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GENERAL DECAY STABILITY OF NONLINEAR DELAYED HYBRID STOCHASTIC SYSTEM WITH SWITCHED NOISES

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Abstract. Hybrid stochastic differential equations (HSDEs) have wide range of real-world applications. In this paper, we study a new kind of nonlinear delayed neutral stochastic differential equations with Markovian switched noises (NHSDE-MSN). Under the assumption that this type of equations satisfies the locally Lipschitz condition and general condition of monotonicity, the existence and uniqueness of global solutions are established. Then, based on the Lyapunov function, M-matrix theory, stochastic analysis techniques, and Barbalat lemma, taking into account the delay as a bounded function, different decay stabilities of solutions are investigated. Finally, a numerical example is given to illustrate the utility of the main results.

Keywords. Decay stability; Markovian switching; Neutral-type systems; Random delays; Switched noises.
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1. INTRODUCTION

Stochastic differential equations have attracted the attention of numerous researchers from various fields due to their usefulness in modeling phenomena in the real world. Many important subclasses of stochastic differential equations have been extensively studied, especially hybrid stochastic differential equations (HSDEs), which are useful in modeling systems from various areas. Moreover, practical experiences have shown that even the subsystems of a hybrid system are stable, and the stability of the switching system is not always guaranteed [1].

In several fields, such as industrial automation, distributed networks, chemical, control, population dynamics, finance, and thermodynamics, almost all the models existing in reality are subject to randomness; see [2, 3, 4]. Researchers are interested in the study of these types of systems with random commutations; see [5, 6, 7, 8]. In dynamical models, delay effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [9, 10, 11, 12]. In [13], Zouine, Bouzahir, and Imzegouan studied a nonlinear stochastic differential equation embodying Markovian jumps

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and Lévy noise, and it did not satisfy linear growth condition. Based on the Lyapunov functional, they set some sufficient criteria to guarantee that stability depends on the factor of delay, as asymptotic and H_∞-stability in L^p. In [14], Xie and Zhang investigated nonlinear delayed and Markovian switched stochastic differential equations when the delay's derivative is strictly less than 1. The authors cited some criteria for moment exponential stability and asymptotic boundness. Recently, Zhand and Chen in [15] investigated the p^{th}-moment and the almost sure stability characterized by a general decay of solution, when the terms of equations are under locally Lipschitz condition and the monotonicity condition, and when the time-varying delay remains a bounded measurable function; see, e.g., [16, 17].

In some areas such as market fluctuations, the importance of Lévy noise could be on its use in modelling the risk management and the option pricing purposes. Therefore, employing Lévy noise is quite different than Gaussian noise [18]. Moreover, Lévy noise is suitable for the model that counts the number of incoming phone calls in a period [19]. Over past few decades, qualitative and quantitative properties of stochastic equations with Lévy noise were extensively studied. In [20], for instance, Wan, Hu, and Chen investigated neutral delayed stochastic differential equations with Lévy noise. More precisely, they established the existence and uniqueness of solutions. They also showed the exponential and almost surely asymptotic stabilities based on the works in [21, 22]. On the other hand, various new results were obtained on the qualitative behaviors of solutions of certain integro-differential equations in [24, 25, 26, 27, 28, 29, 30] recently. Note that random switching, Gaussian noise, time-delays, and Lévy noise are characters that disturb the equilibrium point stability of a dynamical system. So, we can say that a system affected by all these parasites can be more representative of a real system. In this work, we try to adapt the studied model to the phenomena which may not be affected by any noises, but sometimes they can be affected by Gaussian noises like wind, strong heat.... And suddenly, they can also be affected by Levy noises like lightning. And other times, they can be affected by both types of noise. For this reason, we created a random regulator that adapts the model to the type of phenomena that we have discussed. Stimulated by the above discussion, this work focuses on the study of a nonlinear delayed neutral hybrid system taking into account that the model is in noise-free conditions, but it is suddenly influenced some time by a Gaussian noise, some time by Lévy noise, and sometimes by both, in which, we consider the random regulator subjected to a homogeneous Markov process allowing the system to vary randomly, which is never treated in this way. The main points in this work are listed as follows:

- Existence and uniqueness of global solutions to the system is obtained.
- Based on Lyapunov's method, Barbalat lemma, and some stochastic calculation techniques, decay stability of moment and almost surely decay stability of nonlinear delayed NHSDE-MSN are shown.
- The almost surely decay stability and in p^{th} moment are established by using M-matrix theory.

The rest of this paper is structured as follows. Backgrounds, description of the model, and preliminaries which are used in the main results are introduced in Section 2. The main results are introduced in Section 3 and are divided into three subsections. Subsection 3.1 contains the proof of existence and uniqueness of solutions. Subsection 3.2 contains the proof of general decay stability. Subsection 3.3 contains decay stability by M-matrix theory. Finally, Section 4 presents a numerical example illustrating our theoretical results.

2. NOTATION AND PRELIMINARIES

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space with norm $\|\cdot\|$. Thus, for any vector $x=(x_1,x_2,...,x_n)^T$ in $\mathbb{R}^n$, we present $\|x\|=\sqrt{\sum_{i=1}^n |x_i|^2}$. The complete probability space with filtration $\{\mathscr{F}_t\}_{t\geq 0}$ which satisfies the usual conditions is $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$. $\mathbb{E}$ denotes the expectation operator and if $S\subset\Omega$, and its indicator function is denoted by 1_A. $C([-\imath,0],\mathbb{R}^n)\,(\imath>0)$ represents the family of all continuous $\mathbb{R}^n$-valued functions ϕ on $[-\imath,0]$ equipped with norm $\|\phi\|=\sup_{-\imath\leq\flat\leq 0}|\phi(\flat)|$, and, for $a,b\in\mathbb{R}$, $a\vee b=\max\{a,b\}$.

The $\mathscr{F}_t$-adapted $m-$dimensional Brownain motion on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ is denoted by $w(t)$. Let $N(t,y)=\tilde{N}(dt,dy)+\pi(dy)dt$ be a $\mathscr{F}_t$-adapted Poisson process on $[0,+\infty)\times\mathbb{R}^n$ with characteristic measure $\pi(dy)$ satisfies $\pi(Y)<\infty$, where $Y\subset\mathbb{R}^n$, and $\tilde{N}(t,y)$ is the compensator measure satisfies.

Let $\{,\rho(t),\wp(t),t\geq 0\}$ be two Markov chains on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$, with values in $S_2=\{1,2,...,N\}$ and $S_1=\{1,2,3,4\}$, respectively. Their generators are $\Gamma=(\gamma_{ij})_{N\times N}$ and $\Phi=(\varphi_{\ell\varsigma})_{4\times 4}$ given as follows

$$\mathbb{P}\{\rho(t+\Delta)=j|\rho(t)=i\}=\begin{cases}\gamma_{ij}\Delta+o(\Delta), & \text{if } i\neq j,\\ 1+\gamma_{ii}\Delta+o(\Delta), & \text{if } i=j,\end{cases}$$

$$\mathbb{P}\{\wp(t+\Delta)=\varsigma|\wp(t)=\ell\}=\begin{cases}\varphi_{\ell\varsigma}\Delta+o(\Delta), & \text{if } \ell\neq\varsigma,\\ 1+\varphi_{\ell\ell}\Delta+o(\Delta), & \text{if } \ell=\varsigma,\end{cases}$$

where $\gamma_{ij}\geq 0$ is the transition rate from i to j for any $i\neq j$, and $\gamma_{ii}=-\sum_{j\neq i}\gamma_{ij}$. $\varphi_{\ell\varsigma}$ has the same meaning and $\Delta>0$. We further assume in this paper that $\rho(t),\wp(t),w(t)$, and $N(t,y)$ are independent. Let $\hbar_2(t)=\delta_1(t)\delta_2(t)$ and $\hbar_3(t)=\delta_1(t)\delta_3(t)$, with

$$\begin{cases}\delta_1(t)= & \dfrac{(4-\wp(t))^{\wp(t)-1}}{(\wp(t)-1)^{3-\wp(t)}+(3-\wp(t))^{\wp(t)-1}},\\[2ex] \delta_2(t)= & (\wp(t)-1)^{3-\wp(t)},\\[2ex] \delta_3(t)= & (2-\wp(t))^2.\end{cases}$$

Now, we consider the following n-dimensional nonlinear delayed hybrid stochastic system with switched noises on $t\geq 0$

$$\begin{aligned} d\big[u(t)-\mathscr{D}\big(u(t-\sigma_{t,\rho(t)}),\rho(t)\big)\big]=&f\big(t,u(t),u(t-\sigma_{t,\rho(t)}),\rho(t)\big)dt\\ &+\hbar_2(t)g\big(t,u(t),u(t-\sigma_{t,\rho(t)}),\rho(t)\big)dw(t)\\ &+\hbar_3(t)\int_Y h\big(t,u(t),u(t-\sigma_{t,\rho(t)}),y,\rho(t)\big)N(dt,dy), \quad (2.1)\end{aligned}$$

where $u(t)=\big(u_1(t),u_2(t),...,u_n(t)\big)\in\mathbb{R}^n$ is the state vector with $\{u(\flat):-\imath\leq\flat\leq 0\}=\xi\in C([-\imath,0],\mathbb{R}^n)$, and $\wp(0)=\wp_0\in S_1$ and $\rho(0)=\rho_0\in S_2$ are the initial value. The time delay $\sigma:\mathbb{R}^+\longrightarrow[0,\imath]$ is a bounded function, the functions: $\mathscr{D}:\mathbb{R}^n\times S_2\to\mathbb{R}^n$ is the neutral item, $f:\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\times S_2\to\mathbb{R}^n$ is the drift term vector, $g:\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\times S_2\to\mathbb{R}^{n\times m}$ is the diffusion term matrix, and $h:\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\times Y\times S_2\to\mathbb{R}^n$ is the jump term vector. They are all assumed Borel measurable functions. An obvious and simple checking gives us the following

different cases:
If $\wp(t) = 1$, then, $\hbar_2(t) = 0$ and $\hbar_3(t) = 1$,
if $\wp(t) = 2$, then, $\hbar_2(t) = 1$ and $\hbar_3(t) = 0$,
if $\wp(t) = 3$, then, $\hbar_2(t) = 1$ and $\hbar_3(t) = 1$,
and if $\wp(t) = 4$, then, $\hbar_2(t) = 0$ and $\hbar_3(t) = 0$.

Remark 2.1. When process $\wp(t)$ randomly takes its value from S_1, the system also changes its nature into four different types of systems.

It is known that the local Lipschitz condition and the linear growth condition are essential to the classical conditions for the existence and uniqueness of a global solution (see [20]). In this article, the local Lipschitz condition is kept, while the linear growth condition is dropped since we consider the highly nonlinear equation (2.1) generally unsatisfying of the linear growth condition. This paper, therefore, starts from these conditions as an assumption in use.

Assumption 2.1. Let $u_1, u_2, v_1, v_2 \in \mathbb{R}^n$ with $|u_1| \vee |u_2| \vee |v_1| \vee |v_2| \leq b$. For any $b \in \mathbb{N}$, $t \in \mathbb{R}^+$, $i \in S_2$, there exists a positive constant κ_b such that

$$\begin{aligned}\big|f(t,u_1,v_1,i) - f(t,u_2,v_2,i)\big|^2 \vee \big|g(t,u_1,v_1,i) - g(t,u_2,v_2,i)\big|^2 \\ \vee \int_{\mathbb{R}} \big|h(t,u_1,v_1,y,i) - h(t,u_2,v_2,y,i)\big|^2 \pi(dy) \\ \leq \kappa_b\big(\big|u_1 - u_2\big|^2 + \big|v_1 - v_2\big|^2\big).\end{aligned}$$

Assumption 2.1 only guarantees the fact that equation (2.1) has a unique maximal solution; the latter explodes to infinity at a given finite time. To avoid such an explosion, we have to impose one more condition with Lyapunov functions. For this purpose, we need to define some additional notations.

Let $C^{1,2}\big(\mathbb{R}^+ \times \mathbb{R}^n \times S_1 \times S_2, \mathbb{R}^+\big)$ denote the family of all nonnegative functions $\mathscr{V}(t,u,\ell,i)$ defined on $\mathbb{R}^+ \times \mathbb{R}^n \times S_1 \times S_2$, which are continuously once differentiable in t and twice differentiable in u. To simplify, let us denote $\tilde{u}(t) = u(t) - \mathscr{D}\big(u(t - \sigma_{t,\rho(t)}), \rho(t)\big)$. Hence, for each $\mathscr{V} \in C^{1,2}\big(\mathbb{R}^+ \times \mathbb{R}^n \times S_1 \times S_2, \mathbb{R}^+\big)$, we define an operator $\mathscr{L}\mathscr{V} : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times S_1 \times S_2 \to \mathbb{R}$ by

$$\begin{aligned}\mathscr{L}\mathscr{V}(t,u,v,\ell,i) &= \mathscr{V}_t(t,\tilde{u},\ell,i) + \mathscr{V}_u(t,\tilde{u},\ell,i)f(t,u,v,i) \\ &+ \frac{1}{2}\big(\hbar_2(t)\big)^2 trace\big(g^T(t,u,v,i)\mathscr{V}_{uu}(t,\tilde{u},\ell,i)g(t,u,v,i)\big) \\ &+ \int_{\mathbb{R}} \Big[\mathscr{V}\big(t,\tilde{u} + \hbar_3(t)h(t,u,v,y,i),\ell,i\big) - \mathscr{V}(t,\tilde{u},\ell,i)\Big]\pi(dy) \\ &+ \sum_{\varsigma=1}^{4} \varphi_{\ell\varsigma}\mathscr{V}(t,\tilde{u},\varsigma,i) + \sum_{j=1}^{N} \gamma_{ij}\mathscr{V}(t,\tilde{u},\ell,j),\end{aligned}$$

where $\mathscr{V}_t(t,u,i) = \frac{\partial \mathscr{V}(t,u,i)}{\partial t}$, $\mathscr{V}_{uu}(t,u,i) = \Big(\frac{\partial^2 \mathscr{V}(t,u,i)}{\partial u_k \partial u_l}\Big)_{n\times n}$, and

$$\mathscr{V}_u(t,u,i) = \Big(\frac{\partial \mathscr{V}(t,u,i)}{\partial u_1}, \frac{\partial \mathscr{V}(t,u,i)}{\partial u_2}, \ldots, \frac{\partial \mathscr{V}(t,u,i)}{\partial u_n}\Big).$$

Now, we present the following definitions.

Definition 2.1. A function $\Upsilon : \mathbb{R} \to (0,\infty)$ is said to be of $\Upsilon-$type if it satisfies the following three conditions:

(i) Υ is a continuous and nondecreasing function in $\mathbb{R}$ and it is differentiable in $\mathbb{R}^+$;
(ii) $\Upsilon(0) = 1, \Upsilon(\infty) = \infty$ and $\frac{\Upsilon'(t)}{\Upsilon(t)}$ is nonincreasing in $\mathbb{R}^+$;
(iii) $\Upsilon(t) \leq \Upsilon(s)\Upsilon(t-s)$, for any $s,t \geq 0$.

Definition 2.2. Let $\Upsilon \in C(\mathbb{R}^+,\mathbb{R}^+)$ be a $\Upsilon-$type function. Then, the system with initial data ξ is said to be stable in p^{th} $(p \geq 2)$ moment with decay $\Upsilon(t)$ of order $\tilde{\mu} > 0$ if $\limsup\limits_{t\to\infty} \frac{\log \mathbb{E}|u(t)|^p}{\log \Upsilon(t)} \leq -\tilde{\mu}$. Moreover, the system with initial data ξ is said to be stable with decay $\Upsilon(t)$ of order $\frac{\tilde{\mu}}{p} > 0$ almost surely or only a.s. if, there is no ambiguity, $\limsup\limits_{t\to\infty} \frac{\log|u(t)|}{\log \Upsilon(t)} \leq -\frac{\tilde{\mu}}{p}$ a.s.

Remark 2.2. When $\Upsilon(t)$ is equal to e^t or $1+t$, we are in front of the exponential and the usual polynomial stability, respectively. Therefore, we have free choice for Υ-type functions, which gives the generality of our results.

Now we are able to state the following assumptions.

Assumption 2.2. There exists a real number $k \in (0,1)$ such that $\left|\mathscr{D}(u,i) - \mathscr{D}(v,i)\right| \leq k|u-v|$, for $(u,v,i) \in \mathbb{R}^n \times \mathbb{R}^n \times S_2$.

Assumption 2.3. There exist a pair of functions $\mathscr{V} \in C^{1,2}\big(\mathbb{R}^+ \times \mathbb{R}^n \times S_1 \times S_2, \mathbb{R}^+\big)$, $\mathscr{K} \in C\big(\mathbb{R}^n,\mathbb{R}^+\big)$ and positive numbers a_1, a_2, c_3, c_4 and $k_1 > 1, k_2 > 1$ such that

$$\mathscr{K}(\tilde{u}) \leq k_1\big(\mathscr{K}(u) + k\mathscr{K}(v)\big), \qquad \mathscr{K}(u) \leq k\mathscr{K}(v) + k_2\mathscr{K}(\tilde{u}), \tag{2.2}$$

$$a_1\mathscr{K}(u) \leq \mathscr{V}(t,u,\ell,i) \leq a_2\mathscr{K}(u), \tag{2.3}$$

$$\lim_{|u|\to\infty} \mathscr{K}(u) = +\infty, \tag{2.4}$$

and

$$\mathscr{L}\mathscr{V}(t,u,v,\ell,i) \leq \big(-c_3\mathscr{K}(u) + c_4\mathscr{K}(v)\big)\frac{\Upsilon'(t)}{\Upsilon(t)} \tag{2.5}$$

for $(t,u,v,\ell,i) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times S_1 \times S_2$.

We also need the following lemma.

Lemma 2.1. [20] *If g is nonnegative, uniformly continuous, and Lebesgue integrable function on $[0,\infty)$, then $\lim\limits_{t\to\infty} g(t) = 0$.*

3. Main Results

Under the assumptions in Section 2, we now introduce the following lemma.

Lemma 3.1. *Let Assumptions 2.1, 2.2 and 2.3 be satisfied. If $\lambda := \frac{c_3}{k_1 a_2} > 1$ and*

$$c_4 < \frac{a_1}{kk_2}\Upsilon^{-(\lambda+1)}(\imath) - kk_1c_3 \tag{3.1}$$

hold, then

$$\hbar(\varepsilon) := \int_0^{+\infty} \Upsilon^{\varepsilon}(t) \sup_{-\imath\leq\flat\leq 0} \mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\Upsilon'(t)dt < \infty, \tag{3.2}$$

where $\varepsilon \in (0, \varepsilon_0)$ *with* $\varepsilon_0 \in (0, \lambda - 1)$ *being the unique solution of the algebraic equation*

$$k\Upsilon^{\varepsilon+1}(\iota) + \frac{m_2}{\lambda - \varepsilon - 1}\Upsilon^{\lambda}(\iota) - 1 = 0, \tag{3.3}$$

where $m_2 := \frac{k_2(kk_1c_3+c_4)}{a_1}$.

Proof. Define the function $F(\varepsilon) := k\Upsilon^{\varepsilon+1}(\iota) + \frac{m_2}{\lambda-\varepsilon-1}\Upsilon^{\lambda}(\iota) - 1$. It can seen from (3.1) that $F(0) < 0$ and $F(\lambda - 1) > 0$. In addition, $F(\varepsilon)$ is a strictly increasing function on $(0, \lambda - 1)$. Thus there exists a real $\varepsilon_0 \in (0, \lambda - 1)$ with $F(\varepsilon_0) = 0$. It follows that $\mu(\varepsilon) := k\Upsilon^{\varepsilon+1}(\iota) + \frac{m_2}{\lambda-\varepsilon-1}\Upsilon^{\lambda}(\iota) \in (0, 1)$, where $\varepsilon \in (0, \varepsilon_0)$. By using Assumption 2.3, the generalised Itô formula allows us to write for $t \geq 0$

$$\begin{aligned}
&\Upsilon^{\lambda}(t)\mathbb{E}\mathscr{V}(t, \tilde{u}(t), \wp(t), \rho(t)) \\
&= \mathbb{E}\mathscr{V}(0, \tilde{u}(0), \wp_0, \rho_0) + \mathbb{E}\int_0^t \lambda\Upsilon^{\lambda}(s)\frac{\Upsilon'(s)}{\Upsilon(s)}\mathscr{V}\big(s, \tilde{u}(t), \wp(s), \rho(t)\big)ds \\
&\quad + \mathbb{E}\int_0^t \Upsilon^{\lambda}(s)\mathscr{L}\mathscr{V}\big(s, u(s), u(s - \sigma_{s,\rho(s)}), \wp(s), \rho(t)\big)ds \\
&\leq a_2\mathbb{E}\mathscr{K}(\tilde{u}(0)) + \frac{c_3}{k_1}\mathbb{E}\int_0^t \Upsilon^{\lambda}(s)\mathscr{K}\big(\tilde{u}(s)\big)\frac{\Upsilon'(s)}{\Upsilon(s)}ds \\
&\quad - c_3\mathbb{E}\int_0^t \Upsilon^{\lambda}(s)\mathscr{K}\big(u(s)\big)\frac{\Upsilon'(s)}{\Upsilon(s)}ds + c_4\mathbb{E}\int_0^t \Upsilon^{\lambda}(s)\mathscr{K}\big(u(s - \sigma_{s,\rho(s)})\big)\frac{\Upsilon'(s)}{\Upsilon(s)}ds.
\end{aligned} \tag{3.4}$$

Recall that the delay $\sigma : \mathbb{R}^+ \to [0, \iota]$ is a bonded noncontinuous function. It follows from (3.4) and Assumption 2.3 that

$$\Upsilon^{\lambda}(t)\mathbb{E}\mathscr{K}\big(\tilde{u}(t)\big) \leq \frac{a_2}{a_1}\mathbb{E}\mathscr{K}(\tilde{u}(0)) + \frac{(kk_1c_3 + c_4)}{a_1}\mathbb{E}\int_0^t \Upsilon^{\lambda}(s)\sup_{-\iota \leq r \leq 0}\mathscr{K}\big(u(s + r)\big)\frac{\Upsilon'(s)}{\Upsilon(s)}ds.$$

In view of (2.2), we can derive

$$\begin{aligned}
\Upsilon^{\lambda}(t)\mathbb{E}\mathscr{K}\big(u(t)\big) &\leq k\Upsilon^{\lambda}(t)\mathbb{E}\mathscr{K}\big(u(t - \sigma_{t,\rho(t)})\big) + \frac{k_2a_2}{a_1}\mathbb{E}\mathscr{K}(\tilde{u}(0)) \\
&\quad + \frac{k_2(kk_1c_3 + c_4)}{a_1}\mathbb{E}\int_0^t \Upsilon^{\lambda}(s)\sup_{-\iota \leq r \leq 0}\mathscr{K}\big(u(s + r)\big)\frac{\Upsilon'(s)}{\Upsilon(s)}ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}\mathscr{K}\big(u(t)\big) &\leq m_1\Upsilon^{-\lambda}(t) + k\mathbb{E}\mathscr{K}\big(u(t - \sigma_{t,\rho(t)})\big) \\
&\quad + m_2\Upsilon^{-\lambda}(t)\mathbb{E}\int_0^t \Upsilon^{\lambda}(s)\sup_{-\iota \leq r \leq 0}\mathscr{K}\big(u(s + r)\big)\frac{\Upsilon'(s)}{\Upsilon(s)}ds,
\end{aligned}$$

where $m_1 = \frac{k_2a_2}{a_1}\mathbb{E}\mathscr{K}(\tilde{u}(0))$. For any $t \geq \iota$ and $\flat \in [-\iota, 0]$, we can write

$$\begin{aligned}
\mathbb{E}\mathscr{K}\big(u(t + \flat)\big) &\leq m_1\Upsilon^{-\lambda}(t + \flat) + k\mathbb{E}\sup_{-\iota \leq r \leq 0}\mathscr{K}\big(u(t + \flat + r)\big) \\
&\quad + m_2\Upsilon^{-\lambda}(t + \flat)\mathbb{E}\int_0^{t+\flat} \Upsilon^{\lambda}(s)\sup_{-\iota \leq r \leq 0}\mathscr{K}\big(u(s + r)\big)\frac{\Upsilon'(s)}{\Upsilon(s)}ds.
\end{aligned} \tag{3.5}$$

We can easily prove that $\Upsilon^{-\lambda}(t+\flat) \le \Upsilon^{\lambda}(\iota)\Upsilon^{-\lambda}(t)$ and $\Upsilon(s-\flat) \le \Upsilon(s)\Upsilon(\iota)$. Multiplying both sides of (3.5) by $\Upsilon'(t)\Upsilon^{\varepsilon}(t)$, for $\varepsilon \in (0, \lambda-1)$ and integrating on $[\iota, T]$, $(T > \iota)$, we have

$$\begin{aligned}
&\int_{\iota}^{T} \Upsilon^{\varepsilon}(t)\mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\Upsilon'(t)dt \\
&\le m_1\Upsilon^{\lambda}(\iota)\int_{\iota}^{T}\Upsilon^{\varepsilon-\lambda}(t)\Upsilon'(t)dt + k\int_{\iota}^{T}\Upsilon^{\varepsilon}(t)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(t+\flat+r)\big)\Upsilon'(t)dt \\
&\quad + m_2\Upsilon^{\lambda}(\iota)\int_{\iota}^{T}\Upsilon^{\varepsilon-\lambda}(t)\Upsilon'(t)\int_{0}^{t+\flat}\Upsilon^{\lambda}(s)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\frac{\Upsilon'(s)}{\Upsilon(s)}dsdt \\
&:= m_1\Upsilon^{\lambda}(\iota)J_1 + kJ_2 + m_2\Upsilon^{\lambda}(\iota)J_3,
\end{aligned} \tag{3.6}$$

where

$$J_1 := \int_{\iota}^{T}\Upsilon^{\varepsilon-\lambda}(t)\Upsilon'(t)dt = \frac{1}{\varepsilon-\lambda+1}\left[\Upsilon^{\varepsilon-\lambda+1}(t)\right] \le \frac{\Upsilon^{\varepsilon-\lambda+1}(\iota)}{\lambda-\varepsilon-1}. \tag{3.7}$$

Recall that $\frac{\Upsilon'(s)}{\Upsilon(s)}$ is nonincreasing for all $s \ge 0$. Thus we obtain that

$$\begin{aligned}
J_2 &:= \int_{\iota}^{T}\Upsilon^{\varepsilon}(t)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(t+\flat+r)\big)\Upsilon'(t)dt \\
&\le \int_{0}^{T}\Upsilon^{\varepsilon}(s-\flat)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\Upsilon'(s-\flat)ds \\
&\le \Upsilon^{\varepsilon+1}(\iota)\int_{0}^{T}\Upsilon^{\varepsilon+1}(s)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\frac{\Upsilon'(s)}{\Upsilon(s)}ds \\
&\le \Upsilon^{\varepsilon+1}(\iota)\int_{0}^{T}\Upsilon^{\varepsilon}(s)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\Upsilon'(s)ds.
\end{aligned} \tag{3.8}$$

Integrating by parts and using the non-decreasing property of the function Υ on $\mathbb{R}^+$, we can derive that

$$\begin{aligned}
J_3 &:= \int_{\iota}^{T}\Upsilon^{\varepsilon-\lambda}(t)\Upsilon'(t)\int_{0}^{t+\flat}\Upsilon^{\lambda-1}(s)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\Upsilon'(s)dsdt \\
&= \frac{1}{\varepsilon-\lambda+1}\left[\Upsilon^{\varepsilon-\lambda+1}(t)\int_{0}^{t+\flat}\Upsilon^{\lambda-1}(s)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\Upsilon'(s)ds\right] \\
&\quad - \frac{1}{\varepsilon-\lambda+1}\int_{\iota}^{T}\Upsilon^{\varepsilon-\lambda+1}(t)\Upsilon^{\lambda-1}(t+\flat)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(t+\flat+r)\big)\Upsilon'(t+\flat)dt \\
&\le \frac{\Upsilon^{\varepsilon-\lambda+1}(\iota)}{\lambda-\varepsilon-1}\int_{0}^{\iota}\Upsilon^{\lambda-1}(s)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\Upsilon'(s)ds \\
&\quad + \frac{1}{\lambda-\varepsilon-1}\int_{0}^{T}\Upsilon^{\varepsilon}(s)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\Upsilon'(s)ds.
\end{aligned} \tag{3.9}$$

Substituting (3.7)-(3.9) into (3.6), we obtain that

$$\int_0^T \Upsilon^{\varepsilon}(t) \sup_{-\iota\le\flat\le 0} \mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\Upsilon'(t)dt \le M + \Big(k\Upsilon^{\varepsilon+1}(\iota) + \frac{m_2\Upsilon^{\lambda}(\iota)}{\lambda-\varepsilon-1}\Big)\int_0^T \Upsilon^{\varepsilon}(t) \sup_{-\iota\le r\le 0} \mathbb{E}\mathscr{K}\big(u(t+r)\big)\Upsilon'(t)dt,$$

where

$$M := \frac{\Upsilon^{\varepsilon+1}(\iota)}{\lambda-\varepsilon-1}\Big(m_1 + m_2\int_0^{\iota}\Upsilon^{\lambda-1}(s)\sup_{-\iota\le r\le 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\Upsilon'(s)ds\Big) + \int_0^{\iota}\Upsilon^{\varepsilon}(s)\sup_{-\iota\le\flat\le 0}\mathbb{E}\mathscr{K}\big(u(s+\flat)\big)\Upsilon'(s)ds.$$

In view of $\mu(\varepsilon) < 1$, one has

$$\int_0^T \Upsilon^{\varepsilon}(t) \sup_{-\iota\le\flat\le 0} \mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\Upsilon'(t)dt \le \frac{M}{1-\mu(\varepsilon)} < \infty,$$

which implies that

$$\hbar(\varepsilon) = \int_0^{\infty} \Upsilon^{\varepsilon}(t) \sup_{-\iota\le\flat\le 0} \mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\Upsilon'(t)dt < \infty.$$

This completes the proof. □

3.1. **Existence and uniqueness of solutions.**

Theorem 3.1. *Let all the conditions of Lemma 3.1be satisfied. Then, for all $\xi \in C\big[-\iota,0],\mathbb{R}^n\big)$, equation (2.1) admits a unique global solution $u(t)$ on $[-\iota,\infty)$.*

Proof. We divide the proof into two steps as follows.

Step 1. We prove that equation (2.1) has a unique maximal local solution $u(t)$.

For any initial value $\xi \in C([-\iota,0],\mathbb{R}^n)$, one sees that there exists a positive real $\tilde{c}$ such that $\|\xi\| \le \tilde{c}$, almost surly. Then, for each integer $q \ge \tilde{c}$, we define

$$f^q\left(t,u_1,u_2,\rho(t)\right) = f\left(t, \frac{|u_1|\wedge q}{|u_1|}u_1, \frac{|u_2|\wedge q}{|u_2|}u_2, \rho(t)\right),$$

where $\frac{|u_i|\wedge q}{|u_i|} = 0$ if $u_i = 0$ $(i=1,2)$. Similarly, we define $g^q\left(t,u_1,u_2,\rho(t)\right)$ and $h^q\big(t,u_1,u_2,y,\rho(t)\big)$. Consider the following equation

$$\begin{aligned} d\big[u_q(t) - \mathscr{D}\big(u_q(t-\sigma_{t,\rho(t)}),\rho(t)\big)\big] =& f^q\big(t,u_q(t),u_q(t-\sigma_{t,\rho(t)}),\rho(t)\big)dt \\ &+ \hbar_2(t)g^q\big(t,u_q(t),u_q(t-\sigma_{t,\rho(t)}),\rho(t)\big)dw(t) \\ &+ \hbar_3(t)\int_{\mathbb{R}} h^q\big(t,u_q(t),u_q(t-\sigma_{t,\rho(t)}),y,\rho(t)\big)N(dt,dy). \end{aligned} \tag{3.10}$$

Using the property that $\rho(t)$ and $\wp(t)$ are right continuous Markov processes, we have two random constant sequences of stopping times $\{\alpha_k\}_{k\ge 0}$ and $\{\iota_{k^*}\}_{k^*\ge 0}$, with $\wp(t) = \wp(\alpha_k)$ on $t \in [\alpha_k,\alpha_{k+1})$, and $\rho(t) = \rho(\iota_{k^*})$ on $t \in [\iota_{k^*},\iota_{k^*+1})$, where $\{\alpha_k\}_{k\ge 0}$ and $\{\iota_{k^*}\}_{k^*\ge 0}$ are random constants respectively on every interval $[\alpha_k,\alpha_{k+1})$, and $[\iota_{k^*},\iota_{k^*+1})$ for any $k,k^* \ge 0$.

Firstly, taking equation (3.10) on $[a_0^*, a_1^*) = [\alpha_0, \alpha_1) \cap [\iota_0, \iota_1)$, we see that

$$\begin{aligned} d\big[u_q(t) - \mathscr{D}\big(u_q(t-\sigma_{t,\rho_0}), \rho_0\big)\big] =& f^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho_0}), \rho_0\big)dt \\ &+ \hbar_2^0 g^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho_0}), \rho_0\big)dw(t) \\ &+ \hbar_3^0 \int_{\mathbb{R}} h^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho_0}), y, \rho_0\big)N(dt,dy), \end{aligned} \tag{3.11}$$

with initial data $\xi, \wp(t) = \wp_0, \rho(t) = \rho_0$. Here, $\hbar_2^0$ and $\hbar_3^0$ are the initial values of $\hbar_2(t)$ and $\hbar_3(t)$, respectively. By Assumption 2.1, $f^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho_0}), \rho_0\big)$, $g^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho_0}), \rho_0\big)$ and $h^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho_0}), y, \rho_0\big)$ satisfy the global Lipschitz and linear growth conditions. Therefore, [23, Theorem 3.1] allows us to conclude that equation (3.11) has a unique global solution $u_q(t)$ on $t \in [a_0^*, a_1^*)$.

Secondly, without loss of generality, we suppose that $\iota_1 < \alpha_1$ and set $[a_1^*, a_2^*) = [\alpha_0, \alpha_1) \cap [\iota_1, \iota_2)$. Therefore, taking equation (3.10), on $t \in [a_1^*, a_2^*)$, one has

$$\begin{aligned} d\big[u_q(t) - \mathscr{D}\big(u_q(t-\sigma_{t,\rho_1}), \rho_1\big)\big] =& f^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho_1}), \rho_1\big)dt \\ &+ \hbar_2^0 g^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho_1}), \rho_1\big)dw(t) \\ &+ \hbar_3^0 \int_{\mathbb{R}} h^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho_1}), y, \rho_1\big)N(dt,dy), \end{aligned} \tag{3.12}$$

with initial conditions $u_q(t)$ on $t \in [a_0^*, a_1^*)$, $\wp_0$ and ρ_1. Again, [23, Theorem 3.1] shows that equation (3.12) has a unique global solution $u_q(t)$ in $[a_1^*, a_2^*)$.

We repeat this procedure such that equation (3.10) admits a unique global solution $u_q(t)$ on $t \geq -\iota$. Now, letting $\nabla_q = \inf\{t \geq 0; |u_q(t)| \geq q\}$, one sees that $|u_q(t)| \vee |u_q(t-\sigma_{t,\rho_t})| \leq q$ for $0 \leq t \leq \nabla_q$. Thus

$$f^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho(t)}), \rho(t)\big) = f^{q+1}\big(t, u_q(t), u_q(t-\sigma_{t,\rho(t)}), \rho(t)\big),$$

$$g^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho(t)}), \rho(t)\big) = g^{q+1}\big(t, u_q(t), u_q(t-\sigma_{t,\rho(t)}), \rho(t)\big),$$

and

$$h^q\big(t, u_q(t), u_q(t-\sigma_{t,\rho(t)}), y, \rho(t)\big) = h^{q+1}\big(t, u_q(t), u_q(t-\sigma_{t,\rho(t)}), y, \rho(t)\big)$$

are satisfied for any $t \in [0, \nabla_q]$. Hence, $u_q(t) = u_{q+1}(t)$ for $t \in [0, \nabla_q]$. Note that ∇_q is increasing with respect to q from its definition. Also let us define $u(t)$ on $[-\iota, \nabla_\infty)$ with $u(t) = u_q(t)$, where $\nabla_\infty = \lim_{q\to\infty} \nabla_q$. Hence, equation (2.1) has a unique maximal local solution $u(t)$.

Step 2. Let $\nu_0 > 0$ be sufficiently large such that $\|\xi\| = \sup_{-\iota \leq \flat \leq 0} |\xi(\flat)| \leq \nu_0$. For all $\nu \geq (1+k)\nu_0$, we define the stopping time $\iota_\nu = \inf\big\{t \in [0, \nabla_\infty) : |\tilde{u}(t)| \geq \nu\big\}$, $\inf\emptyset = \infty$. It is obvious that ι_ν is increasing. Define $\iota_\infty = \lim_{r\to\infty} \iota_\nu$, whence $\iota_\infty \leq \nabla_\infty$. The fact that $\iota_\infty = \infty$ almost surely gives $\nabla_\infty = \infty$ almost surely means that the solution $\{u(t), t \in [-\iota, \infty)\}$ of equation (2.1) does not explode in finite time.

Now, we prove that $\iota_\infty = \infty$ almost surely. It is equivalent to $\mathbb{P}(\iota_\nu \leq t) \longrightarrow 0$ as $\nu \longrightarrow \infty$ for any $t \geq 0$. Recall that $1_{\{\iota_\nu \leq t\}}$ is the indicator function. From the generalised Itô formula and conditions (2.3) and (2.5), we obtain

$$\begin{aligned}
&\mathbb{E}\Big(1_{\{\iota_\nu\le t\}}\mathscr{V}\big(\iota_\nu,\tilde{u}(\iota_\nu),\wp(\iota_\nu),\rho(\iota_\nu)\big)\Big)\\
&\le \mathbb{E}\mathscr{V}\big(t\wedge\iota_\nu,u(t\wedge\iota_\nu),\wp(t\wedge\iota_\nu),\rho(t\wedge\iota_\nu)\big)\\
&= \mathbb{E}\mathscr{V}(0,\tilde{u}(0),\wp_0,\rho(0))+\mathbb{E}\int_0^{t\wedge\iota_\nu}\mathscr{L}\mathscr{V}\Big(s,u(s),u(s-\sigma_{s,\rho(s)}),\wp(s),\rho(s)\Big)ds\\
&\le a_2\mathbb{E}\mathscr{K}(\tilde{u}(0))+c_4\mathbb{E}\int_0^{t\wedge\iota_\nu}\sup_{-\iota\le\flat\le 0}\mathscr{K}\big(u(s+\flat)\big)\frac{\Upsilon'(s)}{\Upsilon(s)}ds.
\end{aligned}$$

The definition of Υ allows us to write $\frac{1}{\Upsilon(s)}\le 1$ for all $s\ge 0$, which implies that

$$\begin{aligned}
a_1\mathbb{E}(1_{\{\iota_\nu\le t\}}\mathscr{K}(\tilde{u}(\iota_\nu)) \le&\,\mathbb{E}\Big(1_{\{\iota_\nu\le t\}}\mathscr{V}\big(\iota_\nu,\tilde{u}(\iota_\nu),\wp(\iota_\nu),\rho(\iota_\nu)\big)\Big)\\
\le&\, a_2\mathbb{E}\mathscr{K}(\tilde{u}(0))+c_4\int_0^{t\wedge\iota_\nu}\sup_{-\iota\le\flat\le 0}\mathbb{E}\mathscr{K}\big(u(s+\flat)\big)\Upsilon'(s)ds\\
\le&\, a_2\mathbb{E}\mathscr{K}(\tilde{u}(0))+\hbar(0)=cst<+\infty. \qquad (3.13)
\end{aligned}$$

From the definition of ι_ν, we have $\tilde{u}(\iota_\nu)=\nu$. Further, for each $\nu\ge 0$, we define

$$\chi(\nu):=\inf\Big\{\mathscr{K}(u):u\in\mathbb{R}^n,\quad \text{with}\quad |u|\ge\nu\Big\}.$$

Employing (3.13), we can write

$$a_1\chi(\nu)\mathbb{P}\big(\iota_\nu\le t\big)\le a_1\mathbb{E}(1_{\{\iota_\nu\le t\}}\mathscr{K}(\tilde{u}(\iota_\nu))\le cst.$$

Thus

$$\mathbb{P}\big(\iota_\nu\le t\big)\le\frac{cst}{a_1\chi(\nu)}.$$

By using condition (2.4), we note that $\lim_{\nu\to\infty}\chi(\nu)=\infty$. Then $\lim_{\nu\to\infty}\mathbb{P}(\iota_\nu\le t)=0$. Since $t>0$ is arbitrary, then $\mathbb{P}(\iota_\infty=\infty)=1$. Thus equation (2.1) admits a unique global solution $u(t)$ for any $t\ge-\iota$. ☐

3.2. **General decay stability.** In this section, we present and prove the general decay stability of the system under consideration.

Theorem 3.2. *Let the conditions required by Lemma 3.1 be satisfied. Then, the global solutions $u(t)$ of equation (2.1) satisfies*

$$\limsup_{t\to\infty}\frac{\log\mathbb{E}\mathscr{K}(u(t))}{\log\Upsilon(t)}\le-\varepsilon_1,\quad \textit{for any initial data } \xi\in C([-\iota,0],\mathbb{R}^n),$$

with $\varepsilon_1=\tilde{\varepsilon}\wedge(\varepsilon+1)$, where $\tilde{\varepsilon}\in(0,\bar{\vartheta})$ such that $\bar{\vartheta}=\inf_{t\ge\iota}\frac{-t\log(k)}{\iota\log\Upsilon(t)}$ and ε is defined in Lemma 3.1.

Proof. For any $t\ge\iota$ and $\flat\in[-\iota,0]$, it yields from (3.5) together with any $\varepsilon(0,\lambda-1)$ that

$$\begin{aligned}
\mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\le&\, m_1\Upsilon^{-(\varepsilon+1)}(t+\flat)+k\mathbb{E}\sup_{-\iota\le r\le 0}\mathscr{K}\big(u(t+\flat+r)\big)\\
&+m_2\Upsilon^{-\lambda}(t+\flat)\mathbb{E}\int_0^{t+\flat}\Upsilon^{\lambda-1}(s)\sup_{-\iota\le r\le 0}\mathscr{K}\big(u(s+r)\big)\Upsilon'(s)ds.
\end{aligned}$$

Since $\Upsilon^{\lambda-1-\varepsilon}(s)$ is non-decreasing for $s \geq 0$, one sees that

$$\begin{aligned}
\mathbb{E}\mathscr{K}\big(u(t+\flat)\big) &\leq m_1\Upsilon^{-(\varepsilon+1)}(t+\flat)+k\sup_{-\iota\leq r\leq 0}\mathbb{E}\mathscr{K}\big(u(t+\flat+r)\big)\\
&\quad +m_2\Upsilon^{-\lambda}(t+\flat)\Upsilon^{\lambda-1-\varepsilon}(t+\flat)\int_0^{t+\flat}\Upsilon^{\varepsilon}(s)\sup_{-\iota\leq r\leq 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\Upsilon'(s)ds\\
&\leq m_1\Upsilon^{-(\varepsilon+1)}(t+\flat)+k\sup_{-\iota\leq r\leq 0}\mathbb{E}\mathscr{K}\big(u(t+\flat+r)\big)\\
&\quad +m_2\Upsilon^{-(\varepsilon+1)}(t)\Upsilon^{\varepsilon+1}(\iota)\hbar(\varepsilon),
\end{aligned}$$

where the value of $\hbar(\varepsilon)$ is determined by Lemma 3.1, and $\varepsilon\in(0,\varepsilon_0)$ with $\varepsilon_0\in(0,\lambda-1)$. For $t\geq\iota$, we have

$$\mathbb{E}\mathscr{K}\left(u(t+\flat)\right)\leq\Upsilon^{(\varepsilon+1)}(\iota)\big(m_1+m_2\hbar(\varepsilon)\big)\Upsilon^{-(\varepsilon+1)}(t)+k\mathbb{E}\mathscr{K}\big(u(t+\flat-\sigma_{t+\flat,\rho(t+\flat)})\big).$$

For any $t,\flat$, there exists $q_\flat\in[0,\iota]$ such that $\sigma_{t+\flat,\rho(t+\flat)}=q_\flat$. Since $q_\flat\in[0,\iota]$, then there exists $\alpha_\flat\in[0,1]$ such that $q_\flat=\alpha_\flat\iota$. It follows that

$$\begin{aligned}
\mathbb{E}\mathscr{K}\big(u(t+\flat)\big) &\leq\Upsilon^{(\varepsilon+1)}(\iota)\big(m_1+m_2\hbar(\varepsilon)\big)\Upsilon^{-(\varepsilon+1)}(t)+k\mathbb{E}\mathscr{K}\big(u(t+\flat-\alpha_\flat\iota)\big)\\
&\leq L(\varepsilon)\Upsilon^{-(\varepsilon+1)}(t)+k\mathbb{E}\mathscr{K}\big(u(t+s)\big),
\end{aligned}$$

where $L(\varepsilon):=\Upsilon^{(\varepsilon+1)}(\iota)\big(m_1+m_2\hbar(\varepsilon)\big)$ and $-2\iota\leq s=\flat-\alpha_\flat\iota\leq 0$. We separate via two cases

i): If $\alpha_\flat=0$ for any $\flat$, then we can derive that

$$\mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\leq\frac{L(\varepsilon)}{1-k}\Upsilon^{-(\varepsilon+1)}(t)).$$

ii): If $\alpha_\flat\in(0,1]$ for any $\flat\in[-\iota,0]$, then, for any $t\geq\iota$,

$$\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\leq L(\varepsilon)\Upsilon^{-(\varepsilon+1)}(t)+k\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t-\iota+\flat)\big).\tag{3.14}$$

Now, for any $t\geq 2\iota$, we have from (3.14) that

$$\begin{aligned}
&\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t-\iota+\flat)\big)\\
&\leq L(\varepsilon)\Upsilon^{-(\varepsilon+1)}(t-\iota)+k\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t-2\iota+\flat)\big)\\
&\leq L(\varepsilon)\Upsilon^{-(\varepsilon+1)}(t)\Upsilon^{\varepsilon+1}(\iota)+k\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t-2\iota+\flat)\big).
\end{aligned}\tag{3.15}$$

Substituting (3.15) into (3.14), we obtain for any $t\geq 2\iota$ that

$$\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\leq L(\varepsilon)\Upsilon^{-(\varepsilon+1)}(t)\big(1+k\Upsilon^{\varepsilon+1}(\iota)\big)+k^2\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t-2\iota+\flat)\big).$$

Similarly, for any $t\geq\iota$, we can check that

$$\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\leq L(\varepsilon)\Upsilon^{-(\varepsilon+1)}(t)\sum_{j=0}^{n-1}\big(k\Upsilon^{\varepsilon+1}(\iota)\big)^j+k^n\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t-n\iota+\flat)\big),$$

where $n \in \mathbb{N}$-$\{0\}$ such that $n\iota \leq t < (n+1)\iota$. From Lemma 3.1, we have $k\Upsilon^{\varepsilon+1}(\iota) \in (0,1)$. Therefore

$$\begin{aligned}\sup_{-\iota\leq\flat\leq 0} \mathbb{E}\mathscr{K}\big(u(t+\flat)\big) &\leq \frac{L(\varepsilon)}{1-k\Upsilon^{\varepsilon+1}(\iota)}\Upsilon^{-(\varepsilon+1)}(t)+k^{n}\sup_{-\iota\leq\flat\leq 0} \mathbb{E}\mathscr{K}\big(u(t-n\iota+\flat)\big)\\ &\leq \frac{L(\varepsilon)}{1-k\Upsilon^{\varepsilon+1}(\iota)}\Upsilon^{-(\varepsilon+1)}(t)+k^{\frac{t}{\iota}-1}M\\ &\leq \frac{L(\varepsilon)}{1-k\Upsilon^{\varepsilon+1}(\iota)}\Upsilon^{-(\varepsilon+1)}(t)+\Upsilon^{-\tilde{\varepsilon}}(t)\frac{M}{k},\end{aligned}$$

where $\tilde{\varepsilon} \in (0,\bar{\vartheta})$ with $\bar{\vartheta} = \inf_{t\geq\iota} \dfrac{-t\log(k)}{\iota\log\Upsilon(t)}$. Moreover, $\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t+\flat)\big) \leq M_1\Upsilon^{-\varepsilon_1}(t)$, $t \geq \iota$, where $M_1 = \dfrac{L(\varepsilon)}{1-k\Upsilon^{\varepsilon+1}(\iota)}+\dfrac{M}{k} > 0$, and $\varepsilon_1 = \tilde{\varepsilon}\wedge(\varepsilon+1)$. Further, we see from (3.5) that, for any $0 \leq t \leq \iota$ $\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t+\flat)\big) \leq M_2\Upsilon^{-\varepsilon_1}(t)$. Thus, for any $t \geq 0$,

$$\mathbb{E}\mathscr{K}\big(u(t)\big) \leq \max\{M_1, M_2\}\Upsilon^{-\varepsilon_1}(t),$$

which yields our desired result.

□

Lemma 3.2. *Let the assumptions of Lemma 3.1 be fulfilled. Then*

$$\int_0^{\infty}\Upsilon^{\varepsilon}(t)\sup_{-\iota\leq\flat\leq 0}\mathscr{K}\big(u(t+\flat)\big)\Upsilon'(t)dt < \infty, \quad a.s,$$

where $\varepsilon \in (0,\varepsilon_0)$ with ε_0 being determined in Lemma 3.1.

Proof. The detailed proof consists of four steps which are as follows.

Step 1. By following a method similar to the above and using conditions (2.2) and (2.3), for any $t \geq \iota$ and $\flat \in [-\iota,0]$, it follows $(T \geq t)$ that

$$\begin{aligned}&\int_{\iota}^{T}\Upsilon^{\varepsilon}(t)\mathbb{E}\mathscr{V}\big(t+\flat,\tilde{u}(t+\flat),\wp(t+\flat),\rho(t+\flat)\big)\Upsilon'(t)dt\\ &\leq a_2k_1\int_{\iota}^{T}\Upsilon^{\varepsilon}(t)\mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\Upsilon'(t)dt\\ &\quad + a_2k_1k\int_{\iota}^{T}\Upsilon^{\varepsilon}(t)\mathbb{E}\sup_{-\iota\leq r\leq 0}\mathscr{K}\big(u(t+\flat+r)\big)\Upsilon'(t)dt\\ &\leq a_2k_1\int_{\iota}^{T}\Upsilon^{\varepsilon}(t)\sup_{-\iota\leq\flat\leq 0}\mathbb{E}\mathscr{K}\big(u(t+\flat)\big)\Upsilon'(t)dt\\ &\quad + a_2k_1k\Upsilon^{\varepsilon+1}(\iota)\int_{0}^{T}\Upsilon^{\varepsilon}(s)\sup_{-\iota\leq r\leq 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\Upsilon'(s)ds\\ &\leq \big(a_2k_1+a_2k_1k\Upsilon^{\varepsilon+1}(\iota)\big)\hbar(\varepsilon) < \infty,\end{aligned}$$

where $\hbar(\varepsilon)$ is given in (3.2). Furthermore, Lemma 3.1 allows us to write

$$\int_{\iota}^{\infty}\Upsilon^{\varepsilon}(t)\mathbb{E}\mathscr{V}\big(t+\flat,\tilde{u}(t+\flat),\wp(t+\flat),\rho(t+\flat)\big)\Upsilon'(t)dt < \infty. \tag{3.16}$$

Step 2. Note that $\sup_{s\geq 0}\frac{\Upsilon'(s)}{\Upsilon(s)}=\Upsilon'(0)$. Applying Itô formula, Assumption 2.3 and Theorem 3.2, for any $\flat\in[-\iota,0]$ and $t_2>t_1\geq\iota$, we obtain that

$$\begin{aligned}&\mathbb{E}\mathscr{V}\big(t_1+\flat,\tilde{u}(t_1+\flat),\wp(t_1+\flat),\rho(t_1+\flat)\big)-\mathbb{E}\mathscr{V}\big(t_2+\flat,\tilde{u}(t_2+\flat),\wp(t_2+\flat),\rho(t_2+\flat)\big)\\&=\int_{t_1+\flat}^{t_2+\flat}\mathscr{L}\mathscr{V}\Big(s,u(s),u(s-\sigma_{s,\rho(s)}),\wp(s),\rho(s)\Big)ds\\&\leq c_4\int_{t_1+\flat}^{t_2+\flat}\sup_{-\iota\leq r\leq 0}\mathbb{E}\mathscr{K}\big(u(s+r)\big)\frac{\Upsilon'(s)}{\Upsilon(s)}ds\\&\leq c_4L_1\Upsilon'(0)(t_2-t_1)\Upsilon^{-\varepsilon}(t_1+\flat),\end{aligned}$$

where $L_1=\max\{M_1,M_2\}$. Consequently, using the definition of Υ, and applying to it the finite-increments formula on $[t_1,t_2]$, we have

$$\begin{aligned}&\Big|\Upsilon^{\varepsilon}(t_2)\mathbb{E}\mathscr{V}\big(t_1+\flat,\tilde{u}(t_1+\flat),\wp(t_1+\flat),\rho(t_1+\flat)\big)-\Upsilon^{\varepsilon}(t_1)\mathbb{E}\mathscr{V}\big(t_2+\flat,\tilde{u}(t_2+\flat),\wp(t_2+\flat),\rho(t_2+\flat)\big)\Big|\\&\leq(a_2\varepsilon+c_4)\Upsilon'(0)L_1|t_2-t_1|,\end{aligned}$$

which implies that

$$\begin{aligned}&\lim_{t_2\to t_1}\Upsilon^{\varepsilon}(t_2)\mathbb{E}\mathscr{V}\big(t_2+\flat,\tilde{u}(t_2+\flat),\wp(t_2+\flat),\rho(t_2+\flat)\big)\Upsilon'(t_2)\\&=\Upsilon^{\varepsilon}(t_1)\mathbb{E}\mathscr{V}\big(t_1+\flat,\tilde{u}(t_1+\flat),\wp(t_1+\flat),\rho(t_1+\flat)\big)\Upsilon'(t_1).\end{aligned}$$

Using Lemma 2.1, for any $\flat\in[-\iota,0]$, we have

$$\lim_{t\to\infty}\Upsilon^{\varepsilon}(t)\mathbb{E}\mathscr{V}\big(t+\flat,\tilde{u}(t+\flat),\wp(t+\flat),\rho(t+\flat)\big)\Upsilon'(t)=0. \tag{3.17}$$

Step 3. From (3.16), (3.17) and the Fubini theorem [31], for any $\flat\in[-\iota,0]$, we conclude

$$\mathbb{E}\Big\{\int_{\iota}^{\infty}\Upsilon^{\varepsilon}(t)\mathscr{V}\big(t+\flat,\tilde{u}(t+\flat),\wp(t+\flat),\rho(t+\flat)\big)\Upsilon'(t)dt\Big\}<\infty.$$

Therefore, by employing the Chebyshev inequality, one has

$$\mathbb{P}\Big(\int_{\iota}^{\infty}\Upsilon^{\varepsilon}(t)\mathscr{V}\big(t+\flat,\tilde{u}(t+\flat),\wp(t+\flat),\rho(t+\flat)\big)\Upsilon'(t)dt=\infty\Big)=0. \tag{3.18}$$

Now, for any $\flat\in[-\iota,0]$, we derive from (2.3) and (3.18) that

$$\int_{\iota}^{\infty}\Upsilon^{\varepsilon}(t)\mathscr{K}\big(\tilde{u}(t+\flat)\big)\Upsilon'(t)dt<\infty,\quad\text{a.s.} \tag{3.19}$$

Step 4. From (2.2) and the definition of Υ, we can derive, for any $\flat\in[-\iota,0]$ and $T>2\iota$, that

$$\begin{aligned}&\int_{\iota}^{T}\Upsilon^{\varepsilon}(t)\mathscr{K}\big(u(t+\flat)\big)\Upsilon'(t)dt\\&\leq k\int_{\iota}^{T}\Upsilon^{\varepsilon}(t)\sup_{-\iota\leq r\leq 0}\mathscr{K}\big(u(t+\flat+r)\big)\Upsilon'(t)dt+k_2\int_{\iota}^{T}\Upsilon^{\varepsilon}(t)\mathscr{K}\big(\tilde{u}(t+\flat)\big)\Upsilon'(t)dt\\&\leq k\Upsilon^{\varepsilon+1}(\iota)\int_{0}^{T}\Upsilon^{\varepsilon}(s)\sup_{-\iota\leq r\leq 0}\mathscr{K}\big(u(s+r)\big)\Upsilon'(s)ds+k_2\int_{\iota}^{T}\Upsilon^{\varepsilon}(t)\mathscr{K}\big(\tilde{u}(t+\flat)\big)\Upsilon'(t)dt.\end{aligned}$$

It follows from (3.3) that $k\Upsilon^{\varepsilon+1}(\iota)\in(0,1)$. Using condition (3.19), we have

$$\begin{aligned}
&\int_\iota^T \Upsilon^\varepsilon(t)\sup_{-\iota\le\flat\le 0}\mathscr{K}\left(u(t+\flat)\right)\Upsilon'(t)dt\\
&\le \frac{k\Upsilon^{\varepsilon+1}(\iota)}{1-k\Upsilon^{\varepsilon+1}(\iota)}\int_0^\iota \Upsilon^\varepsilon(t)\sup_{-\iota\le\flat\le 0}\mathscr{K}\left(u(t+\flat)\right)\Upsilon'(t)dt\\
&\quad+\frac{k_2}{1-k\Upsilon^{\varepsilon+1}(\iota)}\int_\iota^T \Upsilon^\varepsilon(t)\mathscr{K}\left(\tilde{u}(t+\flat)\right)\Upsilon'(t)dt<\infty. \qquad (3.20)
\end{aligned}$$

Taking $T\to\infty$ in (3.20), we conclude that $\int_0^\infty \Upsilon^\varepsilon(t)\sup_{-\iota\le\flat\le 0}\mathscr{K}\left(u(t+\flat)\right)\Upsilon'(t)dt<\infty$ a.s. Thus, the proof is completed. $\square$

Theorem 3.3. *Let the assumptions of Lemma 3.1 be satisfied. Then, for any initial data $\xi\in C([-\iota,0],\mathbb{R}^n)$, the global solution $u(t)$ of equation (2.1) obeys*

$$\limsup_{t\to\infty}\frac{\log\mathscr{K}(u(t))}{\log\Upsilon(t)}\le-\varepsilon_1,\quad a.s.,$$

where ε_1 is defined in Theorem 3.2.

Proof. For any $\varepsilon\in(0,\varepsilon_0)$, we have, for any $t\ge 0$,

$$\begin{aligned}
\Upsilon^{\varepsilon+1}(t)\mathscr{V}\left(t,\tilde{u}(t),\wp(t),\rho(t)\right)&=\mathscr{V}\left(0,\tilde{u}(0),\wp(0),\rho(0)\right)\\
&\quad+\int_0^t\Big[(\varepsilon+1)\Upsilon^\varepsilon(s)\Upsilon'(s)\mathscr{V}\left(s,\tilde{u}(s),\wp(s),\rho(s)\right)\\
&\quad+\Upsilon^{\varepsilon+1}(s)\mathscr{L}\mathscr{V}\left(s,u(s),u(s-\sigma_{s,\rho(s)}),\wp(s),\rho(s)\right)\Big]ds+M_t,
\end{aligned} \qquad (3.21)$$

where $M_t:=\int_0^t\Upsilon^{\varepsilon+1}(s)dG_s$ with G_t being defined by

$$\begin{aligned}
G_t=&\int_0^t\mathscr{V}_u(s,\tilde{u}(s),\wp(s),\rho(s)).\hbar_2(s)g\left(s,u(s),u(s-\sigma_{s,\rho(s)}),\rho(s)\right)dw(s)\\
&+\int_0^t\int_Y\Big[\mathscr{V}\left(s,\tilde{u}(s)+\hbar_3(s)h\left(s,u(s),u(s-\sigma_{s,\rho(s)}),y,\rho(s)\right),\wp(s),\rho(s)\right)\\
&-\mathscr{V}(s,\tilde{u}(s),\wp(s),\rho(s))\Big]\tilde{N}(ds,dy)\\
&+\int_0^t\int_{\mathbb{R}}\left[\mathscr{V}\left(s,\tilde{u}(s),\wp_0+\bar{c}_1(\wp(s),u),\rho(s)\right)-\mathscr{V}\left(s,\tilde{u}(s),\wp(s),\rho(t)\right)\right]\mu_1(ds,dx)\\
&+\int_0^t\int_{\mathbb{R}}\left[\mathscr{V}\left(s,\tilde{u}(s),\wp(s),\rho_0+\bar{c}_2(\rho(s),u)\right)-\mathscr{V}\left(s,\tilde{u}(s),\wp(s),\rho(t)\right)\right]\mu_2(ds,dx),
\end{aligned}$$

where $\mu_1(ds,dx),\mu_2(ds,dx)$ are two martingale measures (see [15] for more details on the functions $\mu_1,\mu_2,\bar{c}_1$, and $\bar{c}_2$). We can easily prove that G_t is a local martingale with $G_0=0$. Therefore, by using Assumptions 2.3 and (3.21), we conclude

$$\begin{aligned}
a_1\Upsilon^{\varepsilon+1}(t)\mathscr{K}\left(\tilde{u}(t)\right)\le& a_2\mathscr{K}\left(\tilde{u}(0)\right)+\int_0^t\Big[\left(k_1a_2(\varepsilon+1)-c_3\right)\Upsilon^\varepsilon(s)\mathscr{K}\left(u(s)\right)\Upsilon'(s)\\
&+\left((\varepsilon+1)kk_1a_2+c_4\right)\Upsilon^\varepsilon(s)\mathscr{K}\left(u(s-\sigma_{s,\rho(s)})\right)\Upsilon'(s)\Big]ds+M_t.
\end{aligned}$$

From Lemma 3.1, one has $\varepsilon \in (0,\varepsilon_0)$ with $\varepsilon_0 \in (0,\lambda-1)$. It is not difficult to check that $k_1a_2(\varepsilon+1)-c_3<0$. Thus it yields that

$$\begin{aligned}\Upsilon^{\varepsilon+1}(t)\mathscr{K}\left(\tilde{u}(t)\right) \leq & \frac{a_2}{a_1}\mathscr{K}\left(\tilde{u}(0)\right)+\frac{(\varepsilon+1)kk_1a_2+c_4}{a_1}\int_0^t \Upsilon^{\varepsilon}(s)\mathscr{K}\left(u(s-\sigma_{s,\rho(s)})\right)\Upsilon'(s)ds+M_t\\ \leq & \mathscr{H}_0+\mathscr{A}(t)+M_t,\end{aligned} \tag{3.22}$$

where $\mathscr{H}_0 := \frac{a_2}{a_1}\mathscr{K}\left(\tilde{u}(0)\right)$ is a nonnegative bounded $\mathscr{F}_0$-measurable random variable,

$$\mathscr{A}(t) := \frac{(\varepsilon+1)kk_1a_2+c_4}{a_1}\int_0^t \Upsilon^{\varepsilon}(s)\sup_{-\iota\leq\flat\leq 0}\mathscr{K}\left(u(s+\flat)\right)\Upsilon'(s)ds, \quad \text{a.s.},$$

and M_t is a local martingale with $M_0=0$. Using the nonnegative semi-martingale convergence theorem [15], we see from Lemma 3.2 and (3.22) that $\limsup_{t\to\infty}\Upsilon^{\varepsilon+1}(t)\mathscr{K}\left(\tilde{u}(t)\right)<\infty$ a.s. Therefore, there exists a finite positive random variable $\mathscr{H}_1$ such that, for any $t\geq 0$, $\mathscr{K}\left(\tilde{u}(t)\right)\leq \mathscr{H}_1\Upsilon^{-(\varepsilon+1)}(t)$ a.s. From (2.2), we can deduce that

$$\mathscr{K}\left(u(t)\right)\leq k_2\mathscr{H}_1\Upsilon^{-(\varepsilon+1)}(t)+k\mathscr{K}\left(u(t-\sigma_{t,\rho(t)})\right), \quad \text{a.s.}$$

Similar to the proof of Theorem 3.2, we obtain that $\mathscr{K}\left(u(t)\right)\leq \mathscr{H}_2\Upsilon^{-\varepsilon_1}(t)$, where ε_1 is given in Theorem 3.2 and $\mathscr{H}_2$ is a finite positive random variable dependent on $\mathscr{H}_1$, which follows that

$$\limsup_{t\to\infty}\frac{\log\mathscr{K}\left(u(t)\right)}{\log\Upsilon(t)}\leq -\varepsilon_1, \quad \text{a.s.}$$

The proof is completed. □

3.3. **Decay stability with M-matrix theory.** In this section, an M-matrix method is used to prove the decay stability of the system.

Assumption 3.1. Assume that there exist bounded functions $c_{1i}^{\ell}:[0,+\infty[\longrightarrow\mathbb{R}$ and $c_{2i}^{\ell},\beta_{1i}^{\ell},\beta_{2i}^{\ell}:[0,+\infty[\longrightarrow[0,+\infty[$, and bounded functions $h_i^{\ell}(\cdot)$ such that

$$\tilde{u}^T f(t,u,v,i)+\frac{p-1}{2}\left(\hbar_2(t)\right)^2|g(t,u,v,i)|^2\leq c_{1i}^{\ell}(t)|u|^2+c_{2i}^{\ell}(t)|v|^2 \tag{3.23}$$

and

$$\left|\tilde{u}+\hbar_3(t)h(t,u,v,y,i)\right|^p\leq h_i^{\ell}(y)\left(\beta_{1i}^{\ell}(t)|u|^p+\beta_{2i}^{\ell}(t)|v|^p\right), \tag{3.24}$$

where $\ell\times i\in S_1\times S_2$ and $p\geq 2$.

Assumption 3.2. For $(\ell,i)\in S_1\times S_2$, let $\eta_i^{\ell}=\int_Y h_i^{\ell}(y)\pi(dy)<\infty$,

$$\varpi_{1i}^{\ell}(t)=\eta_i^{\ell}\beta_{1i}^{\ell}(t)-(1-k)^{p-1}\pi(Y), \quad \varpi_{2i}^{\ell}(t)=\eta_i^{\ell}\beta_{2i}^{\ell}(t)+k(1-k)^{p-1}\pi(Y),$$

$$e_i^{\ell}=\left[(1+k)^{p-1}(p-2)+2\right]c_{1i}^{\ell}(t)+(1+k)^{p-1}(p-2)c_{2i}^{\ell}(t), \quad \tilde{\Gamma}=\left(|\gamma_{ij}|\right)_{N\times N},$$

and

$$m_i^{\ell}=k(1+k)^{p-1}(p-2)c_{1i}^{\ell}(t)+\left[k(1+k)^{p-1}(p-2)+2\right]c_{2i}^{\ell}(t)+\varpi_{2i}^{\ell}(t).$$

Assume that $\mathscr{A}^\ell := -diag\big(e_1^\ell, e_2^\ell, ..., e_N^\ell\big) - (1+k)^{p-1}\tilde{\Gamma}$ is a nonsingular M-matrix, for all $\ell \in S_1$. Moreover, let $\flat := (\flat_1, ..., \flat_N)^T = \big(\mathscr{A}^\ell\big)^{-1}\vec{1} > 0$ with $\vec{1} = (1, 1, ..., 1)^T$. Suppose that, for each $(\ell, i) \in S_1 \times S_2$, there exist pair positive numbers c_{3i}^ℓ, c_{4i}^ℓ such that

$$1 - \varpi_{1i}^\ell(t) \min_{i \in S_2}(\flat_i) = c_{3i}^\ell \frac{\Upsilon'(t)}{\Upsilon(t)} \quad \text{and} \quad \big(m_i^\ell - ke_i^\ell\big) \max_{i \in S_2}(\flat_i) - k = c_{4i}^\ell \frac{\Upsilon'(t)}{\Upsilon(t)}.$$

Theorem 3.4. *Let Assumptions 2.1, 2.2, 3.1, and 3.2 hold, and assume that*

$$\lambda_i^\ell = \frac{c_{3i}^\ell}{k_1 \max_{i \in S_2}(\flat_i)} > 1 \tag{3.25}$$

and

$$c_{4i}^\ell < \frac{\Upsilon^{-(\lambda_i^\ell + 1)}(\iota)}{kk_2} \min_{i \in S_2}(\flat_i) - kk_1 c_{3i}^\ell \quad \textit{for any } (\ell, i) \in S_1 \times S_2. \tag{3.26}$$

Then, for any initial data $\xi \in C\big([-\iota, 0], \mathbb{R}\big)$, there exists a unique global solution $u(t)$ of equation (2.1) on $t \in [-\iota, \infty)$. Furthermore, the unique solution has the properties

$$\limsup_{t \to \infty} \frac{\log \mathbb{E}|u(t)|^p}{\log \Upsilon(t)} \leq -\varepsilon_1$$

and

$$\limsup_{t \to \infty} \frac{\log |u(t)|^p}{\log \Upsilon(t)} \leq -\varepsilon_1 \quad a.s,$$

where ε_1 is given in Theorem 3.2.

Proof. Consider the function $\mathscr{V}\big(t, \tilde{u}(t), \wp(t), \rho(t)\big) = \flat_{\rho(t)} |\tilde{u}|^p$, and let $a_1 = \min_{i \in S_2}\{\flat_i\}$, $a_2 = \max_{i \in S_2}\{\flat_i\}$. Compute the operator $\mathscr{L}\mathscr{V}$ as follows, for $(\ell, i) \in S_1 \times S_2$,

$$\begin{aligned}
\mathscr{L}\mathscr{V}\big(t, u, v, \ell, i\big) &= p\flat_i |\tilde{u}|^{p-2} \tilde{u}^T f(t, u, v, i) + \frac{p}{2} \flat_i |\tilde{u}|^{p-2} \big(\hbar_2(t)\big)^2 |g(t, u, v, i)|^2 \\
&\quad + \frac{p(p-2)}{2} \flat_i |\tilde{u}|^{p-4} \big(\hbar_2(t)\big)^2 |\tilde{u}^T g(t, u, v, i)|^2 + \sum_{j=1}^N \gamma_{ij} \flat_j |\tilde{u}|^p \\
&\quad + \int_Y \Big[\flat_i \big|\tilde{u} + \hbar_3(t) h(t, u, v, i, y)\big|^p - \flat_i |\tilde{u}|^p\Big] \pi(dy) \\
&\leq p\flat_i |\tilde{u}|^{p-2} \tilde{u}^T f(t, u, v, i) + p\flat_i \frac{p-1}{2} |\tilde{u}|^{p-2} \big(\hbar_2(t)\big)^2 |g(t, u, v, i)|^2 \\
&\quad + \sum_{j=1}^N \gamma_{ij} \flat_j |\tilde{u}|^p + \flat_i \int_Y \Big[\big|\tilde{u} + \hbar_3(t) h(t, u, v, i, y)\big|^p - |\tilde{u}|^p\Big] \pi(dy) \\
&\leq p\flat_i |\tilde{u}|^{p-2} \Big(\tilde{u}^T f(t, u, v, i) + \frac{p-1}{2} \big(\hbar_2(t)\big)^2 |g(t, u, v, i)|^2\Big) \\
&\quad + \sum_{j=1}^N \gamma_{ij} \flat_j |\tilde{u}|^p + \flat_i \int_Y \Big[\big|\tilde{u} + \hbar_3(t) h(t, u, v, i, y)\big|^p - |\tilde{u}|^p\Big] \pi(dy).
\end{aligned}$$

By employing the inequality (3.23) of Assumption 3.1, we have

$$\mathscr{L}\mathscr{V}\left(t,u,v,\ell,i\right) \leq p\flat_i|\tilde{u}|^{p-2}c_{1i}^{\ell}(t)|u|^2 + p\flat_i|\tilde{u}|^{p-2}c_{2i}^{\ell}(t)|v|^2 + \sum_{j=1}^{N}\gamma_{ij}\flat_j|\tilde{u}|^p + \flat_i\int_Y\left[\left|\tilde{u}+\hbar_3(t)h(t,u,v,i,y)\right|^p - |\tilde{u}|^p\right]\pi(dy). \tag{3.27}$$

For any $p \geq 2$, one has

$$|u|^{p-2}|v|^2 \leq \frac{p-2}{p}|u|^p + \frac{2}{p}|v|^p,$$

$$\left|\tilde{u}\right|^p \leq (1+k)^{p-1}\left(|u|^p + k|v|^p\right),$$

and

$$-|\tilde{u}|^p \leq -(1-k)^{p-1}|u|^p + k(1-k)^{p-1}|v|^p.$$

Therefore, the first term of (3.27) is optimized as follows:

$$\begin{aligned}&p\flat_i c_{1i}^{\ell}(t)|\tilde{u}|^{p-2}|\tilde{u}|^2\\&\leq \Big((1+k)^{p-1}(p-2)+2\Big)\flat_i c_{1i}^{\ell}(t)|u|^p + k(1+k)^{p-1}(p-2)\flat_i c_{1i}^{\ell}(t)|v|^p.\end{aligned} \tag{3.28}$$

An upper bound of the second term of (3.27) is

$$\begin{aligned}&p\flat_i c_{2i}^{\ell}(t)|\tilde{u}|^{p-2}|v|^2\\&\leq (1+k)^{p-1}(p-2)\flat_i c_{2i}^{\ell}(t)|u|^p + \Big(k(1+k)^{p-1}(p-2)+2\Big)\flat_i c_{2i}^{\ell}(t)|v|^p.\end{aligned} \tag{3.29}$$

In view of $k \in (0,1)$, the third term has the following upper bound

$$\sum_{j=1}^{N}\gamma_{ij}\flat_j|\tilde{u}|^p \leq (1+k)^{p-1}\sum_{j=1}^{N}|\gamma_{ij}|\flat_j|u|^p + k(1+k)^{p-1}\sum_{j=1}^{N}|\gamma_{ij}|\flat_j|v|^p. \tag{3.30}$$

From inequality (3.24) and Assumption 3.2, one has

$$\begin{aligned}&\flat_i\int_Y\left[\left|\tilde{u}+\hbar_3(t)h(t,u,v,i,y)\right|^p - |\tilde{u}|^p\right]\pi(dy)\\&\leq \Big(\eta_i^{\ell}\beta_{1i}^{\ell}(t) - (1-k)^{p-1}\pi(Y)\Big)\flat_i|u|^p + \Big(\eta_i^{\ell}\beta_{2i}^{\ell}(t) + k(1-k)^{p-1}\pi(Y)\Big)\flat_i|v|^p\\&= \varpi_{1i}^{\ell}(t)\flat_i|u|^p + \varpi_{2i}^{\ell}(t)\flat_i|v|^p.\end{aligned} \tag{3.31}$$

Substituting (3.28)-(3.31) into (3.27) and employing Assumption 3.2, we obtain that

$$
\begin{aligned}
&\mathscr{L}\mathscr{V}\big(t,u,v,\ell,i\big)\\
&\leq \Bigg\{\Bigg[\Big((1+k)^{p-1}(p-2)+2\Big)c_{1i}^{\ell}(t)+(1+k)^{p-1}(p-2)c_{2i}^{\ell}(t)+\varpi_{1i}^{\ell}(t)\Bigg]\flat_i\\
&+(1+k)^{p-1}\sum_{j=1}^{N}|\gamma_{ij}|\flat_j\Bigg\}|u|^p\\
&+\Bigg\{\Bigg[k(1+k)^{p-1}(p-2)c_{1i}^{\ell}(t)+\Big(k(1+k)^{p-1}(p-2)+2\Big)c_{2i}^{\ell}(t)+\varpi_{2i}^{\ell}(t)\Bigg]\flat_i\\
&+k(1+k)^{p-1}\sum_{j=1}^{N}|\gamma_{ij}|\flat_j\Bigg\}|v|^p\\
&\leq \Bigg\{e_i^{\ell}\flat_i+(1+k)^{p-1}\sum_{j=1}^{N}|\gamma_{ij}|\flat_j+\flat_i\varpi_{1i}^{\ell}(t)\Bigg\}|u|^p\\
&+\Bigg\{m_i^{\ell}\flat_i+k(1+k)^{p-1}\sum_{j=1}^{N}|\gamma_{ij}|\flat_j\Bigg\}|v|^p\\
&=\Big[\flat_i\varpi_{1i}^{\ell}(t)-1\Big]|u|^p+\Big[\big(m_i^{\ell}-ke_i^{\ell}\big)\flat_i-k\Big]|v|^p=\Big(-c_{3i}^{\ell}|u|^p+c_{4i}^{\ell}|v|^p\Big)\frac{\Upsilon'(t)}{\Upsilon(t)}.
\end{aligned}
$$

Hence, from conditions (3.25) and (3.26), we derive that all the conditions of Lemma 2.1 are satisfied. From Theorems 3.2 and 3.3, one has

$$
\limsup_{t\to\infty}\frac{\log\mathbb{E}|u(t)|^p}{\log\Upsilon(t)}\leq-\varepsilon_1\quad\text{and}\quad\limsup_{t\to\infty}\frac{\log|u(t)|^p}{\log\Upsilon(t)}\leq-\varepsilon_1\quad a.s.
$$

where ε_1 is given in Theorem 3.2. This completes the proof. □

4. Numerical Simulation

Let $\wp(t)$ and $\rho(t)$ be two right-continuous Markov chain taking values in $S_1=\{1,2,3,4\}$ and $S_2=\{1,2\}$, respectively. Their infinitesimal generators are

$$
\Theta=\begin{bmatrix}-2 & 0.25 & 0.75 & 1\\ 2 & -3 & 0.5 & 0.5\\ 0.3 & 0.25 & -1 & 0.45\\ 1 & 1 & 2 & -4\end{bmatrix}\text{ and }\Gamma=\begin{bmatrix}-1 & 1\\ 8 & -8\end{bmatrix}.
$$

In this example, we take $p=2$, $\Upsilon(t)=e^t$, $t\geq 0$, and consider the scalar nonlinear delayed hybrid stochastic differential equations with Markovian switched noises

$$
\begin{aligned}
&d\big[u(t)-\mathscr{D}\big(u(t-\sigma_{t,\rho(t)}),\rho(t)\big)\big]\\
&=f\big(t,u(t),u(t-\sigma_{t,\rho(t)}),\rho(t)\big)dt\\
&\quad+\hbar_2(t)g\big(t,u(t),u(t-\sigma_{t,\rho(t)}),\rho(t)\big)dw(t)\\
&\quad+\hbar_3(t)\int_0^{+\infty}h\big(t,u(t),u(t-\sigma_{t,\rho(t)}),y,\rho(t)\big)N(dt,dy),
\end{aligned}\tag{4.1}
$$

with initial data $\xi = 1$, where the delay $\sigma_{t,\rho(t)} = \frac{0.15}{\rho(t)}\sin(t) + 0.15 \in [0, 0.3]$, the terms of equation (4.1) are defined by

$$\mathscr{D}(v,i) = \begin{cases} 0 & \text{if } i = 1 \\ \frac{1}{14}v & \text{if } i = 2, \end{cases} \qquad h(t,u,v,i,y) = \begin{cases} yz/2 & \text{if } i = 1 \\ xz/4 & \text{if } i = 2, \end{cases}$$

$$f(t,u,v,i) = \begin{cases} -3u - 2u^3 & \text{if } i = 1 \\ -(u - \frac{1}{14}v)u^4 - 9u & \text{if } i = 2, \end{cases} \qquad g(t,u,v,i) = \begin{cases} 2u^2 & \text{if } i = 1 \\ \sqrt{2}(u - \frac{1}{14}v)u^2 & \text{if } i = 2, \end{cases}$$

$w(t)$ is a one-dimensional Brownian motion and N is a one-dimensional Poisson random measure. The character measure π of the Poisson jump is given by $\pi(dy) := \frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}$. For any $\ell \in S_1, y \in [0,\infty)$ and $t \geq 0$, it is obvious to show that $k = \frac{1}{14}$, $k_1 = \frac{15}{14}$, $k_2 = \frac{14}{13}$,

$$c_{11}^{\ell}(t) = -3, \quad c_{21}^{\ell}(t) = 0, \quad c_{12}^{\ell}(t) = -8.6786, \quad c_{22}^{\ell}(t) = 0.3214,$$
$$\beta_{11}^{\ell}(t) = \beta_{21}^{\ell}(t) = \beta_{12}^{\ell}(t) = \beta_{22}^{\ell}(t) = 1,$$

$h_1^{\ell}(y) = 2 + \frac{y^2}{2}\hbar_3^2(t)$ and $h_2^{\ell}(y) = 4 + \frac{y^2}{4}\hbar_3^2(t) + 2k^2$. Therefore, we can easily check that

$$e_1^{\ell} = -6, \quad e_2^{\ell} = -17.357, \quad \eta_1^1 = \eta_1^3 = 1.25, \quad \eta_1^2 = \eta_1^4 = 1, \quad \eta_2^1 = \eta_2^3 = 2.1301$$
$$\text{and} \quad \eta_2^2 = \eta_2^4 = 1.0051.$$

Then, one has the matrix

$$\mathscr{A}^{\ell} = \begin{bmatrix} 4.9286 & -1.0714 \\ -8.5714 & 8.7857 \end{bmatrix},$$

which is a nonsingular M-matrix for any $\ell \in S_1$, $\flat = (0.28892, 0.39569)^T$ and the coefficients are

$$\varpi_{11}^1 = \varpi_{11}^3 = 0.78571, \quad \varpi_{11}^2 = \varpi_{11}^4 = 0.53571, \quad \varpi_{12}^1 = \varpi_{12}^3 = 1.6658, \quad \varpi_{12}^2 = \varpi_{12}^4 = 0.54082,$$
$$\varpi_{21}^1 = \varpi_{21}^3 = 1.2832, \quad \varpi_{21}^2 = \varpi_{21}^4 = 1.0332, \quad \varpi_{22}^1 = \varpi_{22}^3 = 2.1633, \quad \varpi_{22}^2 = \varpi_{22}^4 = 1.0383,$$
$$m_1^1 = m_1^3 = 1.2832, \quad m_1^2 = m_1^4 = 1.0332, \quad m_2^1 = m_2^3 = 2.8061, \quad m_2^2 = m_2^4 = 1.6811,$$
$$c_{31}^1 = c_{31}^3 = 0.77299, \quad c_{31}^2 = c_{31}^4 = 0.84522, \quad c_{32}^1 = c_{32}^3 = 0.51871, \quad c_{32}^2 = c_{32}^4 = 0.84375,$$
$$c_{41}^1 = c_{41}^3 = 0.60589, \quad c_{41}^2 = c_{41}^4 = 0.50697, \quad c_{42}^1 = c_{42}^3 = 1.5295, \quad c_{42}^2 = c_{42}^4 = 1.0844.$$

Further, it is not difficult to prove that all the conditions of Theorem 3.4 are fulfilled. Thus system (4.1) admits a unique global solution $u(t), t \geq -0.3$. Moreover, these solutions are exponentially stable in mean squares and almost surely exponentially stable.

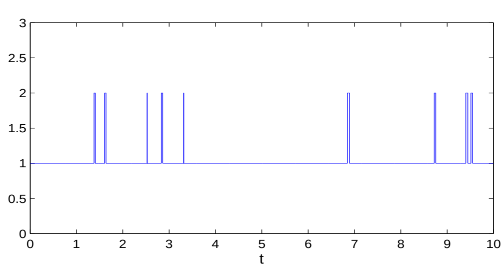

FIGURE 1. Trajectory of Markov chain $\rho(t)$ with $\rho(0) = 1$.

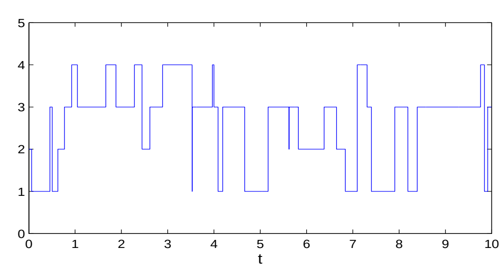

FIGURE 2. Trajectory of Markov chain $\wp(t)$ with $\wp(0) = 2$.

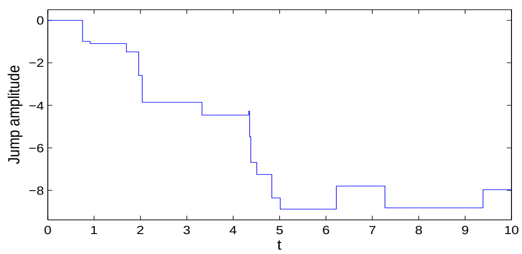

FIGURE 3. Trajectory of Poisson process normally distributed

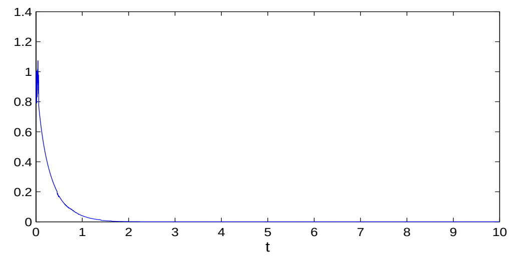

FIGURE 4. Trajectory of the almost surely exponential stability for the solution of system (4.1)

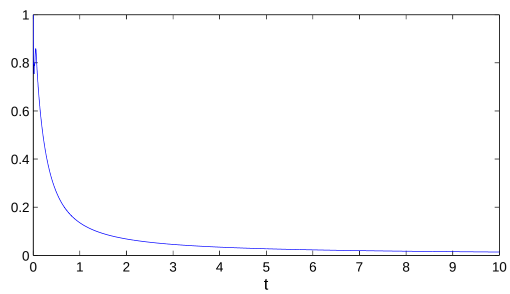

FIGURE 5. Trajectory of the exponential stability in mean square for the solution of system (4.1)

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