Appl. Set-Valued Anal. Optim. 6 (2024), No. 2, pp. 137-155
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.6.2024.2.02

THE METHODS OF FRACTIONAL BACKWARD DIFFERENTIATION FORMULAS FOR SOLVING TWO-TERM FRACTIONAL DIFFERENTIAL SYLVESTER MATRIX EQUATIONS

LAKHLIFA SADEK

Department of Mathematics, Faculty of Sciences, Chouaib Doukkali University, El Jadida, Morocco

Abstract. In this paper, we present the fractional backward differentiation formulas for the numerical solutions of two-term fractional differential Sylvester matrix equations in the Caputo derivative sense, which includes the celebrated two-term fractional differential Lyapunov matrix equations. We give two applications in a two-term time-fractional telegraph equation with illustrative examples. In addition, we also consider two examples to illustrate the effectiveness of the proposed approaches.
Keywords. Fractional differential equation; Fractional backward differentiation formulas method; Grünwald approximation; Two-term time-fractional telegraph equation.
2020 Mathematics Subject Classification. 65F45, 34A08.

1. INTRODUCTION

The fractional derivatives (FD), which acted as a generalization of classical derivatives and dated back to the 17th century, recently gained great interest due to their wide real-world applications, such as, psychology [2], epidemiology [27], biology [6], engineering [18, 20], economics [16], and so on [5, 26, 31]. There are several definitions for the FD. Two of them are the Caputo and Riemann-Liouville fractional operators [2, 3, 6, 8, 15, 16, 18, 20] which are employed in this work. The fractional differential Sylvester matrix equation (FSE) plays an important role in theory control, filter design theory, and model reduction; see, e.g., [1, 21, 22, 24, 28, 29, 30]. The methods of backward differentiation formulas (BDF) gained popularity due to their large absolute stability regions which made them competitive for the treatment of stiff problems [4]. In [9, 14], some authors proposed a method of fractional backward differentiation formulas (FBDF) for solving FDE with delay. In [23], the author proposed the FBDF method for solving the two-term FSE; see [23] for more details.

In this paper, we present the FBDF method order $r \in \{2,3\}$ to solve two-term FSE's and two-term fractional differential Lyapunov matrix equations (FLE's). Let $\alpha, \beta \in \mathbb{R}^+$ and $n_\alpha, n_\beta \in \mathbb{N}^*$ such that $n_\alpha - 1 < \alpha < n_\alpha$ and $n_\beta - 1 < \beta < n_\beta$. We consider the two-term FSE on the interval

*Corresponding author.
E-mail address: lakhlifasadek@gmail.com; sadek.l@ucd.ac.ma
Received 2 August 2022; Accepted 23 May 2023; Published online 27 March 2024.

$[t_0, T]$ of the form

$$\begin{cases} P^{(\alpha)}(t) + P^{(\beta)}(t) = A(t)P(t) + P(t)C(t) + Q(t), \\ P^{(k)}(t_0) = P_k, k \in \{0, 1, \ldots, \max\{n_\beta, n_\alpha\} - 1\}, \end{cases} \tag{1.1}$$

where $C(t) \in \mathbb{R}^{p\times p}$, $A(t) \in \mathbb{R}^{n\times n}$, $Q(t) \in \mathbb{R}^{n\times p}$, $P(t) \in \mathbb{R}^{n\times p}$ is unknown matrix function, and $P^{(\alpha)}(t)$ and $P^{(\beta)}(t)$ are the Caputo derivative (CD) of the matrix function $P(t)$ with order α and β, respectively. In particular,

- If $\alpha = 1$ and $\beta = 1$, then the two-term FSE is called the differential Sylvester matrix equation (see [21]).
- If $C(t) = A^T(t)$ (A^T denotes the transpose of the matrix A), then the two-term FSE is called the two-term FLE.

This paper is organized in this form. In Section 2, we present basic concepts, which are needed in the following sections. We discuss the FBDF method order r for solving the two-term FSE in Section 3. In Sections 4 and 5, we develop the FBDF method order 2 and 3, respectively, for solving the two-term FSE's. In Section 6, we study the error analysis and the convergence. Finally, Section 7 is devoted to numerical experiments for demonstrating the effectiveness of the proposed methods.

2. Basic Concepts

This section provides some important definitions that are useful in obtaining our findings.

Let $\beta \in \mathbb{R}^+$ and $n_\beta \in \mathbb{N}^*$ such that $n_\beta - 1 < \beta < n_\beta$. The Riemann-Liouville derivative (R-LD) and Caputo derivative of order β for function $P(t)$ are defined by

$${}^{\mathrm{RL}}_{t_0}D^{\beta}_{t} P(t) := \frac{1}{\Gamma(n_\beta - \beta)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n_\beta} \int_{t_0}^{t} \frac{P(\tau)}{(t-\tau)^{\beta - n_\beta + 1}} \,\mathrm{d}\tau,$$

and

$$P^{(\beta)}(t) := \frac{1}{\Gamma(n_\beta - \beta)} \int_{t_0}^{t} \frac{P^{(n_\beta)}(\eta)}{(t-\eta)^{\beta - n_\beta + 1}} \,\mathrm{d}\eta,$$

respectively, where the Euler's gamma function $\Gamma]0, +\infty[\longmapsto \mathbb{R}$ is defined by (see [19]),

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} \,\mathrm{d}t.$$

Lemma 2.1. [11] *The relation between CD and the R-LD of function $P(t)$ is*

$$P^{(\beta)}(t) = {}^{\mathrm{RL}}_{t_0}D^{\beta}_{t} \left(P(t) - \sum_{k=0}^{n_\beta - 1} \frac{P^{(k)}(t_0)}{k!} (t - t_0)^k \right), \tag{2.1}$$

where $n_\beta - 1 < \beta < n_\beta$ with $n_\beta \in \mathbb{N}^*$.

3. FBDF Method Order r

In this section, we present the FBDF method order r for solving the two-term FSE. From the relation (2.1) at $t=t_j$, we have

$$\begin{aligned} P^{(\alpha)}(t_j) &= {}_{t_0}^{\mathrm{RL}}D_t^{\alpha}P(t_j) - {}_{t_0}^{RL}D_t^{\alpha}\left(\sum_{k=0}^{n_\alpha-1}\frac{P^{(k)}(t_0)}{k!}(t_j-t_0)^k\right) \\ &= {}_{t_0}^{\mathrm{RL}}D_t^{\alpha}P(t_j) - \sum_{k=0}^{n_\alpha-1}\frac{P^{(k)}(t_0)}{\Gamma(k-\alpha+1)}(t_j-t_0)^{k-\alpha}, \end{aligned}$$

where $N\in\mathbb{N}^*$, step size $h=\frac{T-t_0}{N}$, and

$$P^{(\beta)}(t_j) = {}_{t_0}^{\mathrm{RL}}D_t^{\beta}P(t_j) - \sum_{k=0}^{n_\beta-1}\frac{P^{(k)}(t_0)}{\Gamma(k-\beta+1)}(t_j-t_0)^{k-\beta},$$

with $t_j=t_0+jh$ for $j=0,1,\ldots,N$. From (1.1), we obtain the matrix equation

$$\begin{aligned} &{}_{t_0}^{\mathrm{RL}}D_t^{\alpha}P(t_j) - \sum_{k=0}^{n_\alpha-1}\frac{P^{(k)}(t_0)}{\Gamma(k-\alpha+1)}(jh)^{k-\alpha} + {}_{t_0}^{\mathrm{RL}}D_t^{\beta}P(t_j) \\ &\quad - \sum_{k=0}^{n_\beta-1}\frac{P^{(k)}(t_0)}{\Gamma(k-\beta+1)}(jh)^{k-\beta} = A(t_j)P(t_j)+P(t_j)C(t_j)+Q(t_j). \end{aligned}$$

The Grünwald approximation ([19]) of ${}_{t_0}^{\mathrm{RL}}D_t^{\alpha}P\left(t_j\right)$ and ${}_{t_0}^{\mathrm{RL}}D_t^{\beta}P\left(t_j\right)$ at t_j are

$$\begin{aligned} {}_{t_0}^{\mathrm{RL}}D_t^{\alpha}P\left(t_j\right) &= h^{-\alpha}\sum_{k=0}^{j}\varpi_{k,\alpha}^{r}P\left(t_j-kh\right), \\ {}_{t_0}^{\mathrm{RL}}D_t^{\beta}P\left(t_j\right) &= h^{-\beta}\sum_{k=0}^{j}\varpi_{k,\beta}^{r}P\left(t_j-kh\right), \end{aligned}$$

where $\varpi_{k,\cdot}^{r}$ is determined by the generating function W_β^r for FBDF method order r

$$W_\beta^r(\xi) = \sum_{k=0}^{\infty}\varpi_{k,\beta}^{r}\xi^k. \tag{3.1}$$

The generating functions for FBDF of order $r\in\{2,3\}$ are given below ([14])

$$W_\beta^2(\xi) := \left(\frac{3}{2}-2\xi+\frac{1}{2}\xi^2\right)^{\beta}, \tag{3.2}$$

$$W_\beta^3(\xi) := \left(\frac{6}{11}-3\xi+\frac{3}{2}\xi^2-\frac{1}{3}\xi^3\right)^{\beta}. \tag{3.3}$$

It follows that

$$h^{-\alpha}\sum_{k=0}^{j}\varpi_{k,\alpha}^{r}P\left(t_j-kh\right)-\sum_{m=0}^{n_\alpha-1}\frac{P^{(m)}(t_0)}{\Gamma(m-\alpha+1)}(jh)^{m-\alpha}$$
$$+h^{-\beta}\sum_{k=0}^{j}\varpi_{k,\beta}^{r}P\left(t_j-kh\right)-\sum_{m=0}^{n_\beta-1}\frac{P^{(m)}(t_0)}{\Gamma(m-\beta+1)}(jh)^{m-\beta}$$
$$=A(t_j)P(t_j)+P(t_j)C(t_j)+Q(t_j).$$

Let $n_1:=\min\{n_\alpha,n_\beta\}$ and $n_2:=\max\{n_\alpha,n_\beta\}$. Thus

$$h^{-\alpha}\sum_{k=0}^{j}\varpi_{k,\alpha}^{r}P\left(t_j-kh\right)-\sum_{m=0}^{n_1-1}\left(\frac{(jh)^{m-\alpha}}{\Gamma(m-\alpha+1)}+\frac{(jh)^{m-\beta}}{\Gamma(m-\beta+1)}\right)P^{(m)}(t_0)$$
$$+h^{-\beta}\sum_{k=0}^{j}\varpi_{k,\beta}^{r}P\left(t_j-kh\right)-\sum_{m=n_1}^{n_2-1}\frac{(jh)^{m-\alpha}}{\Gamma(m-\alpha^*+1)}P^{(m)}(t_0)$$
$$=A(t_j)P(t_j)+P(t_j)C(t_j)+Q(t_j),$$

where $\alpha^*=\alpha$ and β whenever $n_2=n_\alpha$ and $n_2=n_\beta$, respectively. Thus

$$h^{-\alpha}\sum_{k=0}^{j}\varpi_{k,\alpha}^{r}P_{j-k}+h^{-\beta}\sum_{k=0}^{j}\varpi_{k,\beta}^{r}P_{j-k}-\sum_{m=0}^{n_1-1}b_{j,m}P^{(m)}(t_0)-\sum_{m=n_1}^{n_2-1}c_{j,m}P^{(m)}(t_0)$$
$$=A(t_j)P(t_j)+P(t_j)C(t_j)+Q(t_j),\qquad j=i+1,i+2,\ldots,$$

where $P_{j-k}=P(t_j-kh)$ and

$$b_{j,m}:=\frac{(jh)^{m-\alpha}}{\Gamma(m-\alpha+1)}+\frac{(jh)^{m-\beta}}{\Gamma(m-\beta+1)},\qquad c_{j,m}:=\frac{(jh)^{m-\alpha}}{\Gamma(m-\alpha^*+1)}.$$

It follows that

$$(h^{-\alpha}\varpi_{0,\alpha}^{r}+h^{-\beta}\varpi_{0,\beta}^{r})P_{i+1}+\sum_{k=1}^{i+1}(h^{-\alpha}\varpi_{k,\alpha}^{r}+h^{-\beta}\varpi_{k,\beta}^{r})P_{i+1-k}$$
$$=\sum_{m=0}^{n_1-1}b_{i+1,m}P^{(m)}(t_0)+\sum_{m=n_1}^{n_2-1}c_{i+1,m}P^{(m)}(t_0)$$
$$+A(t_{i+1})P(t_{i+1})+P(t_{i+1})C(t_{i+1})+Q(t_{i+1}),\tag{3.4}$$

where $\varpi_{k,\cdot}^{r}$ is the coefficient of FBDF with order r. For calculating $\varpi_{k,\cdot}^{r}$ in (3.4), we need the following Lemma.

Lemma 3.1. [34] *Let $\varphi(z):=1+\sum_{k=1}^{\infty}a_kz^k$. Then $(\varphi(z))^\beta=\sum_{k=0}^{\infty}\omega_k^{(\beta)}z^k$, where the coefficients $\omega_k^{(\beta)}$ are recursively evaluated by*

$$\omega_0^{(\beta)}=1\quad \omega_k^{(\beta)}=\sum_{j=1}^{k}\left(\frac{(\beta+1)j}{k}-1\right)a_j\omega_{k-j}^{(\beta)}.$$

Next, we describe the steps for calculating the coefficients $\varpi_{k,\cdot}^{r}$.

Algorithm 1 Calculation of the coefficients $\varpi^r_{k,\cdot}$.

(1) Let $W^r_\beta(z) = d^\beta(\varphi(z))^\beta$.
(2) From Lemma 3.1, we obtain

$$W^r_\beta(z) = d^\beta\left(\sum_{k=0}^{\infty}\omega_k^{(\beta)}z^k\right).$$

(3) $\varpi^r_{k,\beta} = d^\beta\omega_k^{(\beta)}$, (by equation (3.1)).

4. FBDF Method Order 2

In this section, we apply the FBDF method of second order to solve the two-term FSE. Let the approximation P_{i+1} of $P(t_{i+1})$ obtained at step $i+1$ by FBDF method order 2. The next result shows that P_{i+1} is the solution to a Sylvester matrix equation (SME).

Theorem 4.1. *Let $\alpha,\beta \in \mathbb{R}^+$ and $n_\alpha, n_\beta \in \mathbb{N}^*$ such that $n_\alpha - 1 < \alpha < n_\alpha$ and $n_\beta - 1 < \beta < n_\beta$. Let $n_1 := \min\{n_\alpha, n_\beta\}$ and $n_2 := \max\{n_\alpha, n_\beta\}$. Let $P(t)$ be the solution to the FSE (1.1). Then the approximation P_{i+1} of $P(t_{i+1})$ satisfies the following SME:*

$$\mathbb{M}_{i+1}P_{i+1} + P_{i+1}\mathbb{L}_{i+1} + \mathscr{E}_{i+1} = 0, \tag{4.1}$$

where

$$\begin{aligned}\mathbb{M}_{i+1} &= A(t_{i+1}) - \frac{1}{2}\left(h^{-\alpha}\varpi^2_{0,\alpha} + h^{-\beta}\varpi^2_{0,\beta}\right)I_{n\times n},\\ \mathbb{L}_{i+1} &= C(t_{i+1}) - \frac{1}{2}\left(h^{-\alpha}\varpi^2_{0,\alpha} + h^{-\beta}\varpi^2_{0,\beta}\right)I_{p\times p},\end{aligned} \tag{4.2}$$

with $I_{n\times n}$ being $n\times n$ the identity matrix and

$$\begin{aligned}\mathscr{E}_{i+1} = Q(t_{i+1}) &- \sum_{k=1}^{i+1}(h^{-\alpha}\varpi^2_{k,\alpha} + h^{-\beta}\varpi^2_{k,\beta})P_{i+1-k}\\ &+ \sum_{m=0}^{n_1-1} b_{i+1,m}P^{(m)}(t_0) + \sum_{m=n_1}^{n_2-1} c_{i+1,m}P^{(m)}(t_0),\end{aligned} \tag{4.3}$$

where $\varpi^r_{k,\alpha}$ and $\varpi^r_{k,\beta}$ are the coefficients of FBDF with order r.

Proof. At each t_{i+1}, let P_{i+1} be of the approximation of $P(t_{i+1})$, and FBDF with order 2 be defined as

$$\begin{aligned}\left(h^{-\alpha}\varpi^r_{0,\alpha} + h^{-\beta}\varpi^r_{0,\beta}\right)P_{i+1} &+ \sum_{k=1}^{i+1}\left(h^{-\alpha}\varpi^r_{k,\alpha} + h^{-\beta}\varpi^r_{k,\beta}\right)P_{i+1-k}\\ &= \sum_{m=0}^{n_1-1} b_{i+1,m}P^{(m)}(t_0) + \sum_{m=n_1}^{n_2-1} c_{i+1,m}P^{(m)}(t_0)\\ &\quad + A(t_{i+1})P(t_{i+1}) + P(t_{i+1})C(t_{i+1}) + Q(t_{i+1}).\end{aligned}$$

Therefore

$$\left(A(t_{i+1})-\frac{1}{2}(h^{-\alpha}\varpi^2_{0,\alpha}+h^{-\beta}\varpi^2_{0,\beta})I_{n\times n}\right)P_{i+1}+P_{i+1}\Bigg(C(t_{i+1})$$
$$-\frac{1}{2}(h^{-\alpha}\varpi^2_{0,\alpha}+h^{-\beta}\varpi^2_{0,\beta})I_{p\times p}\Bigg)+Q(t_{i+1})$$
$$-\sum_{k=1}^{i+1}\left(h^{-\alpha}\varpi^2_{k,\alpha}+h^{-\beta}\varpi^2_{k,\beta}\right)P_{i+1-k}$$
$$+\sum_{m=0}^{n_1-1}b_{i+1,m}P^{(m)}(t_0)+\sum_{m=n_1}^{n_2-1}c_{i+1,m}P^{(m)}(t_0)=0.$$

We find the following SME $\mathbb{M}_{i+1}P_{i+1}+P_{i+1}\mathbb{L}_{i+1}+\mathscr{E}_{i+1}=0.$ □

We can directly compute the SME (4.1) by using the well-known function `sylvester` in MATLAB. Then, we give the Algorithm 1 of ϖ^2_k in (4.2) and (4.3), we can give an explicit representation of the coefficients of FBDF method order 2 in next Lemma 4.1.

Lemma 4.1. *The coefficients of the FBDF method order* 2 *are obtained explicitly*

$$\varpi^2_{k,\beta}=\left(\frac{3}{2}\right)^{\beta}\omega^{(\beta)}_k,\qquad k=0,1,\ldots,$$

where $\omega^{(\beta)}_0=1$, $\omega^{(\beta)}_1=-\frac{4}{3}\beta$, *and*

$$\omega^{(\beta)}_k=\frac{4}{3}\left(1-\frac{\beta+1}{k}\right)\omega^{(\beta)}_{k-1}+\frac{1}{3}\left(\frac{2(1+\beta)}{k}-1\right)\omega^{(\beta)}_{k-2},\qquad k=2,3,\ldots.$$

Proof. From (3.2) and Algorithm 1, we can give the coefficients of the FBDF method order 2 immediately. □

Corollary 4.1. *Let* $\alpha,\beta\in\mathbb{R}^+$ *and* $n_\alpha,n_\beta\in\mathbb{N}^*$ *be such that* $n_\alpha-1<\alpha<n_\alpha$ *and* $n_\beta-1<\beta<n_\beta$. *Let* $n_1:=\min\{n_\alpha,n_\beta\}$ *and* $n_2:=\max\{n_\alpha,n_\beta\}$. *Let* $P(t)$ *be the solution to the FLE:*

$$\begin{cases}P^{(\alpha)}(t)+P^{(\beta)}(t)=A(t)P(t)+P(t)A^T(t)+Q(t),\\ P^{(k)}(t_0)=P_k,\ k\in\{0,1,\ldots,n_2-1\}.\end{cases}$$

Then the approximation P_{i+1} *of* $P(t_{i+1})$ *satisfies the following Lyapunov matrix equation (LME):*

$$\mathbb{M}_{i+1}P_{i+1}+P_{i+1}\mathbb{M}^T_{i+1}+\mathscr{E}_{i+1}=0,\tag{4.4}$$

where

$$\mathbb{M}_{i+1}=A(t_{i+1})-\frac{1}{2}\left(h^{-\alpha}\varpi^2_{0,\alpha}+h^{-\beta}\varpi^2_{0,\beta}\right)I_{n\times n},$$

and

$$\mathscr{E}_{i+1}=Q(t_{i+1})-\sum_{k=1}^{i+1}\left(h^{-\alpha}\varpi^2_{k,\alpha}+h^{-\beta}\varpi^2_{k,\beta}\right)P_{i+1-k}$$
$$+\sum_{m=0}^{n_1-1}b_{i+1,m}P^{(m)}(t_0)+\sum_{m=n_1}^{n_2-1}c_{i+1,m}P^{(m)}(t_0).$$

Note Theorem 4.1 for $C(t)=A^T(t)$. We can directly compute the LME (4.4) by using the well-known function `lyap` in MATLAB.

5. FBDF METHOD ORDER 3

In this section, we apply the FBDF method order 3 to solve the FSE. Let the approximation P_{i+1} of $P(t_{i+1})$ be obtained at step $i+1$. Theorem 5.1 shows that the matrix P_{i+1} is the solution of a SME.

Theorem 5.1. *Let $\alpha,\beta \in \mathbb{R}^+$ and $n_\alpha, n_\beta \in \mathbb{N}^*$ be such that $n_\alpha - 1 < \alpha < n_\alpha$ and $n_\beta - 1 < \beta < n_\beta$. Let $n_1 := \min\{n_\alpha, n_\beta\}$ and $n_2 := \max\{n_\alpha, n_\beta\}$. Let $P(t)$ be the solution to the FSE (1.1). Then the approximation P_{i+1} of $P(t_{i+1})$ satisfies the following SME:*

$$\mathbb{M}_{i+1}P_{i+1} + P_{i+1}\mathbb{L}_{i+1} + \mathscr{E}_{i+1} = 0, \tag{5.1}$$

where

$$\begin{cases} \mathbb{M}_{i+1} = A(t_{i+1}) - \frac{1}{2}\left(h^{-\alpha}\varpi_{0,\alpha}^2 + h^{-\beta}\varpi_{0,\beta}^2\right) I_{n\times n}, \\ \mathbb{L}_{i+1} = C(t_{i+1}) - \frac{1}{2}\left(h^{-\alpha}\varpi_{0,\alpha}^2 + h^{-\beta}\varpi_{0,\beta}^2\right) I_{p\times p}, \\ \mathscr{E}_{i+1} = Q(t_{i+1}) - \sum_{k=1}^{i+1}\left(h^{-\alpha}\varpi_{k,\alpha}^2 + h^{-\beta}\varpi_{k,\beta}^2\right) P_{i+1-k} \\ \qquad + \sum_{m=0}^{n_1-1} b_{i+1,m}P^{(m)}(t_0) + \sum_{m=n_1}^{n_2-1} c_{i+1,m}P^{(m)}(t_0), \end{cases} \tag{5.2}$$

with the coefficients of FBDF method order 3 being obtained explicitly

$$\varpi_{k,\gamma}^3 = \left(\frac{11}{6}\right)^\gamma \omega_k^{(\gamma)}, \qquad k = 0,1,\ldots, \tag{5.3}$$

where

$$\begin{cases} \omega_0^{(\gamma)} = 1, \\ \omega_1^{(\gamma)} = -\frac{18}{11}\gamma, \\ \omega_2^{(\gamma)} = \frac{9}{11}\gamma + \left(\frac{18}{11}\right)^2 \frac{\gamma(\gamma-1)}{2}, \\ \omega_k^{(\gamma)} = \frac{18}{11}\left(1 - \frac{\gamma+1}{k}\right)\omega_{k-1}^{(\gamma)} + \frac{9}{11}\left(\frac{2(1+\gamma)}{k} - 1\right)\omega_{k-2}^{(\gamma)} \\ \quad + \frac{2}{11}\left(1 - \frac{3(\gamma+1)}{k}\right)\omega_{k-3}^{(\gamma)}, \quad k = 3,4,\ldots, \end{cases} \tag{5.4}$$

with $\gamma \in \{\alpha,\beta\}$.

Proof. In order to prove (5.1), we follow the proof of (4.1) in Theorem 4.1. For the coefficients of FBDF method order 3 in (5.3) and (5.4), the generating function of FBDF method order 3 is (3.3). Applying Algorithm 1, we can give an explicit representation of the coefficients of FBDF method order 3. Hence, the coefficients of the FBDF method order 3 are obtained explicitly by

$$\varpi_{k,\gamma}^3 = \left(\frac{11}{6}\right)^\gamma \omega_k^{(\gamma)}, \qquad k = 0,1,\ldots,$$

where $\omega_0^{(\gamma)} = 1$, $\omega_1^{(\gamma)} = -\frac{18}{11}\gamma$,

$$\begin{aligned}\omega_2^{(\gamma)} &= \frac{9}{11}\gamma + \left(\frac{18}{11}\right)^2 \frac{\gamma(\gamma-1)}{2},\\ \omega_k^{(\gamma)} &= \frac{18}{11}\left(1-\frac{\gamma+1}{k}\right)\omega_{k-1}^{(\gamma)} + \frac{9}{11}\left(\frac{2(1+\gamma)}{k}-1\right)\omega_{k-2}^{(\gamma)}\\ &\quad + \frac{2}{11}\left(1-\frac{3(\gamma+1)}{k}\right)\omega_{k-3}^{(\gamma)},\end{aligned}$$

for $k = 3,4,\ldots$. □

Corollary 5.1. *Let $\alpha,\beta \in \mathbb{R}^+$ and $n_\alpha, n_\beta \in \mathbb{N}^*$ be such that $n_\alpha - 1 < \alpha < n_\alpha$ and $n_\beta - 1 < \beta < n_\beta$. Let $n_1 := \min\{n_\alpha, n_\beta\}$ and $n_2 := \max\{n_\alpha, n_\beta\}$. Let $P(t)$ be the solution to the FLE. Then the approximation P_{i+1} of $P(t_{i+1})$ satisfies the following LME:*

$$\mathbb{M}_{i+1}P_{i+1} + P_{i+1}\mathbb{M}_{i+1}^T + \mathscr{E}_{i+1} = 0, \tag{5.5}$$

where

$$\mathbb{M}_{i+1} = A(t_{i+1}) - \frac{1}{2}\left(h^{-\alpha}\varpi_{0,\alpha}^2 + h^{-\beta}\varpi_{0,\beta}^2\right) I_{n\times n},$$

and

$$\begin{aligned}\mathscr{E}_{i+1} &= Q(t_{i+1}) - \sum_{k=1}^{i+1}\left(h^{-\alpha}\varpi_{k,\alpha}^2 + h^{-\beta}\varpi_{k,\beta}^2\right) P_{i+1-k}\\ &\quad + \sum_{m=0}^{n_1-1} b_{i+1,m}P^{(m)}(t_0) + \sum_{m=n_1}^{n_2-1} c_{i+1,m}P^{(m)}(t_0).\end{aligned}$$

From the Theorem 5.1 with $C(t) = A^T(t)$, we have the desired result immediately.

Next, we summarize the steps of the FBDF method order r for solving the two-term FSE.

Algorithm 2 The FBDF method order r for solving the two-term FSE

Inputs: $A(t)$, $C(t)$, $Q(t)$, $P^{(k)}$ for $k = 0,\ldots,n_2-1$, and t_0, T.

(1) Choose h.
(2) $N = \frac{T-t_0}{h}$.
(3) For $i = 1:N$, compute:
(4) $\varpi_{k,\alpha}^r$ and $\varpi_{k,\beta}^r$,
(5) $\mathbb{M}_{i+1} = A(t_{i+1}) - \frac{1}{2}\left(h^{-\alpha}\varpi_{0,\alpha}^2 + h^{-\beta}\varpi_{0,\beta}^2\right) I_{n\times n}$,
(6) $\mathbb{L}_{i+1} = A(t_{i+1}) - \frac{1}{2}\left(h^{-\alpha}\varpi_{0,\alpha}^2 + h^{-\beta}\varpi_{0,\beta}^2\right) I_{p\times p}$,
(7)

$$\begin{aligned}\mathscr{E}_{i+1} &= Q(t_{i+1}) - \sum_{k=1}^{i+1}\left(h^{-\alpha}\varpi_{k,\alpha}^2 + h^{-\beta}\varpi_{k,\beta}^2\right) P_{i+1-k}\\ &\quad + \sum_{m=0}^{n_1-1} b_{i+1,m}P^{(m)}(t_0) + \sum_{m=n_1}^{n_2-1} c_{i+1,m}P^{(m)}(t_0).\end{aligned}$$

(8) Solve the SME $\mathbb{M}_{i+1}P_{i+1} + P_{i+1}\mathbb{L}_{i+1} + \mathscr{E}_{i+1} = 0$.
(9) End for i.

Next, we summarize the steps of the FBDF method order r for solving the two-term FLE.

Algorithm 3 The FBDF method order r to solve the two-term FLE

Inputs: $A(t)$, $Q(t)$, $P^{(k)}$ for $k=0,\dots,n_2-1$, and t_0,T.

(1) Choose h.
(2) $N=\frac{T-t_0}{h}$.
(3) For $i=1:N$, compute:
(4) $\varpi^r_{k,\alpha}$ and $\varpi^r_{k,\beta}$,
(5) $\mathbb{M}_{i+1}=A(t_{i+1})-\frac{1}{2}\left(h^{-\alpha}\varpi^2_{0,\alpha}+h^{-\beta}\varpi^2_{0,\beta}\right)I_{n\times n}$,
(6)

$$\begin{aligned}\mathscr{E}_{i+1}=Q(t_{i+1})-\sum_{k=1}^{i+1}\left(h^{-\alpha}\varpi^2_{k,\alpha}+h^{-\beta}\varpi^2_{k,\beta}\right)P_{i+1-k}\\+\sum_{m=0}^{n_1-1}b_{i+1,m}P^{(m)}(t_0)+\sum_{m=n_1}^{n_2-1}c_{i+1,m}P^{(m)}(t_0).\end{aligned}$$

(7) Solve the LME $\mathbb{M}_{i+1}P_{i+1}+P_{i+1}\mathbb{M}^T_{i+1}+\mathscr{E}_{i+1}=0$.
(8) End for i.

---

6. The Error Analysis

In this section, we present the error analysis of the error analysis for FBDF method order r. We consider the error $e_j=P(t_j)-P_j$ between the exact solution $P(t_j)$ and the numerical solution P_j.

Lemma 6.1. [12, 13, 14] *Let $\beta\in\mathbb{R}^+$. Then*

$$h^{-\beta}\sum_{k=0}^{j}\varpi^r_kP(t_{j-k})-\sum_{m=0}^{n_\beta-1}b_{j,m}P^{(m)}(t_0)-P^{(\beta)}(t)=O\left(h^r\right),\tag{6.1}$$

and $\varpi^r_k=O(j^{-\beta-1})$.

Lemma 6.2. [12] *Let $M_1,M_2>0$ and $\{\lambda_i\}$ satisfy*

$$|\lambda_l|\le M_1+M_2h\sum_{i=0}^{l-1}|\lambda_i|,\qquad l=k,k+1,\dots,lh\le T.$$

Then $|\lambda_l|\le e^{M_1T}\left(M_1+M_2kh\delta\right)$, for $l\ge k, lh\le T$, where $\delta=\max\{|\lambda_0|,|\lambda_1|,\dots,|\lambda_{k-1}|\}$.

Lemma 6.3. *Let the matrix function F, where $F(P)=A(t)P+PC(t)+Q(t)$. Then*

$$\|F(P_1)-F(P_2)\|\le L\|P_1-P_2\|,\qquad\forall P_1,P_2\in\mathbb{R}^{n\times p},\tag{6.2}$$

where

$$M=\max_{\varsigma\in[t_0,T]}\|A(\varsigma)\|+\max_{\varsigma\in[t_0,T]}\|C(\varsigma)\|.$$

Proof. Letting $P_1, P_2 \in \mathbb{R}^{n\times p}$, we have

$$\begin{aligned} F(P_1) - F(P_2) &= A(t)P_1 + P_1C(t) + Q(t) - (A(t)P_2 + P_2C(t) + Q(t)) \\ &= A(t)(P_1 - P_2) + (P_1 - P_2)C(t), \end{aligned}$$

so

$$\begin{aligned} \|F(P_1) - F(P_2)\| &\le \max_{\eta\in[t_0,T]} \|A(\eta)\|\|P_1 - P_2\| + \max_{\eta\in[t_0,T]} \|C(\eta)\|\|P_1 - P_2\| \\ &\le \left(\max_{\eta\in[t_0,T]} \|A(\eta)\| + \max_{\eta\in[t_0,T]} \|C(\eta)\|\right)\|P_1 - P_2\|. \end{aligned}$$

□

Theorem 6.1. *Let $\alpha, \beta \in \mathbb{R}^+$ and $n_\alpha, n_\beta \in \mathbb{N}^*$ be such that $n_\alpha - 1 < \alpha < n_\alpha$ and $n_\beta - 1 < \beta < n_\beta$. Let the following two-term FSE be given*

$$\begin{cases} P^{(\alpha)}(t) + P^{(\beta)}(t) = A(t)P(t) + P(t)C(t) + Q(t), \\ P^{(k)}(t_0) = P_k,\, k \in \{0, 1, \ldots, \max\{n_\beta, n_\alpha\} - 1\}. \end{cases} \tag{6.3}$$

Then, the FBDF method order r is convergent, and $\|e_j\| = O(h^r)$.

Proof. From (6.3) and (6.1) in Lemma 6.1, we have

$$\begin{aligned} h^{-\alpha}\sum_{k=0}^{j} \varpi^r_{k,\alpha} P(t_{j-k}) + h^{-\beta}\sum_{k=0}^{j} \varpi^r_{k,\beta} P(t_{j-k}) - \sum_{m=0}^{n_1-1} b_{i+1,m} P^{(m)}(t_0) \\ - \sum_{m=n_1}^{n_2-1} c_{i+1,m} P^{(m)}(t_0) - C_j h^r = F(P(t_j)). \end{aligned}$$

In view of (3.4), we obtain

$$\begin{aligned} h^{-\alpha}\sum_{k=0}^{j} \varpi^r_{k,\alpha} P_{j-k} + h^{-\beta}\sum_{k=0}^{j} \varpi^r_{k,\beta} P_{j-k} - \sum_{m=0}^{n_1-1} b_{i+1,m} P^{(m)}(t_0) \\ - \sum_{m=n_1}^{n_2-1} c_{i+1,m} P^{(m)}(t_0) = F(P_j), \end{aligned}$$

for $j = i+1, i+2, \ldots,$. Observe that

$$b_j e_0 - h^{-\alpha}\sum_{k=0}^{j} \varpi^r_{k,\alpha} e_{j-k} - h^{-\beta}\sum_{k=0}^{j} \varpi^r_{k,\beta} e_{j-k} + C_j h^r = -(F(P(t_j)) - F(P_j)).$$

It follows that

$$\begin{aligned} b_j e_0 - h^{-\alpha}\varpi^r_{0,\alpha} e_j - h^{-\beta}\varpi^r_{0,\beta} e_j - h^{-\alpha}\varpi^r_{j,\alpha} e_0 - h^{-\beta}\varpi^r_{j,\beta} e_0 \\ - h^{-\alpha}\sum_{k=1}^{j-1} \varpi^r_{k,\alpha} e_{j-k} - h^{-\beta}\sum_{k=1}^{j-1} \varpi^r_{k,\beta} e_{j-k} + C_j h^r \\ = -(F(P(t_j)) - F(P_j)). \end{aligned}$$

Hence, we have

$$\begin{aligned}\left(h^{-\alpha}\varpi^r_{0,\alpha}+h^{-\beta}\varpi^r_{0,\beta}\right)e_j=&-\left(h^{-\alpha}\varpi^r_{j,\alpha}+h^{-\beta}\varpi^r_{j,\beta}-b_j\right)e_0\\&-\sum_{k=1}^{j-1}\left(h^{-\alpha}\varpi^r_{k,\alpha}+h^{-\beta}\varpi^r_{k,\beta}\right)e_{j-k}\\&+C_jh^r+\left(F(P(t_j))-F(P_j)\right)\end{aligned}$$

and

$$\begin{aligned}\left(h^{-\alpha}\varpi^r_{0,\alpha}+h^{-\beta}\varpi^r_{0,\beta}\right)\|e_j\|\le&\left|h^{-\alpha}\varpi^r_{j,\alpha}+h^{-\beta}\varpi^r_{j,\beta}-b_j\right|\|e_0\|\\&+\sum_{k=1}^{j-1}\left(h^{-\alpha}\varpi^r_{k,\alpha}+h^{-\beta}\varpi^r_{k,\beta}\right)\|e_{j-k}\|\\&+\|C_j\|h^r+\|F(P(t_j))-F(P_j)\|.\end{aligned}$$

Letting $\|C_j\|\le K$ for any j and using inequality (6.2), we have

$$\begin{aligned}\left(h^{-\alpha}\varpi^r_{0,\alpha}+h^{-\beta}\varpi^r_{0,\beta}\right)\|e_j\|\le&\left|h^{-\alpha}\varpi^r_{j,\alpha}+h^{-\beta}\varpi^r_{j,\beta}-b_j\right|\|e_0\|\\&+\sum_{k=1}^{j-1}\left(h^{-\alpha}\varpi^r_{k,\alpha}+h^{-\beta}\varpi^r_{k,\beta}\right)\|e_{j-k}\|+Kh^r+M\|e_j\|.\end{aligned}$$

Now, we estimate e_j. Observe that

$$\begin{aligned}\|e_j\|\le&\frac{Kh^r}{h^{-\alpha}\varpi^r_{0,\alpha}+h^{-\beta}\varpi^r_{0,\beta}-M}+\sum_{k=1}^{j-1}\frac{\left(h^{-\alpha}\varpi^r_{k,\alpha}+h^{-\beta}\varpi^r_{k,\beta}\right)}{h^{-\alpha}\varpi^r_{0,\alpha}+h^{-\beta}\varpi^r_{0,\beta}-M}\|e_{j-k}\|\\&+\frac{\left|h^{-\alpha}\varpi^r_{j,\alpha}+h^{-\beta}\varpi^r_{j,\beta}-b_j\right|}{h^{-\alpha}\varpi^r_{0,\alpha}+h^{-\beta}\varpi^r_{0,\beta}-M}\|e_0\|.\end{aligned}$$

It follows that

$$\|e_j\|\le\frac{Kh^r}{h^{-\alpha}\varpi^r_{0,\alpha}+h^{-\beta}\varpi^r_{0,\beta}-M}+\sum_{k=0}^{j-1}\rho_k\|e_{j-k}\|,$$

where

$$\rho_k=\max\left\{\frac{\left(h^{-\alpha}\varpi^r_{k,\alpha}+h^{-\beta}\varpi^r_{k,\beta}\right)}{h^{-\alpha}\varpi^r_{0,\alpha}+h^{-\beta}\varpi^r_{0,\beta}-M},\frac{\left|h^{-\alpha}\varpi^r_{j,\alpha}+h^{-\beta}\varpi^r_{j,\beta}-b_j\right|}{h^{-\alpha}\varpi^r_{0,\alpha}+h^{-\beta}\varpi^r_{0,\beta}-M}\right\},$$

Thus $\|e_j\|\le\hat{K}h^r+\sum_{k=0}^{j-1}\rho_k\|e_{j-k}\|$. Using Lemma 6.2, we can prove that $e_j=O(h^r)$. □

7. Numerical Illustrations

In this section, four examples are considered ([7, 10, 17, 32, 33]) to illustrate the effectiveness of the approaches using MATLAB software.

Example 7.1. Consider the following two-term FSE:

$$\begin{cases}P^{(\alpha)}(t)+P^{(\beta)}(t)=A(t)P(t)+P(t)C(t)+Q(t), & \alpha,\beta\in]0,1],\\P(0)=P_0,\end{cases}\tag{7.1}$$

with

$$A(t)=\begin{pmatrix}0 & -t\\ 1 & t\end{pmatrix},\quad C(t)=\begin{pmatrix}0 & 0\\ t & 1\end{pmatrix},\quad P_0=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},$$

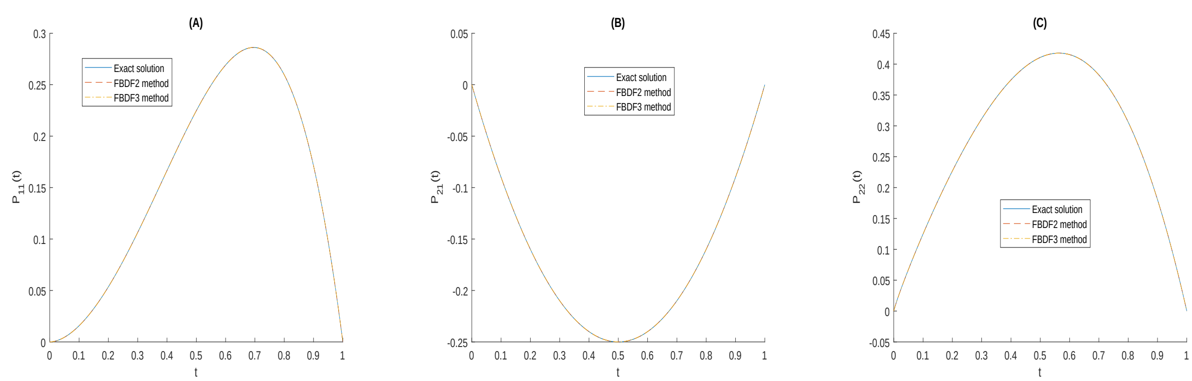
and

$$\begin{cases} Q_{11}(t)=\dfrac{\Gamma(6\beta+1)}{\Gamma(6\beta+1-\alpha)}t^{6\beta-\alpha}-\dfrac{\Gamma(5)}{\Gamma(5-\alpha)}t^{4-\alpha}+\dfrac{\Gamma(6\beta+1)}{\Gamma(5\beta+1)}t^{5\beta}\\ \quad -\dfrac{\Gamma(5)}{\Gamma(5-\beta)}t^{4-\beta}-t^3+t^2,\\ Q_{12}(t)=\dfrac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha}-\dfrac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}+\dfrac{\Gamma(3)}{\Gamma(3-\beta)}t^{2-\beta}\\ \quad -\dfrac{\Gamma(2)}{\Gamma(2-\beta)}t^{1-\beta}+t^{3\beta+1}-t^4-t^2+t,\\ Q_{21}(t)=-t^{6\beta}+2t^4+t^{3\beta+1},\\ Q_{22}(t)=\dfrac{\Gamma(3\beta+1)}{\Gamma(3\beta+1-\alpha)}t^{3\beta-\alpha}-\dfrac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha}+\dfrac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)}t^{2\beta}\\ \quad -\dfrac{\Gamma(4)}{\Gamma(4-\beta)}t^{3-\beta},-t^{3\beta+1}-t^{3\beta}+t^4+t^3-t^2+t. \end{cases}$$

The exact solution of Eq. (7.1) is

$$P(t)=\begin{pmatrix} t^{6\beta}-t^4 & t^2-t\\ 0 & t^{3\beta}-t^3\end{pmatrix}.$$

In Figure 1, the solutions $P_{11}(t)$, $P_{21}(t)$, and $P_{22}(t)$ of FSE (7.1) for $\alpha=0.7$ and $\beta=0.3$ using the FBDF method order 2 (FBDF2) and FBDF method order 3 (FBDF3) for $t\in[0,1]$. In Table 1,



(A) $P_{11}(t)$ (B) $P_{21}(t)$ (C) $P_{22}(t)$

FIGURE 1. The numerical solutions $P_{11}(t)$, $P_{21}(t)$ and $P_{22}(t)$ of FSE (7.1) for $\alpha=0.7,\beta=0.3$ and $h=0.001$.

a comparison of the absolute errors at some selected points with $h=0.001$ are shown.

TABLE 1. Absolute errors at some selected points with $h = 0.001$.

t	FBDF2 method	FBDF3 method
0.2	$9.41381e-05$	$9.35997e-05$
0.4	$8.06323e-05$	$7.99089e-05$
0.6	$7.97658e-05$	$7.87751e-05$
0.8	$8.48741e-05$	$8.35704e-05$
1.0	$9.46905e-05$	$9.30039e-05$

Example 7.2. Consider the two-term FLE:

$$\begin{cases} P^{(\alpha)}(t) + P^{(\beta)}(t) = A(t)P(t) + P(t)A^T(t) + Q(t), \quad \alpha \in]0,1], \beta \in]1,2], \\ P(0) = P^{(1)}(0) = P_0, \end{cases} \tag{7.2}$$

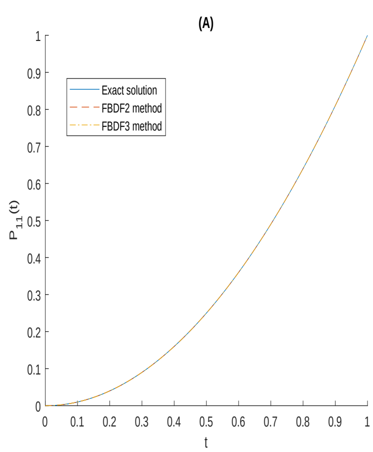
with $A(t) = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$, $P_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and

$$Q(t) = \begin{pmatrix} \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + \frac{2}{\Gamma(3-\beta)} t^{2-\beta} & -t^4 - t^3 \\ -t^3 - t^4 & -\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} - \frac{6}{\Gamma(4-\beta)} t^{3-\beta} \end{pmatrix}.$$

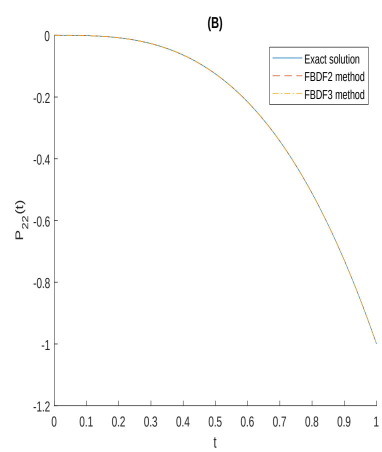
The exact solution of Eq. (7.2) is

$$P(t) = \begin{pmatrix} t^2 & 0 \\ 0 & -t^3 \end{pmatrix}.$$

Figure 2 shows the solutions $P_{11}(t)$ and $P_{22}(t)$ by using methods of FBDF2, FBDF3 and exact solution in functions for $t \in [0,1]$, whenever $\alpha = 0.7$ and $\beta = 1.5$. In Table 2, we show a comparison of the absolute errors at some selected points for $h = 0.001$.



(A) $P_{11}(t)$



(B) $P_{22}(t)$

FIGURE 2. 2D plots of solution $P_{11}(t)$ and $P_{22}(t)$ vs t with $\alpha = 0.7$ and $\beta = 1.5$ and $h = 0.001$.

TABLE 2. Absolute errors at some selected points for $h = 0.001$.

t	FBDF2 method	FBDF3 method
0.1	$3.79945e-06$	$4.70783e-06$
0.2	$5.28709e-06$	$6.12612e-06$
0.3	$6.20277e-06$	$6.97825e-06$
0.4	$6.83201e-06$	$7.54641e-06$
0.5	$7.28863e-06$	$7.94300e-06$
0.6	$7.63194e-06$	$8.22669e-06$
0.8	$8.10886e-06$	$8.58514e-06$
1.0	$8.43033e-06$	$8.79147e-06$

Example 7.3. Consider the following time-fractional telegraph equation:

$$\begin{aligned}\frac{\partial^{\beta} v(x,y,t)}{\partial t^{\beta}} &+ \frac{\partial^{\beta-1} v(x,y,t)}{\partial t^{\beta-1}} + v(x,y,t) \\ &= \Delta v(x,y,t) + \left(\frac{24t^{-\beta}}{\Gamma(5-\beta)} + \frac{24t^{1-\beta}}{\Gamma(6-\beta)}\right)t^4 xy(1-x)(1-y) \\ &\quad + 2x(1-x)t^4 + 2y(1-y)t^4 + xy(1-x)(1-y)t^4, \end{aligned} \tag{7.3}$$

with $\beta \in]1,2]$, the board conditions $v(0,y,t) = v(1,y,t) = v(x,0,t) = v(x,1,t) = 0$, and the initial conditions

$$\frac{\partial v(x,y,0)}{\partial t} = v(x,y,0) = 0,$$

where $(x,y) \in \Omega := [0,1] \times [0,1]$ and $t \in [0,1]$. The exact solution of Eq. (7.3) is $v(x,y,t) = xy(1-x)(1-y)t^4$. Let us discretize as follows: $(x_i, y_j) = (ih_\Omega, jh_\Omega)$ for $1 \leq i,j \leq n$ with $h_\Omega = \frac{1}{n+1}$. Let $P_{ij}(t) = v(x_i, y_j, t)$. Using the central difference discretization defined by:

$$\begin{aligned} P_{ij}^{(\beta)}(t) + P_{ij}^{(\beta-1)}(t) &= \frac{P_{(i-1)j}(t) - 2P_{ij}(t) + P_{(i+1)j}(t)}{h_\Omega^2} \\ &\quad + \frac{P_{i(j-1)}(t) - 2P_{ij}(t) + P_{i(j+1)}(t)}{h_\Omega^2} - P_{ij}(t) + Q_{ij}(t), \end{aligned} \tag{7.4}$$

where

$$\begin{aligned} Q_{ij}(t) &= \left(\frac{24t^{-\beta}}{\Gamma(5-\beta)} + \frac{24t^{1-\beta}}{\Gamma(6-\beta)}\right) t^4 x_i y_j (1-x_i)(1-y_j) \\ &\quad + 2x(1-x_i)t^4 + 2y_j(1-y_j)t^4 + x_i y_j (1-x_i)(1-y_j)t^4, \end{aligned}$$

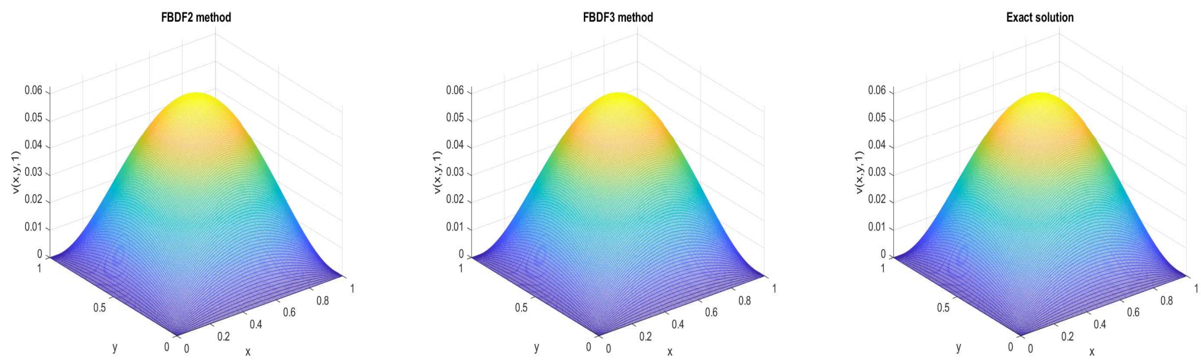
we obtain from system (7.4) that

$$\begin{cases} P^{(\beta)}(t) + P^{(\beta-1)}(t) = AP(t) + P(t)C + Q(t), \quad \beta \in]1,2], \\ P(0) = P^{(1)}(0) = P_0, \end{cases}$$

with

$$A = \text{tridiag}\left(\frac{1}{h_\Omega^2}, \frac{-2}{h_\Omega^2}, \frac{1}{h_\Omega^2}\right) \in \mathbb{R}^{n\times n}, \qquad C = \text{tridiag}\left(\frac{1}{h_\Omega^2}, \frac{-2}{h_\Omega^2} - 1, \frac{1}{h_\Omega^2}\right) \in \mathbb{R}^{n\times n},$$

and $P_0 = 0$. In Figure 3, the solutions v for Equation (7.3) using the FBDF2 method, FBDF3 method and exact solution at $t = 1$ and in Figure 4, the solution $P_{11}(t)$ with $\beta = 1.7, h = 0.01$ and $n = 200$, we can observe that the solutions are identical. We compare the results obtained



(A) FBDF2 method (B) FBDF3 method (C) Exact solution

FIGURE 3. 3D plots of numerical and exact solutions at $t = 1$ with $\beta = 1.7$, $h = 0.01$ and $n = 200$.

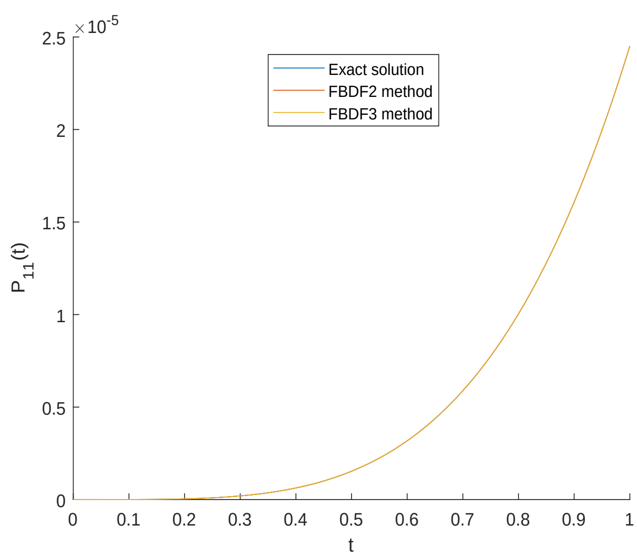


FIGURE 4. 2D plot of numerical $P_{11}(t)$ and exact solutions for $\beta = 1.7, h = 0.01$ and $n = 200$.

from the FBDF2 method and FBDF3 method. In Table 3, we give the obtained runtimes in seconds and the absolute error norms (Abs.error) at $t = 1$, which is the difference between the exact solution and the approximated solution.

TABLE 3. The absolute error norms at $t=1$ and runtimes in seconds with $h=0.01$ and $\beta=1.7$.

	FBDF2 method		FBDF3 method	
n	CPU time	Abs.error	CPU time	Abs.error
200	2.376019	$6.1096e-04$	1.736881	$1.3292e-06$
300	6.442860	$9.1491e-04$	4.575444	$1.9905e-06$
400	13.370125	0.0012	12.358612	$2.6519e-06$

Example 7.4. Consider the following time-fractional telegraph equation [17]:

$$\begin{aligned}\frac{\partial^{\beta}v(x,y,t)}{\partial t^{\beta}}+\frac{\partial^{\beta-1}v(x,y,t)}{\partial t^{\beta-1}} &= \Delta v(x,y,t)+(\frac{24t^{-\beta}}{\Gamma(5-\beta)}\\ &+\frac{24t^{1-\beta}}{\Gamma(6-\beta)})t^4xy(1-x)(1-y)+2x(1-x)t^4+2y(1-y)t^4,\end{aligned} \tag{7.5}$$

where $(x,y)\in[0,1]\times[0,1]$, $t\in[0,1]$, $\beta\in]1,2]$, the board conditions

$$v(0,y,t)=v(1,y,t)=v(x,0,t)=v(x,1,t)=0,$$

and the initial conditions

$$\frac{\partial v(x,y,0)}{\partial t}=v(x,y,0)=0.$$

The exact solution of Eq. (7.5) is $v(x,y,t)=xy(1-x)(1-y)t^4$. Let us discretize as follows: $(x_i,y_j)=(ih_x,jh_y)$ for $1\le i,j\le n$ with $h_x=h_y=\frac{1}{n+1}$. Let $P_{ij}(t)=v(x_i,y_j,t)$. Using the central difference discretization defined by

$$\begin{aligned}P_{ij}^{(\beta)}(t)+P_{ij}^{(\beta-1)}(t) &= \frac{P_{(i-1)j}(t)-2P_{ij}(t)+P_{(i+1)j}(t)}{h_x^2}\\ &+\frac{P_{i(j-1)}(t)-2P_{ij}(t)+P_{i(j+1)}(t)}{h_x^2}+Q_{ij}(t),\end{aligned} \tag{7.6}$$

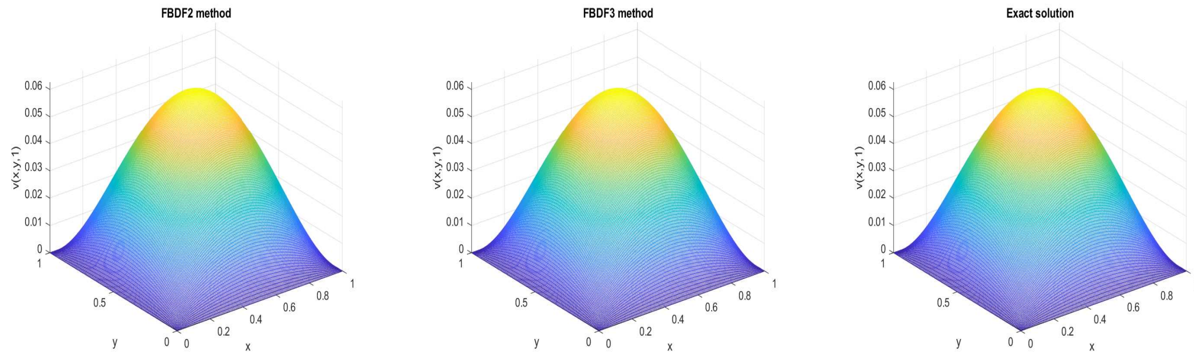
with

$$\begin{aligned}Q_{ij}(t) &= (\frac{24t^{-\beta}}{\Gamma(5-\beta)}+\frac{24t^{1-\beta}}{\Gamma(6-\beta)})t^4x_iy_j(1-x_i)(1-y_j)\\ &+2x(1-x_i)t^4+2y_j(1-y_j)t^4+x_iy_j(1-x_i)(1-y_j)t^4,\end{aligned}$$

we obtain from the system (7.6) that

$$\begin{cases} P^{(\beta)}(t)+P^{(\beta-1)}(t)=AP(t)+P(t)A+Q(t), & \beta\in]1,2],\\ P(0)=P^{(1)}(0)=P_0, \end{cases}$$

where $A=\frac{1}{h_x^2}\text{tridiag}(1,-2,1)\in\mathbb{R}^{n\times n}$ and $P_0=0$. In Figure 5, the solutions $v(x,y,t)$ for Eq. (7.5) using new approaches and exact solution at $t=1$ and in Figure 6, the solution $P_{11}(t)$ with $\beta=1.7$, $h=0.01$ and $n=200$, we can observe that the solutions are identical. We



(A) FBDF2 method (B) FBDF3 method (C) Exact solution

FIGURE 5. 2D plots of numerical and exact solutions at $t = 1$ with $\beta = 1.7$, $h = 0.01$ and $n = 200$.

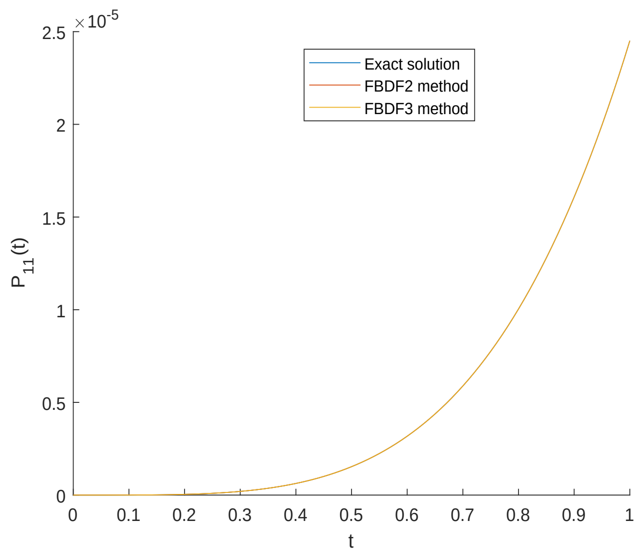


FIGURE 6. 2D plots of numerical and exact solution of $P_{11}(t)$ with $\beta = 1.7$, $h = 0.01$ and $n = 200$.

compare the results obtained from the FBDF2 method and FBDF3 method. In Table 4, we give the obtained runtimes in seconds and the absolute error norms (Abs.error) at $t = 1$, which is the difference between the exact solution and the approximated solution. If the matrices A, C, and Q are constants and large-scale, then Krylov subspaces methods are more efficient and fast; see, e.g., [21, 25, 28, 29, 30].

CONCLUSION

In this paper, we presented the FBDF method of order 2 and 3 for solving the two-term FSEs in the Caputo derivative sense of fractional order. We introduced some theoretical results about convergence, and absolute error norms. We demonstrated the efficiency and accuracy of the proposed method by applying it to four typical examples. It is found that the approximate

TABLE 4. Numerical results of absolute error norms at $t = 1$ and runtimes in seconds for $h = 0.01$.

	FBDF2 method		FBDF3 method	
n	CPU time	Abs.error	CPU time	Abs.error
200	2.350459	$6.4614e-04$	1.787854	$1.2440e-06$
300	7.439960	$9.6760e-04$	6.478860	$1.8629e-06$
400	12.736567	0.0013	9.232403	$2.4819e-06$

solutions produced by our methods are in complete agreement with the corresponding exact solutions. Moreover, these approaches are applicable to other types of two-term fractional differential matrix equations.

Acknowledgments
The author expresses his deep thanks to the anonymous referees for their encouraging and constructive comments, which improved this paper.

REFERENCES

[1] H. Abou-Kandil, G. Freiling, V. Ionescu, G. Jank, Matrix Riccati Equations in Control and Systems Theory, Springer, Birkhäuser Basel, 2012.

[2] W.M. Ahmad, R. El-Khazali, Fractional-order dynamical models of love, Chaos, Solitons & Fractals 33 (2007), 1367–1375.

[3] M.S. Asl, M. Javidi, An improved PC scheme for nonlinear fractional differential equations: Error and stability analysis, J. Comput. Appl. Math. 324 (2017), 101–117.

[4] P. Benner, H. Mena, BDF methods for large-scale differential Riccati equations, Procceeding of Mathematical Theory of Network and Systems 10 (2004), 1–12.

[5] S. Etemad, S. Rezapour, M.E. Samei, On a fractional caputo-hadamard inclusion problem with sum boundary value conditions by using approximate endpoint property, Math. Meth. Appl. Sci. 43 (2017), 9719–9734.

[6] A.D. Freed, K. Diethelm, Fractional calculus in biomechanics: a 3d viscoelastic model using regularized fractional derivative kernels with application to the human calcaneal fat pad, Biomechanics and Modeling in Mechanobiology 5 (2006), 203–215.

[7] M.S. Hashemi, D. Baleanu, Numerical approximation of higher-order time-fractional telegraph equation by using a combination of a geometric approach and method of line, J. Comput. Phys. 316 (2016), 10–20.

[8] A.S. Hendy, J.E. Macías-Díaz, A.J. Serna-Reyes, On the solution of hyperbolic two-dimensional fractional systems via discrete variational schemes of high order of accuracy, J. Comput. Appl. Math. 354 (2019), 612–622.

[9] M.S. Heris, M. Javidi, On fractional backward differential formulas methods for fractional differential equations with delay, Int. J. Appl. Comput. Math. 72 (2018), 4.

[10] M.H. Heydari, M.R. Hooshmandasl, F. Mohammadi, Two-dimensional Legendre wavelets for solving time-fractional telegraph equation, Adv. Appl. Math. Mech. 6 (2014), 247–260.

[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier B.V., Amsterdam, 2006.

[12] M. Kumar, V. Daftardar-Gejji, A new family of predictor-corrector methods for solving fractional differential equations, Appl. Math. Comput. 363 (2019), 124633.

[13] R. Lin, F. Liu, Fractional high order methods for the nonlinear fractional ordinary differential equation, Nonlinear Anal. 66 (2007), 856–869.

[14] C. Lubich, Discretized fractional calculus, SIAM J. Math. Anal. 17 (1986), 704–719.

[15] E.N. Mahmudov, S.S. Yusubov, Nonlocal boundary value problems for hyperbolic equations with a Caputo fractional derivative, J. Comput. Appl. Math. 398 (2021), 113709.

[16] M.M. Meerschaert, E. Scalas, Coupled continuous time random walks in finance, Physica A: Statistical Mechanics and its Applications 370 (2006), 114–118.

[17] O. Nikan, Z. Avazzadeh, J.T. Machado, Numerical approximation of the nonlinear time-fractional telegraph equation arising in neutron transport, Commun. Nonlinear Sci. Numer. Simul. 99 (2021), 105755.

[18] K. Oldham, J. Spanier, The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order, Elsevier, Canada 1974.

[19] I. Podlubny, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Fractional Differential Equations: An Introduction to Fractional Derivatives, 340. Academic Press, New York, 1998.

[20] J. Sabatier, M. Moze, C. Farges, Lmi stability conditions for fractional order systems, Comput. Math. Appl. 59 (2010), 1594–1609.

[21] E.M. Sadek, A.H. Bentbib, L. Sadek, H. Talibi Alaoui, Global extended Krylov subspace methods for large-scale differential Sylvester matrix equations, J. Appl. Math. Comput. 62 (2020), 157–177.

[22] L. Sadek, Controllability and observability for fractal linear dynamical systems, J. Vibration Contr. 29 (2022). DOI: 10.1177/10775463221123354.

[23] L. Sadek, Fractional BDF methods for solving fractional differential matrix equations, Int. J. Appl. Comput. Math. 8 (2022), 238.

[24] L. Sadek, B. Abouzaid, E.M. Sadek, H.T. Alaoui, Controllability, observability and fractional linear-quadratic problem for fractional linear systems with conformable fractional derivatives and some applications, Int. J. Dyn. Contr. 11 (2023), 214–228.

[25] L. Sadek, H.T. Alaoui, Application of MGA and EGA algorithms on large-scale linear systems of ordinary differential equations, J. Comput. Sci. 62 (2022), 101719.

[26] L. Sadek, A.S. Bataineh, H. Talibi Alaoui, I. Hashim, The novel mittag-leffler–galerkin method: Application to a riccati differential equation of fractional order, Fractal and Fractional 7 (2023), 302.

[27] L. Sadek, O. Sadek, H.T. Alaoui, M.S. Abdo, K. Shah, T. Abdeljawad, Fractional order modeling of predicting COVID-19 with isolation and vaccination strategies in Morocc, Computer Modeling in Engineering and Sciences 136 (2023), 1931–1950.

[28] L. Sadek, H. Talibi Alaoui, The extended block arnoldi method for solving generalized differential Sylvester equations, J. Math. Modeling 8 (2020), 189–206.

[29] L. Sadek, H. Talibi Alaoui, The extended nonsymmetric block lanczos methods for solving large-scale differential lyapunov equations, Math. Modeling Comput. 8 (2021), 526–536.

[30] L. Sadek, H. Talibi Alaoui, Numerical methods for solving large-scale systems of differential equations, Ricerche di Matematica 72 (2023), 785–802.

[31] M.E. Samei, M.M. Matar, S. Etemad, S. Rezapour, On the generalized fractional snap boundary problems via g-Caputo operators: Existence and stability analysis, Adv. Diff. Equ. 2021 (2021), 498.

[32] L. Wei, L. Liu, H. Sun, Numerical methods for solving the time-fractional telegraph equation, Taiwanese J. Math. 22 (2018), 1509–1528.

[33] Y.H. Youssri, W.M. Abd-Elhameed, Numerical spectral Legendre-Galerkin algorithm for solving time fractional telegraph equation, Romanian J. Phys. 63 (2018), 107.

[34] J. Zhao, T. Long, Y. Xu, High order fractional backward differentiation formulae, Appl. Anal. 96 (2017), 1669–1680.