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# A REGULARIZED SCHEME FOR THE SPLIT COMMON FIXED POINT PROBLEM OF ASYMPTOTICALLY DEMICONTRACTIVE AND DEMIMETRIC MAPPINGS IN BANACH SPACES

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**Abstract.** In this paper, we introduce a regularized iterative scheme for solving the split common fixed point problem of asymptotically demicontractive and demimetric mappings in real Banach spaces. We then prove that the iterative scheme converges strongly to a solution to the problem, which is also a solution to a variational inequality problem. A split feasibility problem is also considered with the aid of our scheme.

**Keywords.** fixed point problem, asymptotically demicontractive mapping, Hilbert spaces.
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## 1. Introduction

Let $X$ be a nonempty set, and let $T: X \to X$ be a mapping. A point $x \in X$ is called a fixed point of $T$ provide $Tx = x$. One denotes the set of fixed points of $T$ by $Fix(T)$ and denote the identity mapping on $X$ by $I$ in this paper. Let $C$ and $Q$ be convex and closed subsets of real Banach spaces $E_1$ and $E_2$, respectively. Let $A: E_1 \to E_2$ be a linear and bounded operator with the adjoint operator $A^*$. The *split feasibility problem* (SFP) is to find:

$$x \in C \text{ such that } Ax \in Q. \tag{1.1}$$

The SFP, which was introduced by Censor and Elfving [1], has applications in signal processing, radiation therapy, data denoising, and data compression; see, e.g., [2, 3, 4] and the references therein. Recently, various solution methods were introduced and investigated for the SFP; see, e.g., [5, 6, 7, 8, 9] and the references therein.

Let $U: E_1 \to E_1$ and $T: E_2 \to E_2$ be two mappings. The *split common fixed point problem* (SCFP) for mappings $U$ and $T$ is to find

$$x \in Fix(U) \text{ such that } Ax \in Fix(T). \tag{1.2}$$

If, in (1.2), $U$ and $T$ are metric projections onto $C$ and $Q$, respectively, then (1.2) reduces to (1.1). Thus SCFP generalizes the SFP. The SCFP was first studied by Censor and Segal [10] in

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the framework of Hilbert spaces for directed operators. They proposed the following algorithm for its solution:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = U[x_n - \gamma A^*(I-T)Ax_n], \end{cases} \tag{1.3}$$

where $\gamma \in (0, \frac{2}{\lambda})$ and $\lambda$ denotes the spectral radius of the operator $A^*A$. Under some suitable conditions, they proved a weak convergence theorem.

In [11], Moudafi studied the SCFP in infinite dimensional Hilbert spaces for the case that $U$ and $T$ are quasi-nonexpansive mappings such that $I-U$ and $I-T$ are demiclosed at 0 and proposed an iterative scheme which converges weakly to a solution of the problem. We refer the readers to Eslamian and Eslamian [12] and Shehu and Cholamjiak [13] for recent extensions, modifications, and improvements on the results of Moudafi [11].

It is noted that the implementation of algorithm (1.3) and those proposed in [11, 12, 13] require the estimate of the norm of $A$ to find the step size $\gamma$. It known that this is difficult in general, if not impossible; see ([14], Theorem 2.3). To overcome this difficulty, authors considered alternative ways of constructing variable step sizes recently; see, e.g., [15] and the references therein.

Very recently, Taiwo *et al.* [15] studied the SCFP for the class of demicontractive mappings in real Hilbert spaces. They introduced the following self-adaptive, simple, and compact iterative scheme based on an alternative regularization [16] technique, and proved strong convergence theorem:

$$\begin{cases} x_1 \in E_1, \\ x_{n+1} = U_n(\alpha_n f(x_n) + (1-\alpha_n)(x_n - \rho_n A^*(I-T)Ax_n)), n \geq 1, \end{cases} \tag{1.4}$$

where $U_n = (1-\lambda_n)I + \lambda_n U$, $f: E_1 \to E_1$ is a contraction mapping with constant $\nu \in [0, \frac{1}{\sqrt{2}})$, and $\{\lambda_n\}$, $\{\rho_n\}$, and $\{\alpha_n\}$ satisfy appropriate conditions.

The study of the SCFP has recently been extended to Banach spaces for a pair of mappings of different classes. For the case that $E_1$ is uniformly convex and 2-uniformly smooth with smoothness constant $t$ satisfying $0 < t < \frac{1}{\sqrt{2}}$ and $E_2$ a real smooth Banach space, Tang *et al.* [17] studied the SCFP for an asymptotic nonexpansive mapping and a $\tau$-quasi-strict pseudocontractive mapping in the setting of two Banach spaces and proved weak and strong convergence theorems. Also, motivated by Tang *et al.* [18], Wang *et al.* [19] studied the SCFP for asymptotically nonexpansive mappings in the frameworks of two Banach spaces. They proposed the following algorithm and proved strong convergence theorem:

$$\begin{cases} x_1 \in E_1, \\ v_n = x_n - \delta J_{E_1}^{-1} A^* J_{E_2}(I-T)Ax_n \\ x_{n+1} = \alpha_n x_n + \gamma_n f(x_n) + \eta_n U^n v_n, n \geq 1, \end{cases} \tag{1.5}$$

where $\delta \in (0, \frac{1-2t^2}{\|A\|^2})$, $f: E_1 \to E_1$ is a contraction mapping with constant $\nu \in (0,1)$, $\{\alpha_n\}, \{\gamma_n\}$, and $\{\eta_n\} \subset (0,1)$ satisfy appropriate conditions; see also Taiwo *et al.* [20] and Shahzad and Zegeye [21] for recent works on SCFP in Banach spaces.

Motivated by the works above, in this paper, we propose a self adaptive iterative scheme for solving the SCFP for asymptotically demicontractive mappings and demimetric mappings in the case that $E_1$ is 2-uniformly smooth with smoothness constant $0 < t \leq \frac{1}{\sqrt{2}}$ and $E_2$ is a real smooth Banach space. Our iterative scheme extends (1.4) to Banach spaces. The organization of the remaining part of this paper is as follows. In Section 2, we collect some useful definitions, notations, and lemmas, which are needed for our algorithm's analysis. In Section 3, we present our algorithm and prove its strong convergence. We also give some corollaries to our main result. We conclude in Section 4.

## 2. Preliminaries

Let $E$ be a real Banach space, and let $C$ be a nonempty, convex, closed set in $E$. We denote by $E^*$ the dual of $E$ and $\|\cdot\|$ the norm of $E$ or $E^*$. We denote the value of the functional $x^* \in E^*$ at $x \in E$ by $\langle x, x^* \rangle$. For a sequence $\{x_n\}$ of $E$ and $x \in E$, we denote the strong convergence of $\{x_n\}$ and weak convergence of $\{x_n\}$ to $x$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let $S := \{x \in E : \|x\| = 1\}$. The norm of $E$ is said to be *Gâteaux differentiable* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in S$. $E$ is said to be *smooth* if its norm is Gâteaux differentiable for each $y \in S$. $E$ is *strictly convex* if $\|x+y\| < 2$ whenever $x, y \in S$ and $x \neq y$. The *modulus of convexity* of $E$ is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) := \inf\left\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\right\}.$$

$E$ is said to be *uniformly convex* if $\delta_E(\varepsilon) > 0$. It is known that every uniformly convex Banach space is strictly convex and reflexive (see e.g. [22, 23]).

Let $dim(E) \geq 2$. The *modulus of smoothness* of $E$ is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

$E$ is called *uniformly smooth* if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$. The Hilbert spaces, $L^p$(or $\ell_p$) spaces and the Sobolev spaces $W^{k,p}$, $(1 < p < \infty)$ are uniformly convex and uniformly smooth Banach spaces. For more on the geometry of Banach spaces, we refer to [24, 25, 26].

The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

We recall the following properties of the normalized duality mapping (see [25, 27]):

1. $Jx$ is nonempty, convex, bounded, and closed subset of $E^*$ for all $x \in E$.
2. If $E$ is a reflexive, strictly convex, and smooth real Banach space, then $J$ is surjective, injective and single-valued. If $E$ is norm-to-norm uniformly smooth, then $J$ is uniformly continuous on bounded sets.
3. If $E$ is a real Hilbert space, then $J$ is the identity map on $E$.

If $E$ is smooth, $J : E \to E^*$ is said to be weakly sequentially continuous if, for every $y \in E$, $\langle y, Jx_n \rangle \to \langle y, Jx \rangle$ as $x_n \rightharpoonup x$. It is known that $\ell_p(p = 2)$ space has this property but the $L_p(p > 1)$ spaces do not (see [28]).

Let $T : E \to E$ be a mapping. The mapping $(I - T)$ is called demiclosed at zero if, for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to 0$ imply $Tx = x$.

Recall that an operator $T : E \to E$ is said to be:

(i) Lipschitzian if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$;

(ii) nonexpansive if $L = 1$;

(iii) uniformly $L$-Lipschitzian if there exists a positive constant $L$ such that, for all $x,y \in E$ and $n \in \mathbb{N}$, $\|T^n x - T^n y\| \leq L\|x-y\|$;

(iv) asymptotically nonexpansive with sequence $\{\kappa_n\} \subset [1,\infty)$ if $\lim_{n\to\infty} \kappa_n = 1$ and, for all $x,y \in E$, $\|T^n x - T^n y\| \leq \kappa_n \|x-y\|$;

(v) $\kappa$-strictly pseudocontractive if there exists $j(x-y) \in J(x-y)$ and $\kappa \in [0,1)$ such that

$$\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq \frac{1}{2}(1-\kappa)\|(I-T)x-(I-T)y\|^2;$$

(vi) $\kappa$-strictly asymptotically pseudocontractive with sequence $\{\kappa_n\} \subset [1,\infty)$ if $\lim_{n\to\infty} \kappa_n = 1$ and for all $x,y \in E$ there exists $j(x-y) \in J(x-y)$ and $\kappa \in [0,1)$ such that

$$\langle (I-T^n)x-(I-T^n)y, j(x-y)\rangle \geq \frac{1}{2}(1-\kappa)\|(I-T^n)x-(I-T^n)y\|^2 - \frac{1}{2}(\kappa_n^2-1)\|x-y\|^2;$$

(vii) $\kappa$-asymptotically demicontractive with sequence $\{\kappa_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} \kappa_n = 1$ if $Fix(T) \neq \emptyset$ and for all $x \in E$ and $x^* \in Fix(T)$

$$\langle (I-T^n)x, j(x-x^*)\rangle \geq \frac{1}{2}(1-\kappa)\|(I-T^n)x\|^2 - \frac{1}{2}(\kappa_n^2-1)\|x-x^*\|^2;$$

(viii) $\beta$-demimetric where $\beta \in (-\infty,1)$ if $Fix(T) \neq \emptyset$ and for all $x \in E$ and $x^* \in Fix(T)$, we have

$$\langle x-x^*, j(x-Tx)\rangle \geq \frac{1-\beta}{2}\|x-Tx\|^2.$$

If $\beta \in [0,1)$, then $T$ is called demicontractive.

Let $E = H$ (the real Hilbert space), then we can deduce the following definitions.
An operator $T : H \to H$ is said to be:

(1) $\kappa$-strictly pseudocontractive if there exists $j(x-y) \in J(x-y)$ and $\kappa \in [0,1)$ such that

$$\langle (I-T)x-(I-T)y, x-y\rangle \geq \frac{1}{2}(1-\kappa)\|(I-T^n)x-(I-T^n)y\|^2,$$

equivalently

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + \kappa\|(I-T^n)x-(I-T^n)y\|^2;$$

(2) $\kappa$-strictly asymptotically pseudocontractive with sequence $\{\kappa_n\} \subset [1,\infty)$ if $\lim\limits_{n\to\infty} \kappa_n = 1$ and $\forall\, x,y \in E$, there exists $\kappa \in [0,1)$ such that

$$\langle (I-T^n)x-(I-T^n)y, x-y\rangle \geq \frac{1}{2}(1-\kappa)\|(I-T^n)x-(I-T^n)y\|^2 - \frac{1}{2}(\kappa_n^2-1)\|x-y\|^2,$$

equivalently

$$\|T^n x - T^n y\|^2 \leq \kappa_n^2\|x-y\|^2 + \kappa\|(I-T^n)x-(I-T^n)y\|^2;$$

(3) $\kappa$-asymptotically demicontractive with sequence $\{\kappa_n\} \subset [1,\infty)$ such that $\lim\limits_{n\to\infty} \kappa_n = 1$ if $Fix(T) \neq \emptyset$ and for all $x \in E$ and $x^* \in Fix(T)$

$$\langle (I-T^n)x, x-x^*\rangle \geq \frac{1}{2}(1-\kappa)\|(I-T^n)x\|^2 - \frac{1}{2}(\kappa_n^2-1)\|x-x^*\|^2,$$

equivalently

$$\|T^n x - x^*\|^2 \leq \kappa_n^2\|x-x^*\|^2 + \kappa\|T^n x - x\|^2;$$

(4) $\beta$-demimetric where $\beta \in (-\infty, 1)$ if $Fix(T) \neq \emptyset$ and for all $x \in E$ and $x^* \in Fix(T)$, we have

$$\langle x - x^*, (x - Tx)\rangle \geq \frac{1-\beta}{2}\|x - Tx\|^2,$$

equivalently

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \beta\|Tx - x\|^2.$$

The metric projection of $E$ onto $C$, denoted by $P_C$, is the mapping that assigns every point $x \in E$ to its unique nearest point in $C$ i.e., $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The metric projection is characterized by $P_C x \in C$ and $\langle y - P_C x, x - P_C x\rangle \leq 0$ for all $y \in C$. In addition, $P_C$ is nonexpansive and $Fix(P_C) = C$. Since $P_C^2 = P_C$, it follows by induction that $P_C^n = P_C$ for all $n \in \mathbb{N}$.

The class of $\beta$-demimetric mappings contains several important classes of mappings, such as demicontractive, quasi-nonexpansive, and so on. From the above definitions, we see that the class of $k$-asymptotically demicontractive mappings contains the classes of $k$-asymptotically pseudocontractive mappings and asymptotically nonexpansive with nonempty fixed point set. The class of aymptotically demicontractive mappings and the class of demicontractive mappings are independent as illustrated by the following examples.

**Example 2.1.** Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = -3x$. Then $T$ is demicontractive but not asymptotically demicontractive. It is clear that $Fix(T) = \{0\}$. For $\beta \in [\frac{1}{2}, 1)$ and $x \in \mathbb{R}$, we have

$$|Tx - 0|^2 = 9|x|^2 \leq (1 + 16\beta)|x|^2 = |x - 0|^2 + \beta|Tx - x|^2,$$

which shows that $T$ is $\beta$-demicontractive. Suppose that $T$ is $\kappa$-asymptotically demicontractive with sequence $\{\kappa_n\}$. Then there exists some $N_0 \in \mathbb{N}$ such that $\kappa_n < 2$ for $n \geq N_0$. For such $n$ which is even, $|T^n x - 0|^2 = 3^{2n}|x|^2$,

$$\kappa_n^2|x-0|^2 + \kappa|T^n x - x|^2 = \kappa_n^2|x|^2 + \kappa(3^n - 1)^2|x|^2 < (5 + 3^{2n} - 2(3^n))|x|^2 < 3^{2n}|x|^2 = |T^n x - 0|^2.$$

Hence $T$ is not a $\kappa$-asymptotically demicontractive mapping.

**Example 2.2.** Let $H$ be the closed interval $[0, 1]$ with the absolute-value norm. Define $T : H \to H$ by

$$Tx = \begin{cases} \frac{1}{2}\ , & x \in [0, \frac{1}{2}], \\ 0\ , & x \in (\frac{1}{2}, 1]. \end{cases}$$

It is known that $T$ is not demicontractive (see [29, 30]). It is easy to see that $Fix(T) = \frac{1}{2}$. In addition, for $n \geq 2$, $T^n x = \frac{1}{2}$ for all $x \in [0, 1]$. Also for $x \in [0, \frac{1}{2}]$, $\{\kappa_n\} \subset [1, \infty)$, and $\kappa \in [0, 1)$,

$$|T^n x - \frac{1}{2}|^2 = 0 \leq \kappa_n^2|x - \frac{1}{2}|^2 + \kappa|x - \frac{1}{2}|^2 \leq \kappa_n^2|x - \frac{1}{2}|^2 + \kappa|x - T^n x|^2.$$

If $x \in (\frac{1}{2}, 1]$, for $n \geq 2$, we get the inequality above. For $n = 1$, choose $\{\kappa_n\}$ such that $\kappa_1 \geq \frac{2}{2x-1}$. Then

$$\kappa_1^2|x - \frac{1}{2}|^2 + \kappa|x - T^n x|^2 \geq 1 + \kappa x^2 > \frac{1}{4} = |Tx - \frac{1}{2}|^2.$$

Thus $T$ is $\kappa$-asymptotically demicontractive, which is not $\kappa$-demicontractive.

**Example 2.3.** Let $E = \ell_2 := \{x = \{x_i\}_{i=1}^\infty : x_i \in \mathbb{R}, \sum_{i=1}^\infty |x_i|^2 < \infty\}$ and $B = \{x \in \ell_2 : \|x\| \leq 1\}$. Let $T : B \to \ell_2$ be defined by $Tx = (0, x_1^2, a_2x_2, a_3x_3, \dots)$, where $\{a_j\}_{j=1}^\infty$ is a real sequence satisfying: $a_2 > 0, 0 < a_j < 1, j \neq 2$, and $\prod_{j=2}^\infty a_j = \frac{1}{2}$. It is known that $T$ is $\kappa$-strictly asymptotically pseudocontractive but not $k$-strictly pseudocontractive (see [31]). Since $Fix(T) =$

$\{(0,0,0,\dots)\} \neq \emptyset$, it then follows that $T$ is is $\kappa$-asymptotically demicontractive but not $\kappa$-demicontractive.

**Remark 2.1.** Although the classes of demicontractive and asymptotically demicontractive mappings are independent, it is noteworthy that the metric projection $P_C$ is both 0-demicontractive and 0-asymptotically demicontractive.

**Lemma 2.1.** [32] *Let $E$ be a real 2-uniformly smooth Banach space, and let $\kappa > 0$ be its best smooth constant. Let $J$ be the normalized duality mapping on $E$. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|\kappa y\|^2.$$

**Remark 2.2.** The best smoothness constant for Hilbert spaces is $\kappa = \frac{1}{\sqrt{2}}$.

**Lemma 2.2.** [32] *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n \upsilon_n + \theta_n, n \geq 0,$$

*where $\{\sigma_n\}_{n=1}^{\infty}$, $\{\upsilon_n\}_{n=1}^{\infty}$, and $\{\theta_n\}_{n=1}^{\infty}$ satisfy the conditions:*

(i) $\{\sigma_n\}_{n=1}^{\infty} \subset [0,1]$, $\sum_{n=1}^{\infty} \sigma_n = \infty$ *or* $\prod_{n=1}^{\infty}(1-\sigma_n) = 0$*;*

(ii) $\limsup_{n\to\infty} \upsilon_n \leq 0$*;*

(iii) $\theta_n \geq 0$*,* $(n \geq 1)$*,* $\sum_{n=1}^{\infty} \theta_n < \infty$*.*

*Then* $\lim_{n\to\infty} a_n = 0$*.*

**Lemma 2.3.** [33] *Let $\{\Gamma_n\}$ be a sequence of real numbers that never gets monotonically decreasing from a certain $n_0 \in \mathbb{N}$ in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by $\tau(n) := \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}$. Then $\{\tau(n)\}_{n\geq n_0}$ is a non-decreasing sequence verifying $\lim_{n\to\infty} \tau(n) = \infty$, and, for all $n \geq n_0$, $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$ hold.*

## 3. Main Results

In this section, we present our algorithm and do the convergence analysis. We first state some assumptions and notations which will be needed in the sequel.

Let $E_2$ be a real smooth Banach space, and Let $E_1$ be a uniformly convex and 2-uniformly smooth Banach space with smoothness constant $t$ satisfying $0 < t \leq \frac{1}{\sqrt{2}}$ and a weak sequential continuous normalized duality mapping. Let $f : E_1 \to E_1$ be a contraction mapping with coefficient $\nu$ in $(0,1)$. Let $A : E_1 \to E_2$ be a linear and bounded operator with adjoint $A^*$. Let $U : E_1 \to E_1$ be a $\kappa$-asymptotically demicontractive mapping with sequence $\{\kappa_n\}$ satisfying $\sum_{n=1}^{\infty}(\kappa_n^2 - 1) < \infty$, and let $T : E_2 \to E_2$ be $\beta$-demimetric mapping. Assume that $I-U$ and $I-T$ are demiclosed at 0 and $\Gamma = \{x \in E_1 : x \in Fix(U) \text{ and } Ax \in Fix(T)\} \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1 \in E_1$ and

$$x_{n+1} = U_\tau^n(\alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma_n J_1^{-1}A^*J_2(I-T)Ax_n)), n \geq 1, \tag{3.1}$$

where $U_\tau^n = (1-\tau)I + \tau U^n$, $J_1$ and $J_2$ are normalized duality mappings on $E_1$ and $E_2$, respectively, and the following conditions are satisfied:

(a) $\alpha_n \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $0<\tau<1-\kappa$;
(c) $(\kappa_n^2-1)=o(\alpha_n)$ and
(d) For $\mu$ satisfying $2\beta-1<\mu<1$,

$$\gamma_n=\begin{cases}\frac{(1-\mu)\|(I-T)Ax_n\|^2}{2\|A^*J_2(I-T)Ax_n\|^2}, & Ax_n\neq T(Ax_n)\\ 0 & \text{otherwise.}\end{cases}$$

**Lemma 3.1.** *The sequence $\{x_n\}$ generated by (3.1) is bounded.*

*Proof.* Fix $x^*\in\Gamma$ and set $y_n=x_n-\gamma_nJ_1^{-1}A^*J_2(I-T^n)Ax_n$ and $w_n=\alpha_nf(x_n)+(1-\alpha_n)y_n$. Then

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\ &\le\|w_n-x^*\|^2+2\tau\langle U^nw_n-w_n,J_1(w_n-x^*)\rangle+2\tau^2\|t(U^nw_n-w_n)\|^2\\ &\le\|w_n-x^*\|^2+\tau(\kappa_n^2-1)\|w_n-x^*\|^2-(1-\kappa)\tau\|U^nw_n-w_n\|^2+2\tau^2t^2\|U^nw_n-w_n\|^2\\ &\le[1+\tau(\kappa_n^2-1)]\|w_n-x^*\|^2+(\tau^2-\tau(1-\kappa))\|U^nw_n-w_n\|^2,\end{aligned}\tag{3.2}$$

which indicates that

$$\|x_{n+1}-x^*\|^2\le[1+\tau(\kappa_n^2-1)]\|w_n-x^*\|^2,\tag{3.3}$$

and

$$\|w_n-x^*\|\le\alpha_n\nu\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|+(1-\alpha_n)\|y_n-x^*\|.\tag{3.4}$$

Since $T$ is a demimetric mapping, we obtain that

$$\begin{aligned}\|y_n-x^*\|^2&\le\gamma_n^2\|J_1^{-1}A^*J_2(I-T)Ax_n\|^2-2\gamma_n\langle x_n-x^*,A^*J_2(I-T)Ax_n\rangle+2t^2\|x_n-x^*\|^2\\ &\le\gamma_n^2\|A^*J_2(I-T)Ax_n\|^2-\gamma_n(1-\beta)\|Ax_n-TAx_n\|^2+\|x_n-x^*\|^2\\ &=\|x_n-x^*\|^2-\frac{(1-\mu)(1+\mu-2\beta)\|Ax_n-TAx_n\|^4}{4\|A^*J_2(I-T)Ax_n\|^2}\\ &\le\|x_n-x^*\|^2.\end{aligned}\tag{3.5}$$

Substituting (3.5) in (3.4), we obtain

$$\|w_n-x^*\|\le(1-\alpha_n(1-\nu))\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|.\tag{3.6}$$

From (3.3) and (3.6), we obtain by letting $\delta_n=1+\tau(\kappa_n^2-1)$ that

$$\|x_{n+1}-x^*\|\le\delta_n\Big(\big(1-\alpha_n(1-\nu)\big)\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|\Big).\tag{3.7}$$

To this end, let $s_n:=\delta_n\alpha_n(1-\nu)$. Obviously, $s_n<\delta_n$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}\delta_n=1$. The,n for any $\varepsilon\in\mathbb{R}$ such that $0<\varepsilon<1$, there is some $N_0\in\mathbb{N}$ such that $\delta_n\le1+\varepsilon s_n$ for all $n\ge N_0$. Therefore, for such $N_0$, we have that $s_n<\delta_n\le1+\varepsilon s_n$, which together with (3.7) obtains

$$\begin{aligned}\|x_{n+1}-x^*\|&\le(1-s_n(1-\varepsilon))\|x_n-x^*\|+\frac{s_n(1-\varepsilon)}{(1-\varepsilon)(1-\nu)}\|f(x^*)-x^*\|\\ &\le\max\left\{\|x_n-x^*\|,\frac{\|f(x^*)-x^*\|}{(1-\varepsilon)(1-\nu)}\right\}\\ &\cdots\\ &\le\max\left\{\|x_{N_0}-x^*\|,\frac{\|f(x^*)-x^*\|}{(1-\varepsilon)(1-\nu)}\right\},n\ge N_0.\end{aligned}\tag{3.8}$$

Also, for $n < N_0$, we have from (3.7) that

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq \delta_n\Big((1-\alpha_n(1-\nu))\|x_n-x^*\|+\frac{\alpha_n(1-\nu)\|f(x^*)-x^*\|}{(1-\nu)}\Big)\\ &\leq \delta_n \max\Big\{\|x_n-x^*\|, \frac{\|f(x^*)-x^*\|}{(1-\nu)}\Big\}\\ &\cdots\\ &\leq \Pi_{i=1}^n \delta_i \max\Big\{\|x_1-x^*\|, \frac{\|f(x^*)-x^*\|}{(1-\nu)}\Big\}. \end{aligned} \tag{3.9}$$

It can then be deduced from (3.8) and (3.9) that $\{\|x_n-x^*\|\}$ is bounded. Consequently, $\{x_n\},\{y_n\}$ and $\{w_n\}$ are bounded. $\square$

**Theorem 3.1.** *The sequence $\{x_n\}$ generated by (3.1) converges strongly to $x^* \in \Gamma$, where $x^*$ is the unique solution to variational inequality (3.10):*

$$\text{Find } x^* \in \Gamma \text{ such that } \langle f(x^*)-x^*, J_1(x-x^*)\rangle \leq 0, \forall x \in \Gamma. \tag{3.10}$$

*Proof.* We first prove that the solution of (3.10) is unique. Indeed, let $x_1, x_2 \in \Gamma$ be different solutions of (3.10). Then we have that $\langle f(x_1)-x_1, J_1(x_2-x_1)\rangle \leq 0$ and $\langle f(x_2)-x_2, J_1(x_1-x_2)\rangle \leq 0$. Adding the two inequalities, we see that $\langle f(x_2)-x_2+x_1-f(x_1), J_1(x_1-x_2)\rangle \leq 0$, which yields

$$\begin{aligned}\langle x_1-x_2, J_1(x_1-x_2)\rangle &\leq \langle f(x_1)-f(x_2), J_1(x_1-x_2)\\ &\Downarrow\\ \|x_1-x_2\|^2 &\leq \|f(x_1)-f(x_2)\|\|x_1-x_2\|\\ &\Downarrow\\ \|x_1-x_2\|^2 &\leq \nu\|x_1-x_2\|^2.\end{aligned}$$

Since $\nu \in [0,1)$, it then follows that $x_1 = x_2$. Thus the solution of (3.10) is unique.

We next prove that $x_n \to x^*$. Note that

$$\begin{aligned}\|w_n-x^*\|^2 &= \langle \alpha_n f(x_n)+(1-\alpha_n)y_n-x^*, J_1(w_n-x^*)\rangle\\ &= \alpha_n\langle f(x_n)-f(x), J_1(w_n-x^*)\rangle+\alpha_n\langle f(x)-x, J_1(w_n-x^*)\rangle\\ &\quad +(1-\alpha_n)\langle y_n-x^*, J_1(w_n-x^*)\rangle\\ &\leq \frac{1}{2}\alpha_n(\nu^2\|x_n-x^*\|^2+\|w_n-x^*\|^2)+\alpha_n\langle f(x)-x, J_1(w_n-x^*)\rangle\\ &\quad +\frac{1}{2}(1-\alpha_n)(\|y_n-x^*\|^2+\|w_n-x^*\|^2)\\ &= \frac{\alpha_n\nu^2}{2}\|x_n-x^*\|^2+\frac{1}{2}\|w_n-x^*\|^2+\frac{(1-\alpha_n)}{2}\|y_n-x^*\|^2+\alpha_n\langle f(x)-x, J_1(w_n-x^*)\rangle,\end{aligned}$$

which is equivalent to

$$\|w_n-x^*\|^2 \leq \alpha_n\nu^2\|x_n-x^*\|^2+(1-\alpha_n)\|y_n-x^*\|^2+2\alpha_n\langle f(x)-x, J_1(w_n-x^*)\rangle.$$

From (3.5), we obtain that

$$\begin{aligned}\|w_n - x^*\|^2 &\leq \alpha_n \nu^2 \|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n \langle f(x) - x, J_1(w_n - x^*)\rangle \\ &\quad -(1-\alpha_n)\frac{(1-\mu)(1+\mu-2\beta)\|Ax_n - TAx_n\|^4}{4\|A^*J_2(I-T)Ax_n\|^2} \\ &= (1-\alpha_n(1-\nu^2))\|x_n - x^*\|^2 + 2\alpha_n\langle f(x) - x, J_1(w_n - x^*)\rangle \\ &\quad -(1-\alpha_n)\frac{(1-\mu)(1+\mu-2\beta)\|Ax_n - TAx_n\|^4}{4\|A^*J_2(I-T)Ax_n\|^2}. \end{aligned} \tag{3.11}$$

Substituting (3.11) in (3.2), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \delta_n\Bigg((1-\alpha_n(1-\nu^2))\|x_n - x^*\|^2 + 2\alpha_n\langle f(x) - x, J_1(w_n - x^*)\rangle \\ &\quad -(1-\alpha_n)\frac{(1-\mu)(1+\mu-2\beta)\|Ax_n - TAx_n\|^4}{4\|A^*J_2(I-T)Ax_n\|^2}\Bigg) \\ &\quad + (\tau^2 - \tau(1-\kappa))\|U^n w_n - w_n\|^2 \\ &\leq \delta_n\big((1-\alpha_n(1-\nu^2))\|x_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, J_1(w_n - x^*)\rangle\big) \\ &\leq (1-\sigma_n)\|x_n - x^*\|^2 + \sigma_n\frac{2\delta_n\langle f(x) - x, J_1(w_n - x^*)\rangle}{1-\nu^2} + \theta_n, \end{aligned} \tag{3.12}$$

where $\sigma_n = \alpha_n(1-\nu^2)$, $\delta_n = 1 + \tau(\kappa_n^2 - 1)$ and $\theta_n = \tau(\kappa_n^2 - 1)\sup_{n\in\mathbb{N}}\{\|x_n - x^*\|^2\}$.

We further divide the proof into the following two cases.

**Case 1**: Suppose that $\{\|x_n - x^*\|\}$ is monotone decreasing for some large $N_0 \in \mathbb{N}$. Then by the boundedness of $\{\|x_n - x^*\|\}$, it implies that $\lim_{n\to\infty}(\|x_{n+1} - x^*\| - \|x_n - x^*\|) = 0$. From (3.12), we obtain that

$$\begin{aligned}&(1-\alpha_n)\frac{(1-\mu)(1+\mu-2\beta)\|Ax_n - TAx_n\|^4}{4\|A^*J_2(I-T)Ax_n\|^2} + (\tau(1-\kappa) - \tau^2)\|U^n w_n - w_n\|^2 \\ &\leq (1-\sigma_n)\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \sigma_n\frac{2\delta_n\langle f(x) - x, J_1(w_n - x^*)\rangle}{1-\nu^2} + \theta_n.\end{aligned}$$

Taking the limit in the inequality above as $n \to \infty$, we obtain that

$$\frac{\|Ax_n - TAx_n\|^2}{\|A^*J_2(I-T)Ax_n\|} \to 0 \tag{3.13}$$

and

$$\|U^n w_n - w_n\| \to 0. \tag{3.14}$$

From (3.13), we see that

$$\|Ax_n - TAx_n\| \leq \frac{\|A^*\|\|Ax_n - TAx_n\|^2}{\|A^*J_2(I-T)Ax_n\|} \to 0 \text{ as } n \to \infty. \tag{3.15}$$

Using (3.15), we obtain that

$$\|y_n - x_n\| = \|\gamma_n J_1^{-1} A^* J_2(I-T)Ax_n\| \leq \gamma_n\|A^*J_2(I-T)Ax_n\| \leq \|(I-T)Ax_n\|^2 \to 0$$

as $n \to \infty$, which indicates that

$$\|w_n - x_n\| \leq \alpha_n\|f(x_n) - x_n\| + (1-\alpha_n)\|y_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.16}$$

From (3.14) and (3.16), we assert that

$$\|x_{n+1}-x_n\| \leq \|w_n-x_n\| + \tau\|U^n w_n - w_n\| \to 0, \text{ as } n \to \infty. \tag{3.17}$$

It then follows from (3.16) and (3.17) that $\|w_n - w_{n-1}\| \to 0$ as $n \to \infty$. Moreover, since $U$ is uniformly $L$-Lipschitzian, we have that

$$\begin{aligned}
\|w_n - Uw_n\| &\leq \|w_n - U^n w_n\| + L\|U^{n-1}w_n - w_n\| \\
&\leq \|w_n - U^n w_n\| + L\big(\|U^{n-1}w_n - U^{n-1}w_{n-1}\| + \|U^{n-1}w_{n-1} - w_{n-1}\| \\
&\quad + \|w_{n-1} - w_n\|\big) \\
&\leq \|w_n - U^n w_n\| + L\big(L\|w_n - w_{n-1}\| + \|U^{n-1}w_{n-1} - w_{n-1}\| \\
&\quad + \|w_{n-1} - w_n\|\big).
\end{aligned} \tag{3.18}$$

Therefore taking the limit of (3.18) and using (3.14), we arrive at $\|w_n - Uw_n\| \to 0$ as $n \to \infty$. Since $E_1$ is a reflexive Banach space, by the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x} \in E_1$. Also, we can conclude that $w_{n_k} \rightharpoonup \bar{x} \in E_1$. Therefore from the demiclosedness of $I-U$, we obtain that $\bar{x} \in Fix(U)$. Since $A$ is linear it implies that $Ax_{n_k} \rightharpoonup A\bar{x} \in E_2$. It also follows from (3.15) and the demiclosedness of $I-T$ that $A\bar{x} \in Fix(T)$. Thus $\bar{x} \in \Gamma$. Observe that $\langle f(x^*) - x^*, J_1(\bar{x} - x^*)\rangle \leq 0$. By the weak sequential continuity of $J_1$, it then yields that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(x^*) - x^*, J_1(w_n - x^*)\rangle \\
&= \limsup_{k\to\infty}\langle f(x^*) - x^*, J_1(w_{n_k} - x^*)\rangle = \langle f(x^*) - x^*, J_1(\bar{x} - x^*)\rangle \leq 0.
\end{aligned}$$

Recall from (3.12) that

$$\|x_{n+1} - x^*\|^2 \leq (1-\sigma_n)\|x_n - x^*\|^2 + \sigma_n \upsilon_n + \theta_n, \tag{3.19}$$

where $\sigma_n = \alpha_n(1-\nu^2)$,

$$\upsilon_n = \frac{2\delta_n\langle f(x^*) - x^*, J_1(w_n - x^*)\rangle}{1-\nu^2},$$

and

$$\theta_n = \tau(\kappa_n^2 - 1)\sup_{n\in\mathbb{N}}\{\|x_n - x^*\|^2\}.$$

We then can deduce

i. $\sigma_n \in [0,1]$, $\sum_{n=1}^{\infty}\sigma_n = \infty$;
ii. $\limsup_{n\to\infty}\upsilon_n \leq 0$;
iii. $\theta_n \geq 0$ and $\sum_{n=1}^{\infty}\theta_n < \infty$.

Bapplying Lemma 2.2 to (3.19), we obtain that $\{x_n\}$ converges strongly to $x^*$.

**Case 2**: Assume that $\{\|x_n - x^*\|\}$ is not eventually monotonically decreasing. Then there is $m \in \mathbb{N}$ such that $\|x_m - x^*\|^2 \leq \|x_{m+1} - x^*\|^2$, $m \geq N_0$. Let $\varphi_n = \|x_n - x\|^2$ and $\rho : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq N_0$ by $\rho(n) := \max\{k \in \mathbb{N} : k \leq n, \varphi_k \leq \varphi_{k+1}\}$. Clearly, $\rho$ is a non-decreasing sequence satisfying $\rho(n) \to \infty$ as $n \to \infty$ and $\varphi_{\rho(n)} \leq \varphi_{\rho(n)+1}$ for all $n \geq N_0$. By following similar argument as in Case 1, we obtain that, as $n \to \infty$,

$$\|w_{\rho(n)} - Uw_{\rho(n)}\| \to 0,\ \|Ax_{\rho(n)} - TAx_{\rho(n)}\| \to 0, \text{ and } \limsup_{n\to\infty}\langle f(x^*) - x^*, J_1(w_{\rho(n)} - x^*)\rangle \leq 0.$$

Since $\varphi_{\rho(n)} \leq \varphi_{\rho(n)+1}$, it follows from (3.19) that

$$0 \leq (1-\sigma_{\rho(n)})\varphi_{\rho(n)} - \varphi_{\rho_n+1} + 2\sigma_{\rho(n)}\delta_{\rho(n)}\langle f(x^*)-x^*, J_1(w_{\rho(n)}-x^*) + \tau(\kappa^2_{\rho(n)}-1)M$$
$$\leq -\sigma_{\rho(n)}\varphi_{\rho_n+1} + 2\sigma_{\rho(n)}\delta_{\rho(n)}\langle f(x^*)-x^*, J_1(w_{\rho(n)}-x^*) + \tau(\kappa^2_{\rho(n)}-1)M,$$

which implies that

$$\varphi_{\rho(n)+1} \leq 2\delta_{\rho(n)}\langle f(x^*)-x^*, J_1(w_{\rho(n)}-x^*)\rangle + \frac{\tau(\kappa^2_{\rho(n)}-1)M}{\sigma_{\rho(n)}}. \tag{3.20}$$

Taking the limit of (3.20) as $n\to\infty$, we obtain that $\lim_{n\to\infty}\varphi_{\rho(n)} = 0$. Then by Lemma 2.3, we obtain

$$0 \leq \varphi_n \leq \max\{\varphi_n, \varphi_{\rho(n)}\} \leq \varphi_{\rho(n)+1} \to 0.$$

Thus $\lim_{n\to\infty}\varphi_n = 0$ and $\{x_n\}$ converges strongly to $x^*$. □

We next give some of the consequences of our main results.

**Corollary 3.1.** *Let $E_1$ be a uniformly convex and 2-uniformly smooth Banach space with smoothness constant $t$ satisfying $0 < t \leq \frac{1}{\sqrt{2}}$ and a weak sequential continuous normalized duality mapping, and let $E_2$ be a real Hilbert space. Let $f : E_1 \to E_1$ be a contraction mapping with coefficient $\nu$ in $(0,1)$, and let $A : E_1 \to E_2$ be a bounded and linear operator with adjoint $A^*$. Let $U : E_1 \to E_1$ be an asymptotically quasi-nonexpansive mapping with sequence $\{\kappa_n\}$ satisfying $\sum_{n=1}^{\infty}(\kappa_n^2-1) < \infty$, and let $T : E_2 \to E_2$ be a quasi-nonexpansive mapping. Assume that $I-U$ and $I-T$ are demiclosed at 0 and $\Gamma_1 = \{x \in E_1 : x \in Fix(U) \text{ and } Ax \in Fix(T)\} \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1 \in E_1$ and*

$$x_{n+1} = U_\tau^n(\alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma_n J_1^{-1}A^*J_2(I-T)Ax_n)), n \geq 1, \tag{3.21}$$

*where $U_\tau^n = (1-\tau)I + \tau U^n$ and the following conditions are satisfied:*

(a) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$*;*
(b) $0 < \tau < 1$*;*
(c) $(\kappa_n^2 - 1) = o(\alpha_n)$ *and*
(d) *for $\mu$ satisfying $-1 < \mu < 1$,*

$$\gamma_n = \begin{cases} \frac{(1-\mu)\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2}, & Ax_n \neq T(Ax_n) \\ 0 & otherwise. \end{cases}$$

*Then, the sequence $\{x_n\}$ generated by (3.21) converges strongly to $x^* \in \Gamma_1$, where $x^*$ is the unique solution to the variational inequality*

$$\text{Find } x^* \in \Gamma_1 \text{ such that } \langle f(x^*)-x^*, J_1(x-x^*)\rangle \leq 0, \forall x \in \Gamma_1.$$

*Proof.* Since $U : E_1 \to E_1$ is asymptotically quasi-nonexpansive, it is easy to see that $U$ is 0-asymptotically demicontractive. Also, as $T$ is quasi-nonexpansive, it follows from the definition that $T$ is 0-demimetric. Therefore the result follows from the proof of Theorem 3.1 immediately. □

**Corollary 3.2.** *Let $E_1$ and $E_2$ be real Hilbert spaces. Let $f : E_1 \to E_1$ be a contraction mapping with coefficient $\nu$ in $(0,1)$, and let $A : E_1 \to E_2$ be a linear and bounded operator with adjoint $A^*$. Let $U : E_1 \to E_1$ be a $\kappa$-asymptotically demicontractive mapping with sequence $\{\kappa_n\}$ satisfying*

$\sum_{n=1}^{\infty}(\kappa_n^2-1)<\infty$, *and let* $T:E_2\to E_2$ *be a* $\beta$*-demicontractive mapping. Assume that* $I-U$ *and* $I-T$ *are demiclosed at* $0$ *and* $\Gamma_2=\{x\in E_1 : x\in Fix(U) \text{ and } Ax\in Fix(T)\}\neq\emptyset$*. Let* $\{x_n\}$ *be the sequence generated by* $x_1\in E_1$ *and*

$$x_{n+1}=U_\tau^n(\alpha_n f(x_n)+(1-\alpha_n)(x_n-\gamma_n A^*(I-T)Ax_n)), n\geq 1, \tag{3.22}$$

*where* $U_\tau^n=(1-\tau)I+\tau U^n$ *and the following conditions are satisfied:*

(a) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(b) $0<\tau<1-\kappa$;
(c) $(\kappa_n^2-1)=o(\alpha_n)$ *and*
(d) *for* $\mu$ *satisfying* $2\beta-1<\mu<1$,

$$\gamma_n=\begin{cases}\frac{(1-\mu)\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2}, & Ax_n\neq T(Ax_n)\\ 0 & otherwise.\end{cases}$$

*Then, the sequence* $\{x_n\}$ *generated by* (3.22) *converges strongly to* $x^*\in\Gamma_2$*, where* $x^*$ *is the unique solution of the variational inequality*

$$\textit{Find } x^*\in\Gamma_2 \textit{ such that } \langle f(x^*)-x^*, x-x^*\rangle\leq 0, \forall x\in\Gamma_2.$$

*Proof.* Since $E_1$ is a uniformly convex and 2-unifomly smooth Banach space, $E_2$ is a smooth Banach space, and $T$ is $\beta$-demimetric for $\beta\in[0,1)$, then the result follows from the proof of Theorem 3.1 immediately. □

The next corollary extends the work of Taiwo *et al.* [15] from the framework of real Hilbert spaces to Banach spaces.

**Corollary 3.3.** *Let* $E_1$ *be a uniformly convex and* 2*-uniformly smooth Banach space with smoothness constant* $t$ *satisfying* $0<t\leq\frac{1}{\sqrt{2}}$ *and a weak sequential continuous normalized duality mapping, and* $E_2$ *be a real smooth Banach space. Let* $f:E_1\to E_1$ *be a contraction mapping with coefficient* $\nu$ *in* $(0,1)$*, and let* $A:E_1\to E_2$ *be a bounded linear operator with adjoint* $A^*$*. Let* $U:E_1\to E_1$ *be a* $\kappa$*-demicontractive mapping, and let* $T:E_2\to E_2$ *be a* $\beta$*-demicontractive mapping. Assume that* $I-U$ *and* $I-T$ *are demiclosed at* $0$ *and* $\Gamma_3=\{x\in E_1 : x\in Fix(U) \text{ and } Ax\in Fix(T)\}\neq\emptyset$*. Let* $\{x_n\}$ *be the sequence generated by* $x_1\in E_1$ *and*

$$x_{n+1}=U_\tau(\alpha_n f(x_n)+(1-\alpha_n)(x_n-\gamma_n J_1^{-1}A^*J_2(I-T)Ax_n)), n\geq 1, \tag{3.23}$$

*where* $U_\tau=(1-\tau)I+\tau U$ *and the following conditions are satisfied:*

(a) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(b) $0<\tau<1-\kappa$ *and*
(c) *for* $\mu$ *satisfying* $2\beta-1<\mu<1$,

$$\gamma_n=\begin{cases}\frac{(1-\mu)\|(I-T)Ax_n\|^2}{2\|A^*J_2(I-T)Ax_n\|^2}, & Ax_n\neq T(Ax_n)\\ 0 & otherwise.\end{cases}$$

*Then, the sequence* $\{x_n\}$ *generated by* (3.23) *converges strongly to* $x^*\in\Gamma_3$*, where* $x^*$ *is the unique solution of the variational inequality*

$$\textit{Find } x^*\in\Gamma_3 \textit{ such that } \langle f(x^*)-x^*, J_1(x-x^*)\rangle\leq 0, \forall x\in\Gamma_3.$$

*Proof.* Similar to (3.2) and (3.3), using the definition of demicontactive mappings and Lemma 2.1, we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|U_\tau w_n - x^*\|^2 \\ &\leq \|w_n - x^*\|^2 + (\tau^2 - \tau(1-\kappa))\|Uw_n - w_n\|^2 \\ &\leq \|w_n - x^*\|^2.\end{aligned}$$

Note that $T$ is $\beta$-demimetric with $\beta \in [0,1)$. Thus the proof follows similar procedure as in the proof of Theorem 3.1 immediately. □

**Split Feasibility Problem.** Let $C$ and $Q$ be convex and closed subsets of the real Hilbert spaces $H_1$ and $H_2$, respectively, and let $A : H_1 \to H_2$ be a bounded and linear operator with the adjoint operator $A^*$. We recall the *split feasibility problem* (SFP): Find

$$x \in C \text{ such that } Ax \in Q. \tag{3.24}$$

We note that (3.24) is equivalent to: Finding

$$x \in Fix(P_C) \text{ such that } Ax \in Fix(P_Q).$$

We shall denote the solution of the SFP (3.24) by $SFP(C,Q)$. From Remark 2.1, by taking $U = P_C$ and $T = P_Q$, we obtain the following theorem for solving the SFP (3.24).

**Theorem 3.2.** *Let $C$ and $Q$ be convex and closed subsets of the real Hilbert spaces $H_1$ and $H_2$, respectively. Let $f : H_1 \to H_1$ be a contraction mapping with coefficient $\nu$ in $(0,1)$, and let $A : H_1 \to H_2$ be a bounded linear operator with adjoint $A^*$. Assume that $SFP(C,Q) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1 \in H_1$ and*

$$x_{n+1} = P_{C_\tau}(\alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma_n A_2^*(I-P_Q)Ax_n)), n \geq 1, \tag{3.25}$$

*where $P_{C_\tau} = (1-\tau)I + \tau P_C$ and the following conditions are satisfied:*

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $0 < \tau < 1$; *and*

(c) *for $\mu$ satisfying* $2\beta - 1 < \mu < 1$,

$$\gamma_n = \begin{cases} \frac{(1-\mu)\|(I-P_Q)Ax_n\|^2}{2\|A^*(I-P_Q)Ax_n\|^2}, & Ax_n \neq P_Q(Ax_n) \\ 0 & otherwise. \end{cases}$$

*Then, the sequence $\{x_n\}$ generated by (3.25) converges strongly to $x^* \in SFP(C,Q)$, where $x^*$ is the unique solution to the variational inequality*

$$\text{Find } x^* \in SFP(C,Q) \text{ such that } \langle f(x^*) - x^*, x - x^* \rangle \leq 0, \forall x \in SFP(C,Q).$$

## 4. Conclusion

In this paper, we studied the split common fixed point problem with the classes of asymptotically demicontractive and demimetric mappings in real Banach spaces. We introduced a new iterative scheme that does not require a prior estimate of the operator norms for its implementation and proved that it converges strongly to a solution to the problem, which is also a solution to some variational inequality problem.

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