $V(\cdot)$ remains continuous and monotone (thus maximal monotone, too). So, we have $u_0 \le \theta_+$. Thus

$$u_0 \in [0,\theta_+],\ u_0 \ne 0. \tag{3.4}$$

From (3.4), (3.1) and (3.3), it follows that $u_0 \in W_0^{1,p(z)}(\Omega)\setminus\{0\}$ is a positive solution to problem (1.1). From Fan [12], we know that $u_0 \in C_+\setminus\{0\}$. Also, using hypothesis $\mathbf{H_1}(iii)$, we have

$$-\triangle_{p(z)}u_0-\triangle_{q(z)}u_0+\widehat{\xi}u_0^{p(z)-1} \ge 0 \text{ in } \Omega.$$

Using Proposition 4 of Papageorgiou-Qin-Radulescu [13], we obtain that $u_0 \in int\ C_+$. We have

$$\begin{aligned}&-\triangle_{p(z)}u_0-\triangle_{q(z)}u_0+\widehat{\xi}u_0^{p(z)-1}\\ &=f(z,u_0)+\widehat{\xi}u_0^{p(z)-1}\\ &\le f(z,\theta_+)+\widehat{\xi}\theta_+^{p(z)-1}\text{(see (3.4) and hypothesis } \mathbf{H_1}(iii))\\ &\le -\triangle_{p(z)}\theta_+-\triangle_{q(z)}\theta_++\widehat{\xi}\theta_+^{p(z)-1}.\end{aligned}$$

We know that

$$f(z,\theta_+) \le -C_0 < 0 \text{ for a.a. } z\in\Omega \text{ (see hypothesis } \mathbf{H_1}(ii)).$$

Then using Proposition 5 of Papageorgiou-Qin-Radulescu [13] , we infer that

$$0 \le u_0(z) < \theta_+ \text{ for all } z \in \overline{\Omega}.$$

For the negative solution, we start with the Carathéodory function $g_-(z,x)$ defined by

$$g_-(z,x)=\begin{cases} f(z,\theta_-) & \text{if } x \le \theta_- \\ f(z,-x^-) & \text{if } \theta_- < x.\end{cases} \tag{3.5}$$

We set

$$G_-(z,x)=\int_0^x g_-(z,s)\,ds$$

and consider the C^1- functional $\psi_- : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\psi_-(u)=\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz+\int_\Omega\frac{1}{q(z)}|Du|^{q(z)}dz-\int_\Omega G_-(z,u)\,dz$$
$$\text{for all } u \in W_0^{1,p(z)}(\Omega).$$

Working with $\psi_{-}(u)$ as above, we obtain a negative solution $v_0 \in -int\ C_+$ with $\theta_{-} < v_0(z)$ for all $z \in \overline{\Omega}$. $\square$

In fact, we can show that (1.1) has extremal constant sign solutions, that is, a smallest positive solutions and a biggest negative solution (barrier solutions). We will use these solutions in order to produce a nodal one (a sign-changing solution).

To this end, motivated by hypothesis $\mathbf{H_1}(ii)$, we consider the following auxiliary anisotropic Dirichlet problem

$$-\triangle_{p(z)}u(z) - \triangle_{q(z)}u(z) = C_1 |u(z)|^{\tau(z)-2} u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{3.6}$$

Proposition 3.2. *With $\tau \in C(\overline{\Omega})$ as in hypothesis $\mathbf{H_1}(ii)$, problem (3.6) has a unique positive solution $\overline{u} \in int\ C_+$, and since the problem is odd, $\overline{v} = -\overline{u} \in -int\ C_+$ is the unique negative solution to (3.6).*

Proof. Consider the C^1- functional $\sigma_+ : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz - \int_\Omega \frac{C_1}{\tau(z)} \left(u^+\right)^{\tau(z)} dz$$
$$\text{for all } u \in W_0^{1,p(z)}(\Omega).$$

We want to show the coercivity of $\sigma_+(\cdot)$. So, we may assume that $\|u\|_{\tau(z)} \geq 1$ is large enough (recall that $W_0^{1,p(z)}(\Omega) \hookrightarrow L^{\tau(z)}(\Omega)$ compactly, cf. Proposition 2.2). We have

$$\begin{aligned}
\sigma_+(u) &\geq \frac{1}{p_+}\rho_p(Du) - \frac{C_1}{\tau_+}\rho_\tau(u) \\
&\geq \frac{1}{p_+}\|u\|^{p_-} - \frac{C_1}{\tau_+}\|u\|_{\tau(z)}^{\tau_+} \text{ (see Proposition 2.1)} \\
&\geq \frac{1}{p_+}\|u\|^{p_-} - C_5\|u\|^{\tau_+} \text{ for some } C_5 > 0,
\end{aligned}$$

but $\tau_+ < q_+ < p_+$ (see hypotheses $\mathbf{H_0}, \mathbf{H_1}(ii)$). It follows that

$$\sigma_+(\cdot) \text{ is coercive.}$$

Also, using Proposition 2.2 (the anisotropic Sobolev embedding theorem), we see that $\sigma_+(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $\overline{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\sigma_+(\overline{u}) = \inf\left\{\sigma_+(u) : u_0 \in W_0^{1,p(z)}(\Omega)\right\}. \tag{3.7}$$

As in the proof of Proposition 3.1, since $\tau_+ < q_-$, we have

$$\sigma_+(\overline{u}) < 0 = \sigma_+(0).$$

Hence,

$$\overline{u} \neq 0. \tag{3.8}$$

From (3.7), we have

$$\left\langle \sigma_+'(\overline{u}), h\right\rangle = 0 \text{ for all } h \in W_0^{1,p(z)}(\Omega).$$

Hence,

$$\langle V(\overline{u}), h\rangle = \int_\Omega C_1 (\overline{u}_+)^{\tau(z)-1} h dz \text{ for all } h \in W_0^{1,p(z)}(\Omega). \tag{3.9}$$

In (3.9), we choose the test function $h=-\overline{u}^{-}\in W_0^{1,p(z)}(\Omega)$. We obtain $\rho_p(D\overline{u}^{-})\leq 0$. Hence,

$$\overline{u}\geq 0,\ \overline{u}\neq 0$$

(see Proposition 2.1 and (3.8)). From (3.9), we see that $\overline{u}$ is a positive solution of (3.6). From Theorem 4.1 of Fan-Zhao [10] (see also [14]), it follows that $\overline{u}\in L^{\infty}(\Omega)$. Then the anisotropic regularity theory of Fan [12] and the anisotropic maximum principle of Papageorgiou-Radulescu-Zhang [14] (Proposition A2) imply that $\overline{u}\in int\ C_+$.

Next, we prove the uniqueness of this positive solution. Suppose that $\widetilde{u}$ is another positive solution to (3.6). Again we show that $\widetilde{u}\in int\ C_+$. Using Proposition 4.1.22, p.274, of [11], we conclude that

$$\frac{\widetilde{u}}{\overline{u}}\in L^{\infty}(\Omega) \text{ and } \frac{\overline{u}}{\widetilde{u}}\in L^{\infty}(\Omega). \tag{3.10}$$

We introduce the integral functional $j:L^1(\Omega)\to\overline{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$ defined by

$$j(u)=\begin{cases}\displaystyle\int_\Omega \frac{1}{p(z)}\left|Du^{\frac{1}{q_-}}\right|^{p(z)}dz+\int_\Omega \frac{1}{q(z)}\left|Du^{\frac{1}{q_-}}\right|^{q(z)}dz & \\ \qquad\qquad\qquad\qquad \text{if } u\geq 0,\ u^{\frac{1}{q_-}}\in W_0^{1,p(z)}(\Omega) & \\ +\infty \qquad\qquad\qquad\qquad\qquad \text{otherwise} & \end{cases}$$

Let

$$\operatorname{dom}(j)=\left\{u\in L^1(\Omega):j(u)<\infty\right\}$$

(the effective domain of $j(\cdot)$). From Takǎc-Giacomoni ([15], Theorem 2.2), we know that $j(\cdot)$ is convex. Let $h=\overline{u}^{q_-}-\widetilde{u}^{q_-}\in W_0^{1,p(z)}(\Omega)$. On account of (3.10), we see that for $t\in(0,1)$ small, we have

$$\begin{aligned}&\overline{u}^{q_-}+th\in dom(j),\\&\widetilde{u}^{q_-}+th\in dom(j).\end{aligned}$$

Then the convexity of $j(\cdot)$ implies that the directional derivatives of $j(\cdot)$ at $\overline{u}^{q_-}$ and at $\overline{v}^{q_-}$ in the direction h exists, and using the nonlinear Green's identity, we have

$$j'(\overline{u}^{q_-})(h)=\frac{1}{q_-}\int_\Omega\frac{-\triangle_{p(z)}\overline{u}-\triangle_{q(z)}\overline{u}}{\overline{u}^{q_--1}}hdz=\frac{1}{q_-}\int_\Omega\frac{C_1}{\overline{u}^{q_--\tau(z)}}hdz,$$

$$j'(\widetilde{u}^{q_-})(h)=\frac{1}{q_-}\int_\Omega\frac{-\triangle_{p(z)}\widetilde{u}-\triangle_{q(z)}\widetilde{u}}{\widetilde{u}^{q_--1}}hdz=\frac{1}{q_-}\int_\Omega\frac{C_1}{\widetilde{u}^{q_--\tau(z)}}hdz.$$

The convexity of $j(\cdot)$ implies the monotonicity of the directional derivative. Hence

$$0\leq C_1\int_\Omega\left[\frac{1}{\overline{u}^{q_--\tau(z)}}-\frac{1}{\widetilde{u}^{q_--\tau(z)}}\right](\overline{u}^{q_-}-\widetilde{u}^{q_-})dz\leq 0.$$

Thus

$$\overline{u}=\widetilde{u}\text{ (recall that }\tau_+<q_-),$$

which proves the uniqueness of the positive solution of (3.6). The equation is odd, therefore

$$\overline{v}=-\overline{u}\in -int\ C_+$$

is the unique negative solution to (3.6). $\square$

Having these unique constant sign solutions of (3.6), we can generate the extremal (barrier) solutions of problem (1.1). In what follows, by $\mathscr{S}_+$ (resp. $\mathscr{S}_-$), we denote the set of positive (resp. negative) solutions of problem (1.1). We know that

$$\varnothing \neq \mathscr{S}_+ \cap [0,\theta_+] \subseteq int\ C_+,\ \varnothing \neq \mathscr{S}_- \cap [\theta_-,0] \subseteq -int\ C_+.$$

Proposition 3.3. *If hypotheses* $\mathbf{H_0}$ *and* $\mathbf{H_1}$ *hold, then problem* (1.1) *has a smallest positive solution*

$$u_* \in \mathscr{S}_+ \cap [0,\theta_+] \subseteq int\ C_+,$$

and a biggest negative solution

$$v_* \in \mathscr{S}_- \cap [\theta_-,0] \subseteq -int\ C_+.$$

Proof. As in Lemma 4.1 of Filippakis-Papageorgiou [16], we infer that $\mathscr{S}_+ \cap [0,\theta_+]$ is downward directed (that is, if $u_1,\ u_2 \in \mathscr{S}_+ \cap [0,\theta_+]$, then we can find $u \in \mathscr{S}_+ \cap [0,\theta_+]$ such that $u \leq u_1,\ u \leq u_2$). Then, Lemma 3.10, p. 178 of Hu-Papageorgiou [17] implies that we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{S}_+ \cap [0,\theta_+]$ such that $\inf \mathscr{S}_+ = \inf_{n\in\mathbb{N}} u_n$. Evidently, $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded. Then the anisotropic regularity theory (see [12]) implies that there exist $\alpha \in (0,1)$ and $C_6 > 0$ such that

$$u_n \in C_0^{1,\alpha}\left(\overline{\Omega}\right),\ \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq C_6 \text{ for all } n \in \mathbb{N}. \tag{3.11}$$

Recall that $C_0^{1,\alpha}\left(\overline{\Omega}\right) \hookrightarrow C_0^1\left(\overline{\Omega}\right)$ compactly. Then by (3.11), we may assume that

$$u_n \to u_* \text{ in } C_0^1\left(\overline{\Omega}\right). \tag{3.12}$$

If $u_* = 0$, it follows from (3.12) that we can find $n_0 \in \mathbb{N}$ such that

$$0 \leq u_n(z) \leq \delta \text{ for all } z \in \overline{\Omega},\ \text{all } n \geq n_0$$

(here $\delta > 0$ is as in hypothesis $\mathbf{H_1}(ii)$). Therefore

$$C_1 u_n(z)^{\tau(z)-1} \leq f(z,u_n(z)) \text{ for a. a. } z \in \overline{\Omega},\ \text{all } n \geq n_0 \tag{3.13}$$

(see hypothesis $\mathbf{H_1}(ii)$). Fix $n \geq n_0$ and consider the Carathéodory function $l_n^+(z,x)$ defined by

$$l_n^+(z,x) = \begin{cases} C_1\left(x^+\right)^{\tau(z)-1} & \text{if} \quad x \leq u_n(z) \\ C_1 u_n(z)^{\tau(z)-1} & \text{if} \quad u_n(z) < x. \end{cases} \tag{3.14}$$

We set

$$L_n^+(z,x) = \int_0^x l_n^+(z,s)\,ds$$

and introduce the C^1- functional $\mu_+ : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\mu_+(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz - \int_\Omega L_{n-}^+(z,u)\,dz$$
$$\text{for all } u \in W_0^{1,p(z)}(\Omega).$$

From (3.14) and Proposition 2.1, it is clear that $\mu_+(\cdot)$ is coercive. Also Proposition 2.2 (the anisotropic Sobolev embedding theorem) implies that $\mu_+(\cdot)$ is sequentially weakly lower semi-continuous. So, we can find $\widetilde{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\mu_+(\widetilde{u}) = \inf\left\{\mu_+(u) : u_0 \in W_0^{1,p(z)}(\Omega)\right\}. \tag{3.15}$$

As before, since $\tau_+ < q_-$, we have

$$\mu_+(\widetilde{u}) < 0 = \mu_+(0),$$

therefore $\widetilde{u} \neq 0$. From (3.15), we have

$$\left\langle \mu_+'(\widetilde{u}), h\right\rangle = 0 \text{ for all } h \in W_0^{1,p(z)}(\Omega).$$

Hence,

$$\langle V(\widetilde{u}), h\rangle = \int_\Omega l_n^+(z,\widetilde{u})\, h dz \text{ for all } h \in W_0^{1,p(z)}(\Omega). \tag{3.16}$$

In (3.16), we use the test function $h = -\widetilde{u}^- \in W_0^{1,p(z)}(\Omega)$. Then $\rho_p(D\widetilde{u}^-) \leq 0$. Thus

$$\widetilde{u} \geq 0,\ \widetilde{u} \neq 0$$

(see Proposition 2.1). Also, in (3.16), we let $h = [\widetilde{u} - u_n]^+ \in W_0^{1,p(z)}(\Omega)$. We obtain

$$\begin{aligned}
\left\langle V(\widetilde{u}), [\widetilde{u}-u_n]^+\right\rangle &= \int_\Omega C_1 u_n(z)^{\tau(z)-1} [\widetilde{u}-u_n]^+ dz \text{ (see } (3.14))\\
&\leq \int_\Omega f(z,u_n)[\widetilde{u}-u_n]^+ dz \text{ (see } (3.13))\\
&= \left\langle V(u_n), [\widetilde{u}-u_n]^+\right\rangle \text{ (since } u_n \in \mathscr{S}_+).
\end{aligned}$$

Thus $\widetilde{u} \leq u_n$ (see Proposition 2.3). So, we have proved that

$$\widetilde{u} \in [0, u_n],\ \widetilde{u} \neq 0. \tag{3.17}$$

Then (3.17), (3.14) and (3.16) imply that $\widetilde{u}$ is a positive solution to (3.6). Hence by Proposition 3.2, we have $\widetilde{u} = \overline{u}$. Consequently,

$$\overline{u} \leq u_n \text{ for all } n \geq n_0 \text{ (see } (3.17)),$$

a contradiction to our assumption that $u_* = 0$. Then $u_* \neq 0$ and we have

$$\langle V(u_*), h\rangle = \int_\Omega f(z,u_*)\, h dz \text{ for all } h \in W_0^{1,p(z)}(\Omega) \text{ (see } (3.12)).$$

Hence,

$$u_* \in \mathscr{S}_+ \cap [0, \theta_+] \subseteq \mathit{int}\, C_+\ u_* = \inf \mathscr{S}_+.$$

Similarly, we produce

$$v_* \in \mathscr{S}_- \cap [\theta_-, 0] \subseteq -\mathit{int}\, C_+$$

such that $v \leq v_*$ for all $v \in \mathscr{S}_-$. Note that the latter is upward directed (that is, if v_1, $v_2 \in \mathscr{S}_-$, then we can find $v \in \mathscr{S}_-$ such that $v_1 \leq v$, $v_2 \leq v$). ☐

Using these extremal constant sign solutions, we can generate a nodal (sign changing) one. The proof is based on the following idea. We look for nontrivial solutions of (1.1) in $[v_*, u_*]$. Then any such nontrivial solution of (1.1) distinct from u_* and v_*, will be nodal, since u_* and v_* are extremal constant sign solutions of (1.1). For this approach to work, we need to use truncations and comparison arguments.

Proposition 3.4. *If hypotheses* $\mathbf{H_0}$, $\mathbf{H_1}$ *hold, then problem* (1.1) *admits a nodal solution*

$$y_0 \in [v_*, u_*] \cap C_0^1(\overline{\Omega}).$$

Proof. With $u_* \in int\ C_+$ and $v_* \in -int\ C_+$ being the two extremal constant sign solutions of (1.1) produced in Proposition 3.3, we introduce the following truncation of $f(z,\cdot)$

$$e(z,x) = \begin{cases} f(z,v_*(z)) & \text{if} \quad x < v_*(z) \\ f(z,x) & \text{if} \quad v_*(z) \le x \le u_*(z) \\ f(z,u_*(z)) & \text{if} \quad u_*(z) < x. \end{cases} \tag{3.18}$$

This is a Carathéodory function. We also consider the positive and the negative truncations of $e(z,\cdot)$, namely the Carathéodory functions

$$e_\pm(z,x) = e\left(z, \pm x^\pm\right). \tag{3.19}$$

We set

$$E(z,x) = \int_0^x e(z,s)\,ds, \qquad E_\pm(z,x) = \int_0^x e_\pm(z,s)\,ds,$$

and consider the C^1- functionals $\gamma,\ \gamma_\pm : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\gamma(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz - \int_\Omega E(z,u)\,dz$$

$$\gamma_\pm(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz - \int_\Omega E_\pm(z,u)\,dz$$

$$\text{for all } u \in W_0^{1,p(z)}(\Omega).$$

Using (3.18), (3.19), and the anisotropic regularity theory, we can easily check that

$$K_\gamma \subseteq [v_*,u_*] \cap C_0^1(\overline{\Omega}),\ K_{\gamma_+} \subseteq [0,u_*] \cap C_+,\ K_{\gamma_-} \subseteq [v_*,0] \cap (-C_+).$$

The extremality of u_* and v_* implies that

$$K_\gamma \subseteq [v_*,u_*] \cap C_0^1(\overline{\Omega}),\ K_{\gamma_+} = \{0,u_*\},\ K_{\gamma_-} = \{0,v_*\}. \tag{3.20}$$

Claim: $u_* \in int\ C_+$ and $v_* \in -int\ C_+$ are local minimizers to $\gamma(\cdot)$.

From (3.18), (3.19) and Proposition 2.1, it is clear that $\gamma_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_* \in W_0^{1,p(z)}(\Omega)$ such that

$$\gamma_+(\widetilde{u}_*) = \inf\left\{\gamma_+(u) : u \in W_0^{1,p(z)}(\Omega)\right\}. \tag{3.21}$$

Let $u \in C_+ \backslash \{0\}$. Since $u \in int\ C_+$, using Proposition 4.1.22, p. 274 of [11], we can find $t \in (0,1)$ such that $tu \le u_*$. Choosing $t \in (0,1)$ even smaller if necessary, we can also insure that $tu(z) \le \delta$ for all $z \in \overline{\Omega}$, with $\delta > 0$ as in hypothesis $\mathbf{H_1}(ii)$. Then using hypothesis $\mathbf{H_1}(ii)$ and the fact that $\tau_+ < q_-$, we obtain $\gamma_+(tu) < 0$. Hence,

$$\gamma_+(\widetilde{u}_*) < 0 = \gamma_+(0) \text{ (see (3.21))}.$$

Thus

$$\widetilde{u}_* \neq 0. \tag{3.22}$$

From (3.21) we have $\widetilde{u}_* \in K_{\gamma_+}$ and so, from (3.22) and (3.20), it follows that

$$\widetilde{u}_* = u_* \in int\ C_+. \tag{3.23}$$

From (3.18), (3.19) we see that

$$\gamma\,|_{C_+} = \gamma_+\,|_{C_+}.$$

Then from (3.23) and (3.21), we infer that u_* is a local $C_0^1\left(\overline{\Omega}\right)$ minimizer of $\gamma(\cdot)$, therefore u_* is a local $W_0^{1,p(z)}(\Omega)$ minimizer of $\gamma(\cdot)$ (see Papageorgiou-Radulescu-Zhang [14], Proposition A3). Similarly, for $v_* \in -int\ C_+$, using this time, the functional $\gamma_-(\cdot)$. This proves the Claim. Without any loss of generality, we may assume that $\gamma(v_*) \leq \gamma(u_*)$. The reasoning is similar if the opposite inequality is true. Moreover, on account of (3.20), we may assume that K_γ is finite. Otherwise we already have an infinity of smooth nodal solutions and so, we are done. Using Theorem 5.7.6, p. 449, of [11], we can find $\rho \in (0,1)$ small such that

$$\begin{gathered} \gamma(v_*) \leq \gamma(u_*) \\ < \inf\{\gamma(u) : \|u-u_*\| = \rho\} = m_\rho, \\ \|v_* - u_*\| > \rho. \end{gathered} \tag{3.24}$$

The coercivity of $\gamma(\cdot)$ (see (3.18)) implies that

$$\gamma(\cdot) \text{ satisfies the } C-\text{condition} \tag{3.25}$$

(see [11], p.369). Then (3.24) and (3.25) permit the use of the mountain pass theorem (see [11], p.401).. So, we can find $y_0 \in W_0^{1,p(z)}(\Omega)$ such that $y_0 \in K_\gamma$ and $m \leq \gamma(y_0)$, therefore

$$y_0 \in [v_*, u_*] \cap C_0^1\left(\overline{\Omega}\right) \text{ (see (3.20))},\ y_0 \notin \{v_*, u_*\} \text{ (see (3.24))}. \tag{3.26}$$

Since y_0 is a critical point of $\gamma(\cdot)$ of mountain pass type, from Theorem 6.5.8, p.527, of [11], we have

$$C_1(\gamma, y_0) \neq 0. \tag{3.27}$$

On the other hand, hypothesis $\mathbf{H_1}(ii)$ and Proposition 6 of Leonardi-Papageorgiou [18] imply that

$$C_k(\gamma, 0) = 0 \text{ for all } k \in \mathbb{N}_0. \tag{3.28}$$

Comparing (3.28) and (3.27), we conclude that $y_0 \neq 0$. Thus $y_0 \in C_0^1\left(\overline{\Omega}\right)$ is a nodal solution to (1.1). □

Summarizing, we can state the following multiplicity theorem for problem (1.1) (three solutions theorem). Note that we provide sign information for all the solutions, which are also ordered.

Theorem 3.1. *If hypotheses $\mathbf{H_0}$, $\mathbf{H_1}$ hold, then problem (1.1) has at least three nontrivial solutions*

$$\begin{aligned} &u_0 \in int\ C_+,\ v_0 \in -int\ C_+,\ y_0 \in C_0^1\left(\overline{\Omega}\right) \text{ nodal, with} \\ &v_0(z) \leq y_0(z) \leq u_0(z) \text{ for all } z \in \overline{\Omega}. \end{aligned}$$

4. Infinitely Many Nodal Solutions

In this section, by introducing a local symmetry condition for $f(z,\cdot)$, we produce a whole sequence of distinct smooth nodal solutions which converges to zero in $C_0^1(\overline{\Omega})$. A similar result can be found in Aizicovici-Papageorgiou-Staicu ([1], Theorem 3.6) for anisotropic Neumann problems. However, in [1], the reaction satisfies a sign condition. In contrast, here $f(z,\cdot)$ is sign changing on each semiaxis.

The new hypotheses on $f(z,x)$ are the following.

$\mathbf{H_1'}$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that, for a. a. $z\in\Omega$, $f(z,0)=0$, and

(i) there exist $\theta_-<0<\theta_+$ and $a_0\in L^\infty(\Omega)$ such that

$$f(z,\theta_+)\le -C_0<0<C_0\le f(z,\theta_-) \text{ for a. a. } z\in\Omega,$$

$$|f(z,x)|\le a_0(z) \text{ for a. a. } z\in\Omega, \text{ all } |x|\le\widehat{\theta}:=\max\{\theta_+,-\theta_-\},$$

and for some $\widehat{s}\in(0,\min\{\theta_+,-\theta_-\})$ one has that for a. a. $z\in\Omega$, $f(z,\cdot)\,|_{[-\widehat{s},\widehat{s}]}$ is odd;

(ii) there exist $\tau\in C(\overline{\Omega})$ with $\tau_+<q_-$ and $\delta>0$ such that

$$C_1|x|^{\tau(z)}\le f(z,x)x \text{ for a. a. } z\in\Omega, \text{ all } |x|\le\delta;$$

(iii) there exists $\widehat{\xi}>0$ such that for a. a. $z\in\Omega$, the function

$$x\to f(z,x)+\widehat{\xi}\,|x|^{p(z)-2}x$$

is nondecreasing on $\left[-\widehat{\theta},\widehat{\theta}\right]$.

From Proposition 3.3, we know that problem (1.1) has extremal constant sign solutions

$$u_*\in int\ C_+ \text{ and } v_*\in -int\ C_+.$$

Then

$$0\in int_{C_0^1(\overline{\Omega})}[v_*,u_*]\neq\varnothing. \tag{4.1}$$

Let $V\subseteq W_0^{1,p(z)}(\Omega)\cap L^\infty(\Omega)$ be a finite dimensional subspace. Also, let $k:\Omega\times\mathbb{R}\to\mathbb{R}$ be the Carathéodory function defined by

$$k(z,x)=\begin{cases} f(z,-\widehat{s}) & \text{if} \quad x<-\widehat{s},\\ f(z,x) & \text{if} \quad -\widehat{s}\le x\le\widehat{s},\\ f(z,\widehat{s}) & \text{if} \quad \widehat{s}<x.\end{cases} \tag{4.2}$$

We set

$$K(z,x)=\int_0^x k(z,s)\,ds$$

and consider the C^1- functional $\psi_0:W_0^{1,p(z)}(\Omega)\to\mathbb{R}$ defined by

$$\psi_0(u)=\int_\Omega\frac{1}{p(z)}|Du|^{p(z)}dz+\int_\Omega\frac{1}{q(z)}|Du|^{q(z)}dz-\int_\Omega K_{.}(z,u)\,dz$$

for all $u\in W_0^{1,p(z)}(\Omega)$.

Evidently $\psi_0 \in C^1\left(W_0^{1,p(z)}(\Omega)\right)$, it is even (see hypothesis $\mathbf{H}_1'(i)$) and on account of (4.2) and Proposition 2.1, $\psi_0(\cdot)$ is coercive, and so it satisfies the $C-$condition.

Proposition 4.1. *If hypotheses $\mathbf{H_0}$, $\mathbf{H}_1'$ hold, then there exists $\rho_V > 0$ such that*

$$\sup\{\psi_0(u) : u \in V, \|u\| = \rho_V\} < 0.$$

Proof. Evidently, we can always assume that $\widehat{s} \leq \delta$. Since V is finite dimensional, all norms are equivalent (see Papageorgiou-Winkert ([19], p.183). So we can find $\rho_V \in (0,1)$ small such that

$$u \in V, \ \|u\| \leq \rho_V \Longrightarrow |u(z)| \leq \widehat{s} \text{ for a.a. } z \in \Omega.$$

Since $W_0^{1,p(z)}(\Omega) \hookrightarrow L^{\tau(z)}(\Omega)$ continuously (in fact compactly) and

$$W_0^{1,p(z)}(\Omega) \hookrightarrow W_0^{1,q(z)}(\Omega)$$

continuously, we can also have

$$\|u\| \leq \rho_V \Longrightarrow \|u\|_{\tau(z)} \leq 1, \ \|u\|_{1,q(z)} \leq 1.$$

Then, for $u \in V$ with $\|u\| = \rho_V$, it follows that

$$\begin{aligned}\psi_0(u) &\leq \frac{1}{q_-}\left[\rho_p(Du) + \rho_q(Du)\right] - \frac{C_1}{\tau_+}\rho_\tau(u) \\ &\leq \frac{1}{q_-}\left[\|u\|^{p_-} + C_7\|u\|^{q_-}\right] - C_8\|u\|^{\tau_+}\end{aligned}$$

for some $C_7, C_8 > 0$. Here we have used once more that on V all the norms are equivalent. Since $\tau_+ < q_-$, we see that by taking $\rho_V \in (0,1)$ even smaller if necessary, we will have

$$\sup\{\psi_0(u) : u \in V, \|u\| = \rho_V\} < 0.$$

□

Then we can prove the following multiplicity result for problem (1.1).

Theorem 4.1. *If hypotheses $\mathbf{H_0}$, $\mathbf{H}_1'$ hold, then problem (1.1) has a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^1(\overline{\Omega})$ of nodal solutions such that $u_n \to 0$ in $C_0^1(\overline{\Omega})$.*

Proof. On account of Proposition 4.1, we can use Theorem 1 of Kajikiya [20] and obtain $\{u_n\}_{n\in\mathbb{N}} \subseteq K_{\psi_0}$ such that

$$u_n \to 0 \text{ in } W_0^{1,p(z)}(\Omega). \tag{4.3}$$

The anisotropic regularity theory implies that there exist $\alpha \in (0,1)$ and $C_9 > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}), \ \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq C_9 \text{ for all } n \in \mathbb{N}. \tag{4.4}$$

The compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$, (4.4) and (4.3) imply that

$$u_n \to 0 \text{ in } C_0^1(\overline{\Omega}).$$

Hence

$$\{u_n\}_{n\in\mathbb{N}} \subseteq int_{C_0^1(\overline{\Omega})}[v_*, u_*] \cap [-\widehat{s}, \widehat{s}],$$

and we conclude that $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^1(\overline{\Omega})$ are nodal solutions of problem (1.1). □

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