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STABILITY AND EXISTENCE OF SOLUTIONS FOR WEAK BILEVEL OPTIMIZATION PROBLEMS

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Abstract. In this paper, we are interested in finding sufficient conditions, which ensure the existence of solutions to a weak (pessimistic) bilevel optimization problem (S) in a topological sequential setting. To this aim, we consider an approach of (S) by means of a family of regularized perturbed bilevel optimization problems (S_n) of (S). Unlike (S), such problems (S_n) have the privilege to admit solutions under mild assumptions. Using the notion of variational convergence, after establishing stability results, we prove the existence of solutions to problem (S).
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1. INTRODUCTION

Let U and V be two Hausdorff topological spaces, and let X and Y be two nonempty subsets of U and V, respectively. Let $F, f : U \times V \to \mathbb{R}$ be real valued functions. We are concerned with the following weak (in the sense of [11] and [24]) nonlinear bilevel optimization problem

$$(S): \qquad \min_{x\in X} \sup_{y\in \mathscr{M}(x)} F(x,y),$$

where $\mathscr{M}(x)$ is the solution set of the parameterized problem

$$\mathscr{P}(x): \qquad \min_{y\in Y} f(x,y).$$

Bilevel optimization problems have various applications, such as, economics, optimal taxation, engineering design, ecology, system planning and transportation, and so on; see, e.g., [9, 13, 29] and the references therein. In terms of game theory, problem (S) corresponds to a non-zero-sum noncooperative game where a leader plays against a follower. The leader disposing of full information about the follower's constraints and objective function f announces first a strategy $x \in X$ to minimize his objective function F, and the follower responds rationally by choosing a strategy $y(x) \in Y$ to minimize his objective function f. When $\widehat{X} = \{x \in X :$

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$\mathscr{M}(x)$ is not a singleton$\}$ is nonempty, then, for $x \in \widehat{X}$, any strategy $y(x) \in \mathscr{M}(x)$ is suitable for the follower, but it can generate a worst case scenario for the leader. Assume that the leader adopts a pessimistic attitude in the game. Then, having the possibility to anticipate in the game but not in the follower's choice, he provides himself by minimizing the marginal function $\sup_{y\in\mathscr{M}(x)} F(x,y)$ over his constraint set X. From the viewpoint of the leader's attitude, (S) is also called a pessimistic bilevel optimization problem. In terms of hierarchical game, the leader's problem (S) is called the first or upper level problem, and for an announced strategy x by the leader, the follower's problem $\mathscr{P}(x)$ is called the lower level problem.

As it is well known, weak nonlinear bilevel optimization problems present difficulties in their theoretical and numerical studies; see, e.g., [14, 16, 23, 26]. Next, we give a survey on these two aspects of studies. Concerning the theoretical aspect, we will especially focus our attention on the works dealing with the existence of solutions. We note that the search for sufficient conditions ensuring the existence of solutions for such a class of problems is a difficult task. The difficulty is mainly due to the presence of the constraint solution set $\mathscr{M}(x)$ (which is an output of problem $\mathscr{P}(x)$) in the first level. In contrast, bilevel optimization problems with the formulation

$$\min_{x\in X} \inf_{y\in\mathscr{M}(x)} F(x,y)$$

can admit solutions under mild assumptions (see [9] and [13]). Such problems are called strong (or optimistic) bilevel optimization problems and correspond to the case where both players cooperate. For results on the existence of solutions to weak bilevel optimization problems, we refer to [2, 3, 4, 5, 6, 25]. Let us summarize the main results obtained in these works. In the finite dimensional case, in [2] and [3], sufficient conditions based on convexity properties were provided to guarantee the existence of solutions to problem (S). For $x \in X$, consider in the first level the following relaxed auxiliary problem

$$\mathscr{P}_F(x): \qquad \max_{y\in Y} F(x,y).$$

Then, in [3], it was shown that the problem of existence of solutions to (S) can be reduced to the existence of a common solution of the two convex parameterized problems $\mathscr{P}(x)$ and $\mathscr{P}_F(x)$ for every strategy $x \in X$. The convexity of the two problems results from the data. In [4], the existence of solutions was established for some classes of weak linear bilevel programming problems via a penalty method. In [5] and [6], the existence of solutions to weak bilevel optimization problems was established respectively via the existence of solutions to "MinSup and D.C. problems" and "reverse convex and convex maximization problems". In [25], sufficient conditions were given for the existence of solutions to a class of weak bilevel optimization problems, where the follower's objective function was weakly analytic. Other works dealing with optimality conditions and approximation for weak bilevel optimization problems can be found in [1, 12, 13, 14, 19, 20, 24].

Likewise, the numerical study of the class of weak bilevel optimization problems is also a difficult task. Let us summarize some numerical approaches in the literature for this class of problems. Note first that the most of them concern only the linear case. In [4], using an exact penalty method, Aboussoror and Mansouri transformed the linear pessimistic bilevel optimization problem into a single-level optimization one. Then, under appropriate assumptions, they showed the existence of solutions to the bilevel problem and proposed an algorithm. However, no numerical results were given. Dempe et al. investigated in [15] a weak bilevel linear optimization problem. By means of duality and the follower's value function, they transformed the bilevel problem to a single one. Then, they proposed algorithms for computing global and local solutions. A numerical example was given for illustration. In [16], the authors considered the weak formulation of a bilevel electricity tariff optimization problem for demand response management. For such a problem, they provided an algorithm with numerical examples. A comparison with the strong (optimistic) formulation was given. In [27], Malyshev and Strekalovsky were interested in solving a quadratic-linear weak bilevel optimization problem. They first reduced it to a sequence of strong bilevel optimization problems. Then, replacing the lower level problem by the corresponding Karush-Kuhn-Tucker conditions and using a penalty method, they reduced each of these strong bilevel problems to a sequence of nonconvex single level problems. Finally, by means of this approach, they developed global and local search algorithms with numerical examples. Following the study and the penalty method considered in [4], Zheng et al. presented in [30] a new variant of the penalty method to solve the linear weak bilevel programming problem. An algorithm was then given with numerical examples for illustration. Zheng et al. in [31] studied a weak bilevel programming problem in which the follower's solution set was discrete. After transforming it into a one level optimization problem, they presented a maximum entropy approach for its resolution. Numerical examples were given for illustration.

Let us return to the theoretical framework. In this paper, via a theoretical approach, we give sufficient conditions, which guarantee the existence of solutions to a class of weak bilevel optimization problems of type (S). To this aim, we consider a family of weak regularized perturbed bilevel optimization problems (S_n) of (S). These problems, which have the privilege of admitting solutions under mild assumptions, are constructed by means of a perturbation and a regularization. The perturbation is done on the objective functions of the two players, and the regularization consists in substituting the constraint solution set $\mathscr{M}(x)$ in the first level, by the sequential closure of the set of approximate strict ε-solutions ($\varepsilon > 0$) of the perturbed lower level problem. Then, we establish some fundamental convergence results concerning the perturbed and the original lower levels. Similarly, other intermediate stability results are established for the first level. Finally, using these results and the notion of variational convergence, we show under additional appropriate assumptions that any accumulation point of a sequence of solutions of regularized perturbed problems is a solution to original problem (S). The obtained result is an extension of those given in [7, 21, 22, 26, 28] for Stackelberg problems with unique lower level solutions. Moreover, our result gives an extension with improvement of the previous results established in [2] and [3] where convexity assumptions are required. The extension is from the finite dimensional case to a general topological one, and our result is obtained without resorting to convexity. Throughout the paper, illustrative examples are also given.

The outline of the paper is as follows. In Section 2, we recall some definitions and results that will be used in our study. In Section 3, we introduce the family of regularized perturbed problems and establish some fundamental results concerning the lower level, that will be used in the sequel. In Section 4, we establish our main result on the existence of solutions to the original problem (S). Finally, we end this paper by Section 5.

2. Preliminaries

Throughout the paper, sets X and Y are endowed with the induced topologies of U and V respectively. We recall the following definition concerning sequential limits of sets ([17]).

Definition 2.1. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of nonempty subsets of a Hausdorff topological space $\mathscr{W}$. The sequential $\liminf$ and $\limsup$ of the sequence $(A_n)_n$ are the sets defined as follows

1) $\liminf_{n\to+\infty} A_n = \Big\{y \in \mathscr{W} \,/\, \exists y_n \to y, \text{ as } n \to +\infty, y_n \in A_n, \forall n \in \mathbb{N}\Big\}$,
2) $\limsup_{n\to+\infty} A_n = \Big\{y \in \mathscr{W} \,/\, \exists y_{n_k} \to y, \text{ as } k \to +\infty, y_{n_k} \in A_{n_k}, \forall k \in \mathbb{N}\Big\}$.

In the sequel, for a subset A of a Hausdorff topological space $\mathscr{W}$, $\overline{A}$ and $\overline{A}^{seq}$ denote respectively the topological and the sequential closure of A in $\mathscr{W}$, where we recall that

$$\overline{A}^{seq} = \Big\{x \in \mathscr{W} : \exists (x_k) \subset A, x_k \to x, \text{ as } k \to +\infty\Big\},$$

i.e., the set of limits of all converging sequences of A. Set A is called sequentially closed if $A = \overline{A}^{seq}$. Then, we have $\overline{A}^{seq} \subset \overline{A}$. When the topological space is first countable, the two notions of closure coincide (see [17]). Set A is said to be sequentially compact if any sequence in A admits a subsequence converging to a point of A.

Remark 2.1. We recall that, in a Hausdorff topological space $\mathscr{W}$, the topological $\liminf$ and $\limsup$ of sets are closed ([10]) (and hence sequentially closed). When $\mathscr{W}$ is a first countable topological space, the topological and sequential definitions of $\liminf$ and $\limsup$ of sets coincide ([17]).

For the convenience of the reader, we recall the following definition and some fundamental results on the convergence of solution sets of optimization problems based on the use of variational convergence.

Definition 2.2. ([8, 32]) Let $\varphi_n, \varphi : Y \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, be functions. We say that the sequence (φ_n) variationally converges to φ, if the following properties are satisfied

i) For any $y \in Y$ and any sequence (y_n) converging to y in Y, we have $\liminf_{n\to+\infty} \varphi_n(y_n) \geq \varphi(y)$,
ii) For any $y \in Y$, there exists a sequence (y_n) in Y, such that $\limsup_{n\to+\infty} \varphi_n(y_n) \leq \varphi(y)$.

Consider the following minimization problems

$$(\mathscr{R}_n): \quad \min_{y\in Y} \varphi_n(y), \qquad (\mathscr{R}): \quad \min_{y\in Y} \varphi(y).$$

Let $\operatorname{Argmin}\mathscr{R}_n$ and $\operatorname{Argmin}\mathscr{R}$ denote, respectively, their solution sets. For $\varepsilon > 0$, let ε-$\operatorname{Argmin}\mathscr{R}_n$ and ε-$\operatorname{Argmin}\mathscr{R}$ denote, respectively, their sets of approximate ε-solutions. Then, we have the following well known fundamental convergence results.

Theorem 2.1. ([8, 32]) Assume that the sequence (φ_n) variationally converges to φ. Then,
i) $\limsup_{n\to+\infty} \operatorname{Argmin}\mathscr{R}_n \subset \operatorname{Argmin}\mathscr{R}$,
ii) for any sequence $\varepsilon_n \searrow 0^+$, $\limsup_{n\to+\infty} \varepsilon_n$-$\operatorname{Argmin}\mathscr{R}_n \subset \operatorname{Argmin}\mathscr{R}$.

Remark 2.2. In the case that V is a first countable Hausdorff topological space, the result of Theorem 2.1 means that any accumulation point of a sequence of solutions or approximate ε_n-solutions of problems $(\mathscr{R}_n)$, $n \in \mathbb{N}$, is a solution to problem $(\mathscr{R})$.

3. A Regularization and Perturbation Approach

In this section, we first introduce a regularized perturbed problem (S_n) of (S), and then under appropriate mild assumptions, we prove the existence of solutions to problem (S_n). This result is fundamental to establish the existence of solutions to original problem (S).

3.1. **The regularization and perturbation approach.** Let us consider sequences of perturbations (F_n) and (f_n) of leader's and follower's objective functions F and f respectively, with $F_n, f_n : X \times Y \to \mathbb{R}$, $n \in \mathbb{N}$. Let $\mathscr{M}_n(x)$ and $\mathscr{M}_n^s(x,\varepsilon)$, $\varepsilon > 0$, denote respectively the set of solutions and the set of approximate strict ε-solutions of the following perturbed parameterized problem

$$\mathscr{P}_n(x): \qquad \min_{y \in Y} f_n(x,y),$$

and let

$$v_n(x) = \inf_{y \in Y} f_n(x,y)$$

denote its infimal value. If $v_n(x)$ is a finite real number, then $\mathscr{M}_n^s(x,\varepsilon)$ has the following expression

$$\mathscr{M}_n^s(x,\varepsilon) = \left\{ y \in Y \,/\, f_n(x,y) < v_n(x) + \varepsilon \right\}.$$

Let $\varepsilon_n \searrow 0^+$ be a fixed sequence in the rest of the paper, and consider the following regularized perturbed problem of (S)

$$(S_n): \qquad \min_{x \in X} \sup_{y \in \overline{\mathscr{M}_n^s(x,\varepsilon_n)}^{seq}} F_n(x,y).$$

The regularization consists in substituting the solution set $\mathscr{M}(x)$ by the set $\overline{\mathscr{M}_n^s(x,\varepsilon_n)}^{seq}$.

We will use the following assumptions:

- **(3.1)** For any $n \in \mathbb{N}$, f_n satisfies the following property:
  For any $(x,y) \in X \times Y$, and any sequence (x_k) converging to x in X, there exists a sequence (y_k) converging to y in Y such that $\limsup_{k \to +\infty} f_n(x_k,y_k) \leq f_n(x,y)$.
- **(3.2)** For any $n \in \mathbb{N}$, f_n is sequentially lower semicontinuous on $X \times Y$.
- **(3.3)** For any $n \in \mathbb{N}$, F_n is sequentially lower semicontinuous on $X \times Y$.
- **(3.4)** For any $(x,y) \in X \times Y$, and any sequence (x_n) converging to x in X, there exists a sequence (y_n) converging to y in Y such that $\limsup_{n \to +\infty} f_n(x_n,y_n) \leq f(x,y)$.
- **(3.5)** For any $(x,y) \in X \times Y$, and any sequence (x_n,y_n) converging to $(x,y) \in X \times Y$, we have

$$\liminf_{n \to +\infty} f_n(x_n,y_n) \geq f(x,y).$$

Let us give the following remarks concerning the above assumptions which are useful in the sequel.

Remark 3.1. *i)* Under assumption (3.2) and the sequential compactness of Y, for any $(n,x) \in \mathbb{N} \times X$, $v_n(x)$ is a finite real number, and $\mathscr{M}_n(x)$ is a sequentially compact set.

ii) Let (x_n) be a sequence converging to x in X. Then, assumptions (3.4) and (3.5) imply that $(f_n(x_n,.))_n$ epiconverges to $f(x,.)$ ([8]). Such assumptions were used in several works in different contexts (see, for example, [1, 18, 19, 21, 22, 26]). They were principally used in a sequential setting and they served in general to show stability results.

iii) In a first countable Hausdorff topological space, the notions of topological and sequential "continuity, lower, and upper semicontinuities" coincide ([17]).

iv) It is easy to see that if a function is sequentially upper semicontinuous, then it satisfies assumption (3.1). The converse is not true in general as we see in the following example.

Example 3.1. Let $X = Y = [1,2]$, and h be the function defined on $\mathbb{R} \times \mathbb{R}$ by

$$h(x,y) = \begin{cases} -x^2 - 1, & \text{if } y = 1, \\ -(y-x)^2, & \text{if } y \neq 1. \end{cases}$$

Then, we can easily verify that h is not upper semicontinuous at $(x,1)$ for any $x \in X$, but it satisfies assumption (3.1). In fact

i) if $y = 1$, we choose $y_k = 1$ for all $k \in \mathbb{N}$,

ii) if $y \in]1,2]$, then any sequence $(y_k) \subset [1,2]$ converging to y is suitable.

3.2. **Convergence results of the lower level.** In this subsection, we establish some convergence results concerning principally the solution and approximate solution sets of the original and the perturbed lower level problems $\mathscr{P}(x)$ and $\mathscr{P}_n(x)$, respectively.

Proposition 3.1. *Let $n \in \mathbb{N}$. Assume that assumptions (3.1) and (3.2) hold. If moreover, set Y is sequentially compact, then, for any $x \in X$ and any sequence (x_k) converging to x in X,*

1) $\lim_{k\to+\infty} v_n(x_k) = v_n(x)$,

2) $\overline{\mathscr{M}_n^s(x,\varepsilon_n)}^{seq} \subset \liminf_{k\to+\infty} \overline{\mathscr{M}_n^s(x_k,\varepsilon_n)}^{seq}$, i.e., the multifunction $\overline{\mathscr{M}_n^s(.,\varepsilon_n)}^{seq}$ is sequentially lower semicontinuous on X.

Proof. 1) First, note that $v_n(x)$ is a finite real number (Remark 3.1). Let $y \in Y$. Assumption (3.1) implies that there exists a sequence $(y_k) \subset Y$ converging to y in Y such that

$$\limsup_{k\to+\infty} f_n(x_k,y_k) \le f_n(x,y).$$

It follows that

$$\limsup_{k\to+\infty} v_n(x_k) \le \limsup_{k\to+\infty} f_n(x_k,y_k) \le f_n(x,y).$$

Since y is arbitrary in Y, one sees that $\limsup_{k\to+\infty} v_n(x_k) \le v_n(x)$. Now, let us show that $\liminf_{k\to+\infty} v_n(x_k) \ge v_n(x)$. Assume the contrary that there exists $\alpha \in \mathbb{R}$ such that

$$\liminf_{k\to+\infty} v_n(x_k) < \alpha < v_n(x). \tag{1}$$

Set $\liminf_{k\to+\infty} v_n(x_k) = \lim_{\substack{k\to+\infty \\ k\in\mathscr{N}}} v_n(x_k)$, where $\mathscr{N}$ is an infinite subset of $\mathbb{N}$. Then, there exists $k_0 \in \mathscr{N}$ such that $v_n(x_k) < \alpha$, for all $k \ge k_0$, $k \in \mathscr{N}$. Hence, for all $k \ge k_0$ and $k \in \mathscr{N}$, there exists $y_k \in Y$ such that $f_n(x_k,y_k) < \alpha$. Since Y is sequentially compact, then there exists an infinite subset $\mathscr{N}_1 \subset \mathscr{N}_0$ such that $(y_k)_{k\in\mathscr{N}_1}$ converges to a point $\bar{y} \in Y$. Then, using assumption (3.2), we obtain

$$v_n(x) \le f_n(x,\bar{y}) \le \liminf_{k\to+\infty} f_n(x_k,y_k) \le \alpha,$$

which gives a contradiction to the last strict inequality in (1). Hence $\liminf_{k\to+\infty} v_n(x_k) \ge v_n(x)$. We conclude that $\lim_{k\to+\infty} v_n(x_k) = v_n(x)$.

2) Let $y \in \mathscr{M}_n^s(x,\varepsilon_n)$, i.e., $f_n(x,y) - v_n(x) < \varepsilon_n$. From assumption (3.1), there exists a sequence $(y_k) \subset Y$, converging to y in Y such that $\limsup_{k\to+\infty} f_n(x_k,y_k) \le f_n(x,y)$. Assumption (3.2) implies that $\lim_{k\to+\infty} f_n(x_k,y_k) = f_n(x,y)$. Besides, we have $\lim_{k\to+\infty} v_n(x_k) = v_n(x)$. Then

$$\lim_{k\to+\infty} (f_n(x_k,y_k) - v_n(x_k)) = f_n(x,y) - v_n(x) < \varepsilon_n,$$

and hence $f_n(x_k,y_k) < v_n(x_k)+\varepsilon_n$ for large k. That is, $y_k \in \mathscr{M}_n^s(x,\varepsilon_n)$ for large k. It follows that

$$\mathscr{M}_n^s(x,\varepsilon_n) \subset \liminf_{k\to+\infty} \mathscr{M}_n^s(x_k,\varepsilon_n) \subset \liminf_{k\to+\infty} \overline{\mathscr{M}_n^s(x_k,\varepsilon_n)}^{seq}.$$

Since $\liminf_{k\to+\infty} \overline{\mathscr{M}_n^s(x_k,\varepsilon_n)}^{seq}$ is closed (Remark 2.1) and hence sequentially closed, then

$$\overline{\mathscr{M}_n^s(x,\varepsilon)}^{seq} \subset \liminf_{k\to+\infty} \overline{\mathscr{M}_n^s(x_k,\varepsilon_n)}^{seq}.$$

□

Recall that $\varepsilon_n \searrow 0^+$ is a sequence fixed throughout the paper. Define on X the marginal function $w_n(.)$ by

$$w_n(x) = \sup_{y\in\overline{\mathscr{M}_n^s(x,\varepsilon_n)}^{seq}} F_n(x,y).$$

Then, problem (S_n) is written as

$$(S_n): \qquad \min_{x\in X} w_n(x).$$

We obtain the following result on the existence of solutions to problem (S_n).

Theorem 3.1. *Let $n\in\mathbb{N}$. Assume that X and Y are sequentially compact and that assumptions (3.1)-(3.3) are satisfied. Then, problem (S_n) has at least one solution.*

Proof. Let us show that the marginal function $w_n(.)$ is sequentially lower semicontinuous on X sequentially compact. Let $x\in X$ and (x_k) be a sequence converging to x in X. Let $y \in \overline{\mathscr{M}_n^s(x,\varepsilon_n)}^{seq}$. From property 2) of Proposition 3.1, there exists a sequence (y_k) converging to y, with $y_k \in \overline{\mathscr{M}_n^s(x_k,\varepsilon_n)}^{seq}$, for large k. Then, $w_n(x_k)\geq F_n(x_k,y_k)$ for large k. By using assumption (3.3), we obtain

$$\liminf_{k\to+\infty} w_n(x_k) \;\; \geq \;\; \liminf_{k\to+\infty} F_n(x_k,y_k) \geq F_n(x,y).$$

Since y is arbitrary in $\overline{\mathscr{M}_n^s(x,\varepsilon_n)}^{seq}$, we deduce that

$$\liminf_{k\to+\infty} w_n(x_k) \;\; \geq \;\; \sup_{y\in\overline{\mathscr{M}_n^s(x,\varepsilon_n)}^{seq}} F_n(x,y) = w_n(x).$$

That is, marginal function $w_n(.)$ is sequentially lower semicontinuous on X. Then, the result follows from the sequential compactness of X immediately. □

Let $(x,n)\in X\times\mathbb{N}$. Let $v(x)$ denote the infimal value of the lower level problem $\mathscr{P}(x)$, and let $\mathscr{M}_n(x,\varepsilon_n)$ denote the set of approximate ε_n-solutions of problem $\mathscr{P}_n(x)$. Assume that its infimal value $v_n(x)$ is a finite real number. In this case, $\mathscr{M}_n(x,\varepsilon_n)$ has the following expression

$$\mathscr{M}_n(x,\varepsilon_n) = \left\{ y\in Y : f_n(x,y) \leq v_n(x)+\varepsilon_n \right\}.$$

Then, we have the following convergence result.

Proposition 3.2. *Assume that assumptions (3.4) and (3.5) hold. If, moreover, the set Y is sequentially compact, then, for any $x\in X$ and any sequence (x_n) converging to x in X,*

1) $\lim_{n\to+\infty} v_n(x_n) = v(x)$,

2) $\limsup_{n\to+\infty} \mathscr{M}_n(x_n,\varepsilon_n) \subset \mathscr{M}(x)$, *i.e., the sequence of multifunctions $(\mathscr{M}_n(.,\varepsilon_n))_n$ is sequentially upper convergent to the multifunction $\mathscr{M}(.)$ ([18]).*

Proof. 1) The proof is similar to the proof of 1) of Proposition 3.1. So it is omitted here.

2) Let $y \in \limsup_{k\to+\infty} \mathscr{M}_n(x_n,\varepsilon_n)$. There exists a subsequence $(y_n)_{n\in\mathscr{N}}$ converging to y, with $y_n \in \mathscr{M}_n(x_n,\varepsilon_n)$, for all $n\in\mathscr{N}$, where $\mathscr{N}$ is an infinite subset of $\mathbb{N}$. Then

$$f_n(x_n,y_n) \le v_n(x_n)+\varepsilon_n,\ \forall n\in\mathscr{N}.$$

Assumption (3.5) implies that

$$f(x,y) \quad \le \quad \liminf_{n\to+\infty} f_n(x_n,y_n) \le \limsup_{n\to+\infty} f_n(x_n,y_n) \le \limsup_{n\to+\infty}(v_n(x_n)+\varepsilon_n) \le v(x).$$

That is $y\in\mathscr{M}(x)$. □

Corollary 3.1. *Assume that Y is sequentially compact and that assumptions (3.4), (3.5) and the following assumption are satisfied*

(3.6) *for any $(x,n)\in X\times\mathbb{N}$, function $f_n(x,.)$ is sequentially lower semicontinuous on Y.*

Then, for any $x\in X$ and any sequence (x_n) converging to x in X,

$$\limsup_{n\to+\infty} \overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq} \subset \mathscr{M}(x).$$

Proof. Assumption (3.6) implies that set $\mathscr{M}_n(x_n,\varepsilon_n)$ is sequentially closed. Hence, $\overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq} \subset \mathscr{M}_n(x_n,\varepsilon_n)$. Then, Proposition 3.2 implies that

$$\limsup_{n\to+\infty} \overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq} \subset \mathscr{M}(x).$$

□

Note that assumption (3.2) implies assumption (3.6). For $x\in X$, consider in the upper level the following relaxed problem

$$\mathscr{P}_F(x): \quad \max_{y\in Y} F(x,y),$$

and let $\mathscr{M}_F(x)$ denote its solution set. The term relaxed is used in the sense that the constraint solution set $\mathscr{M}(x)$ is substituted by the set Y.

Proposition 3.3. *Assume that assumption (3.6) and the following assumptions are satisfied*

(3.7) *for any $(x,y)\in X\times Y$, and any sequence (x_n) converging to x in X, there exists a sequence (y_n) in Y such that $\limsup_{n\to+\infty} f_n(x_n,y_n) \le -F(x,y)$,*

(3.8) *for any $(x,y)\in X\times Y$, and any sequence (x_n,y_n) converging to $(x,y)\in X\times Y$,*

$$\liminf_{n\to+\infty} f_n(x_n,y_n) \ge -F(x,y).$$

Then, for any $x\in X$ and any sequence (x_n) converging to x in X,

$$\limsup_{n\to+\infty} \overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq} \subset \mathscr{M}_F(x).$$

Proof. By assumptions (3.7) and (3.8), one sees that $(f_n(x_n,.))_n$ variationally converges to $\widehat{F}(x,.) = -F(x,.)$. Let $\mathscr{M}_{\widehat{F}}(x)$ denote the solution set of problem

$$\mathscr{P}_{\widehat{F}}(x): \qquad \min_{y\in Y} \widehat{F}(x,y).$$

Then, Theorem 2.1 implies that

$$\limsup_{n\to+\infty} \mathscr{M}_n(x_n,\varepsilon_n) \subset \mathscr{M}_{\widehat{F}}(x) = \mathscr{M}_F(x).$$

Since $\mathscr{M}_n^s(x_n,\varepsilon_n)\subset\mathscr{M}_n(x_n,\varepsilon_n)$, which is sequentially closed, then $\overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq}\subset\mathscr{M}_n(x_n,\varepsilon_n)$. It follows that

$$\limsup_{n\to+\infty}\overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq}\subset\limsup_{n\to+\infty}\mathscr{M}_n(x_n,\varepsilon_n)\subset\mathscr{M}_F(x).$$

□

Remark 3.2. *i*) Note that, in the rule of this game, it is assumed that the leader has full information about the follower. Moreover, assume that the leader knows the perturbation on the data f and F. Then, he has the possibility to verify the fulfillment of the properties in (3.7) and (3.8).

ii) The role of assumptions (3.7) and (3.8) is twofold. On the one hand, they link the first and the second levels. On the other hand, they serve to establish stability results. They are similar to assumptions (3.4) and (3.5), except that (3.7) is weaker than (3.4) concerning the fact that this latter requires the convergence of the sequence (y_n). Note that assumptions (3.4) and (3.5), which were widely used (see, e.g., [1, 18, 21, 22, 26]), are in particular satisfied in the case of barrier and penalty methods (see [22, 28]).

We consider the following example borrowed from [22] where assumptions (3.7) and (3.8) are satisfied.

Example 3.2. Let $f_n,F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $n\in\mathbb{N}^*$ be the functions defined by

$$\text{if } x\le 0,\quad f_n(x,y)=0,\ \text{for } y\in\mathbb{R},$$

$$\text{if } x>0,\quad f_n(x,y)=\begin{cases}0, & \text{for } y\le 0,\\ -ny\exp\left[-\left(\dfrac{ny}{x}\right)^2\right], & \text{for } y>0,\end{cases}$$

and

$$F(x,y)=\begin{cases}\dfrac{x}{\sqrt{2\exp(1)}}, & \text{for } y=0 \text{ and } x>0,\\ 0, & \text{otherwise.}\end{cases}$$

Then, (f_n) and $-F$ satisfy assumptions (3.7) and (3.8).

4. Existence of Solutions to Problem (S)

In this section, using the convergence results concerning the first and second levels, we establish our main result on the existence of solutions to the original weak bilevel optimization problem (S). The notion of variational convergence plays an important role in establishing this result.

In the following, for a subset A of U, ψ_A denotes the indicator function of the set A, i.e.,

$$\psi_A(x)=\begin{cases}0, & \text{if } x\in A,\\ +\infty, & \text{if } x\notin A.\end{cases}$$

The following results of Proposition 4.1 and Corollary 4.1 will be established for arbitrary $x\in X$ and $(x_n)\subset X$, with (x_n) a sequence converging to x in X. Such results are fundamental to obtain our main result on the existence of solutions to problem (S).

Let $x \in X$, and $(x_n)_n$ be a sequence converging to x in X. Consider the following auxiliary maximization problems in the first level

$$S(x): \quad \max_{y \in \mathscr{M}(x)} F(x,y), \qquad \text{and} \qquad S_n(x_n): \quad \max_{y \in \overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq}} F_n(x_n,y),$$

which are respectively equivalent to the following minimization problems

$$\mathscr{Q}(x): \qquad \min_{y \in Y}\{-F(x,y) + \psi_{\mathscr{M}(x)}(y)\},$$

and

$$\mathscr{Q}_n(x_n): \qquad \min_{y \in Y}\{-F_n(x_n,y) + \psi_{\overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq}}(y)\}.$$

The equivalence is in the sense that $S(x)$ (resp. $S_n(x_n)$) and $\mathscr{Q}(x)$ (resp. $\mathscr{Q}_n(x_n)$) have the same solution set and opposite optimal values. Define the functions $\phi(.)$ and $\phi_n(.)$ on Y by

$$\phi(y) = -F(x,y) + \psi_{\mathscr{M}(x)}(y),$$

and

$$\phi_n(y) = -F_n(x_n,y) + \psi_{\overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq}}(y).$$

Then we have the following convergence result.

Proposition 4.1. *Let Y be sequentially compact and assume that assumptions (3.4)-(3.8) and the following assumption are satisfied*

(4.1) *$(F_n)_n$ sequentially continuously converges to F.*

Then, (ϕ_n) variationally converges to ϕ.

Proof. 1) Let $\bar{y} \in Y$ and $(\bar{y}_n)$ be a sequence converging to $\bar{y}$ in Y. Let us show that

$$\liminf_{n \to +\infty} \phi_n(\bar{y}_n) \geq \phi(\bar{y}).$$

We prove in the following cases.

i) If $\bar{y} \in \mathscr{M}(x)$, then $\phi(\bar{y}) = -F(x,\bar{y})$ and

$$\begin{aligned}
\liminf_{n \to +\infty} \phi_n(\bar{y}_n) &= \liminf_{n \to +\infty}\{-F_n(x_n,\bar{y}_n) + \psi_{\overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq}}(\bar{y}_n)\} \\
&\geq \liminf_{n \to +\infty}(-F_n(x_n,\bar{y}_n)) \\
&= \lim_{n \to +\infty}(-F_n(x_n,\bar{y}_n)) \\
&= -F(x,\bar{y}) = \phi(\bar{y}).
\end{aligned}$$

ii) If $\bar{y} \notin \mathscr{M}(x)$, then $\phi(\bar{y}) = +\infty$. Moreover, we have $\bar{y}_n \notin \overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq}$ for large $n \in \mathbb{N}$. Otherwise, there exists an infinite subset $\mathscr{N}_1 \subset \mathbb{N}$ such that $\bar{y}_n \in \overline{\mathscr{M}_n^s(\varepsilon_n,x_n)}^{seq}$ for all $n \in \mathscr{N}_1$. Then, from Corollary 3.1, we have

$$\bar{y} \in \limsup_{k \to +\infty} \overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq} \subset \mathscr{M}(x),$$

which gives a contradiction to the fact that $\bar{y} \notin \mathscr{M}(x)$, which implies $y_n \notin \overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq}$ for large $n \in \mathbb{N}$. Hence, $\phi_n(\bar{y}_n) = +\infty$ for large $n \in \mathbb{N}$, and the result follows.

2) Now, let $\bar{y} \in Y$. We show that there exists a sequence (y_n^*) in Y such that $\limsup_{n \to +\infty} \phi_n(y_n^*) \leq \phi(\bar{y})$. Then, we have two cases to consider:

i) If $\bar{y} \notin \mathscr{M}(x)$, then, $\phi(\bar{y}) = +\infty$, and the result is obvious.

ii) If $\bar{y} \in \mathscr{M}(x)$, then $\phi(\bar{y}) = -F(x,\bar{y})$. For $n \in \mathbb{N}$, let $y_n^* \in \overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq}$. Then, $\phi_n(y_n^*) = -F_n(x_n,y_n^*)$. Let $\mathscr{N}_2$ be an infinite subset of $\mathbb{N}$ such that

$$\limsup_{n\to+\infty} \phi_n(y_n^*) = \limsup_{n\to+\infty}(-F_n(x_n,y_n^*)) = \lim_{\substack{n\to+\infty\\ n\in\mathscr{N}_2}} (-F_n(x_n,y_n^*)).$$

Using the sequential compactness of Y, there exists an infinite subset $\mathscr{N}_3$ of $\mathscr{N}_2$ such that $y_n^* \to y^*$ as $n \to +\infty$, $n \in \mathscr{N}_3$. Then, $y^* \in \limsup_{n\to+\infty} \overline{\mathscr{M}_n^s(x_n,\varepsilon_n)}^{seq}$, and Proposition 3.3 implies that $y^* \in \mathscr{M}_F(x)$. That is, y^* solves the problem

$$\mathscr{P}_F(x): \quad \max_{y\in Y} F(x,y).$$

It follows that

$$\begin{aligned}\limsup_{n\to+\infty} \phi_n(y_n^*) &= \lim_{\substack{n\to+\infty\\ n\in\mathscr{N}_2}} (-F_n(x_n,y_n^*)) \\ &= \lim_{\substack{n\to+\infty\\ n\in\mathscr{N}_3}} (-F_n(x_n,y_n^*)) = -F(x,y^*) \\ &\leq -F(x,\bar{y}) = \phi(\bar{y}),\end{aligned}$$

where the inequality follows from the fact that y^* solves the problem $\mathscr{P}_F(x)$ and $\bar{y}$ is a feasible point to $\mathscr{P}_F(x)$. □

Corollary 4.1. *Let the assumptions of Proposition 4.1 hold. Then*

$$\limsup_{n\to+\infty} Argmin\,\mathscr{Q}_n(x_n) \subset Argmin\,\mathscr{Q}(x).$$

Proof. The result follows by using Proposition 4.1 and Theorem 2.1. □

For $x \in X$, set $w(x) = \sup_{y\in\mathscr{M}(x)} F(x,y)$. Then, the original weak bilevel optimization problem is written as

$$(S): \qquad \min_{x\in X} w(x).$$

Let $\mathrm{Argmin}\,S$ and $\mathrm{Argmin}\,S_n$ denote the solution sets of problems (S) and (S_n), respectively.

Now, we are able to state the following theorem on the existence of solutions to problem (S).

Theorem 4.1. *Let X and Y be sequentially compact and assume that assumptions (3.1)-(3.5), (3.7), (3.8), and (4.1) and the following assumption are satisfied*

(4.2) *for any $(x,n) \in X \times \mathbb{N}$, $F_n(x,.)$ is sequentially upper semicontinuous on Y.*

Then

i) $\limsup_{n\to+\infty} Argmin\,S_n \neq \emptyset$,

ii) $\limsup_{n\to+\infty} Argmin\,S_n \subset Argmin\,S$.

Then the original weak bilevel optimization problem (S) admits at least one solution.

Proof. *i*) According to Theorem 3.1, for every $n \in \mathbb{N}$, regularized perturbed problem (S_n) admits a solution x_n. Since X is sequentially compact, then (x_n) admits a subsequence converging to a point $x \in X$. It follows that $x \in \limsup_{n\to+\infty} \mathrm{Argmin}\,S_n$.

ii) Let $\bar{x} \in \limsup_{n\to+\infty} \mathrm{Argmin}\,S_n$. Let us show that $\bar{x} \in \mathrm{Argmin}\,S$. There exists a subsequence $(\bar{x}_n)_{n\in\mathscr{N}_0}$, converging to $\bar{x}$, with $\bar{x}_n \in \mathrm{Argmin}\,S_n$, for all $n \in \mathscr{N}_0$, and $\mathscr{N}_0$ is an infinite subset of $\mathbb{N}$. Assume that $\bar{x} \notin \text{Argmin}\, S$. Then, there exists $x^* \in X$, such that $w(x^*) < w(\bar{x})$, i.e., $\sup_{y \in \mathscr{M}(x^*)} F(x^*, y) < w(\bar{x})$. Hence

$$F(x^*, y) < w(\bar{x}), \quad \forall\, y \in \mathscr{M}(x^*). \tag{2}$$

For $n \in \mathscr{N}_0$, let $\bar{y}_n$ be a solution to

$$\max_{y \in \overline{\mathscr{M}_n^s(\varepsilon_n, \bar{x}_n)}^{seq}} F_n(\bar{x}_n, y).$$

That is $\bar{y}_n \in \text{Argmin}\mathscr{Q}_n(\bar{x}_n)$ (such a point exists from the sequential compactness of the set $\overline{\mathscr{M}_n^s(\varepsilon_n, \bar{x}_n)}^{seq}$ and the sequential upper semicontinuity of the function $F_n(\bar{x}_n, .)$). Hence

$$w_n(\bar{x}_n) = F_n(\bar{x}_n, \bar{y}_n). \tag{3}$$

From the sequential compactness of Y, there exists an infinite subset $\mathscr{N}_1$ of $\mathscr{N}_0$ such that $\bar{y}_n \to \bar{y}$ as $n \to +\infty$, $n \in \mathscr{N}_1$. Hence, $\bar{y} \in \limsup_{n \to +\infty} \text{Argmin}\mathscr{Q}_n(\bar{x}_n)$ and then Corollary 4.1 implies $\bar{y} \in \text{Agrmin}\mathscr{Q}(\bar{x})$, that is,

$$w(\bar{x}) = F(\bar{x}, \bar{y}), \tag{4}$$

and $\bar{y} \in \mathscr{M}(\bar{x})$. For $n \in \mathscr{N}_1$, let y_n^* be a solution to $\max_{y \in \overline{\mathscr{M}_n^s(x^*, \varepsilon_n)}^{seq}} F_n(x^*, y)$ (such a point exists since $\overline{\mathscr{M}_n^s(x^*, \varepsilon_n)}^{seq}$ is a sequentially compact set and the function $F_n(x^*, .)$ is sequentially upper semicontinuous on Y), i.e.,

$$w_n(x^*) = F_n(x^*, y_n^*), \tag{5}$$

and $y_n^* \in \overline{\mathscr{M}_n^s(x^*, \varepsilon_n)}^{seq}$. From the sequential compactness of Y, there exists an infinite subset $\mathscr{N}_2 \subset \mathscr{N}_1$ such that $y_n^* \to y^*$ as $n \to +\infty$, $n \in \mathscr{N}_2$. Since $y_n^* \in \overline{\mathscr{M}_n^s(x^*, \varepsilon_n)}^{seq}$ for all $n \in \mathscr{N}_2{}^*$, it follows that (Corollary 3.1)

$$y^* \in \limsup_{n \to +\infty} \overline{\mathscr{M}_n^s(x^*, \varepsilon_n)}^{seq} \subset \mathscr{M}(x^*).$$

Then, from properties (2) and (4), we have $F(x^*, y^*) < w(\bar{x}) = F(\bar{x}, \bar{y})$. Therefore, assumption (4.1) implies that $F_n(x^*, y_n^*) < F_n(\bar{x}_n, \bar{y}_n)$ for large $n \in \mathscr{N}_2$. Finally, using (5) and (3), we obtain $w_n(x^*) < w_n(\bar{x}_n)$, for large $n \in \mathscr{N}_2$. This strict inequality contradicts the optimality of $\bar{x}_n$ to problem (S_n), for large $n \in \mathscr{N}_2$. We conclude that $\bar{x}$ is a solution to problem (S). □

Moreover, if space U satisfies the first axiom of countability, we obtain an improvement of the result of Theorem 4.1.

Theorem 4.2. *Let the assumptions of Theorem 4.1 hold. If, moreover, topological space U is first countable, then any accumulation point of a sequence of solutions of the regularized perturbed problems (S_n) solves the original weak bilevel optimization problem (S).*

Proof. For $n \in \mathbb{N}$, let $\tilde{x}_n$ be a solution of the regularized perturbed problem (S_n), i.e., $\tilde{x}_n \in \text{Argmin}\, S_n$. Let $\tilde{x} \in X$ be an accumulation point of the sequence $(\tilde{x}_n)_n$. Since U is first countable, then there exists a subsequence $(\tilde{x}_n)_{n \in \mathscr{N}}$, converging to $\tilde{x}$, where $\mathscr{N}$ is an infinite subset of $\mathbb{N}$. It follows that $\tilde{x} \in \limsup_{n \to +\infty} \text{Argmin}\, S_n$. Then, by *ii*) of Theorem 4.1 we deduce that $\tilde{x} \in \text{Argmin}\, S$. That is $\tilde{x}$ solves the original weak bilevel optimization problem (S). □

We present the following example where all assumptions of Theorem 4.2 are satisfied.

Example 4.1. Let $X=[1,2]$, $Y=[0,3]\times[1,2]$, F, f, F_n, and $f_n, n\in\mathbb{N}^*$, be the functions defined on $\mathbb{R}^2$ by

$$f_n(x,y) = -y_1 e^{-(\frac{y_1}{xy_2})^2} + \frac{1}{n}, \qquad f(x,y) = -y_1 e^{-(\frac{y_1}{xy_2})^2},$$

and

$$F_n(x,y) = \frac{(xy_2 - \frac{1}{n})}{\sqrt{2e}} e^{\frac{1}{n}}, \qquad F(x,y) = \frac{xy_2}{\sqrt{2e}}.$$

Then, we have that F is continuous on $X\times Y$, and X and Y are compact sets. Moreover, we can verify via simple calculus that assumptions (3.1)-(3.5), (3.7), (3.8), and (4.1)-(4.2) are satisfied.

5. Conclusions

It is known that the study of existence of solutions to weak bilevel optimization problems is a difficult task in general. In this paper, in order to provide sufficient conditions to guarantee the existence of solutions to the weak bilevel optimization problem (S), we considered an approach with two operations, a perturbation and a regularization. The perturbation is done on the objective functions of the two players, and the regularization consists in replacing the constraint solution set $\mathscr{M}(x)$ in the first level by the sequential closure of the approximate strict ε-solutions of the perturbed lower level problem. Under mild assumptions we demonstrated that every regularized perturbed problem admits solutions. Using the notion of variational convergence and the results obtained for the regularized perturbed problems, we proved the existence of solutions to original problem (S). More precisely, we shown that any accumulation point of a sequence of solutions to the regularized perturbed problem is a solution to problem (S). The obtained result is an extension of those given in [7, 21, 22, 26, 28] for Stackelberg problems with unique lower level solution. Moreover, it gives an extension with improvement of the previous results established under convexity assumptions in [2] and [3]. In fact, our result is established without resorting to convexity, and it gives an extension from the finite dimensional case to a general topological one. Finally, we note that this approach raises the question of finding adaptable approximation methods to solve problem (S).

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