Appl. Set-Valued Anal. Optim. 6 (2024), No. 2, pp. 193-215
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.6.2024.2.05

GENERALIZED VISCOSITY INERTIAL TSENG'S METHOD WITH ADAPTIVE STEP SIZES FOR SOLVING PSEUDOMONOTONE VARIATIONAL INEQUALITIES WITH FIXED POINT CONSTRAINTS

O.T. MEWOMO[1,*], O.J. OGUNSOLA[2], T.O. ALAKOYA[1]

[1]*School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa*
[2]*College of Physical Sciences, Federal University of Agriculture, Abeokuta, Nigeria*

Abstract. In this paper, we study the problem of finding a solution of a pseudomonotone variational inequality problem with the constraints of fixed points of a finite family of demicontractive multivalued mappings. We introduce a new generalized viscosity inertial Tseng's extragradient method which uses self-adaptive step sizes. Unlike some existing results in this direction, we prove our strong convergence theorems without the sequentially weakly continuity condition of the pseudomonotone operator and without the knowledge of Lipschitz constants. Moreover, our strong convergence results do not follow the conventional "two cases" approach, which was often employed in proving strong convergence. Finally, we apply our result to convex minimization problems and present several numerical experiments to illustrate the performance of the proposed algorithms in comparison with other existing methods in the literature.
Keywords. Demicontractive multivalued mappings; Inertial algorithm; Self-adaptive step size; Variational inequality problem.
2020 Mathematics Subject Classification. 65K15, 90C30.

1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. In this paper, we consider the *variational inequality problem* (VIP) of finding a point $p \in C$ such that

$$\langle Ap, x-p \rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

where C is a convex and closed set in H, and $A : H \to H$ is a nonlinear operator. We denote by $VI(C,A)$ the solution set of the VIP (1.1).

Variational inequality theory is a vital tool that are often used in physics, economics, engineering, optimization theory, operator theory, and many others. Solution method of variational inequalities is an active area of research both in theory and applications. The two common approaches to solve VIP are regularized methods and projection methods. In this work, the projection method is adopted and the class of pseudomonotone operators is considered.

For any elements $x, y \in H$, we recall that a mapping $A : H \to H$ is said to be:

*Corresponding author.
E-mail address: mewomoo@ukzn.ac.za (O.T. Mewomo).
Received 14 February 2023; Accepted 3 May 2023; Published online 27 March 2024.

(1) η-*strongly monotone* if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta \|x - y\|^2;$$

(2) η-*inversely strongly monotone* if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta \|Ax - Ay\|^2;$$

(3) *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0$;

(4) η-*strongly pseudomonotone* if there exists a constant $\eta > 0$ such that

$$\langle Ay, x - y\rangle \geq 0 \Rightarrow \langle Ax, x - y\rangle \geq \eta \|x - y\|^2,$$

(5) *pseudomonotone* if $\langle Ay, x - y\rangle \geq 0 \Rightarrow \langle Ax, x - y\rangle \geq 0$;

(6) *Lipschitz-continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$; A is said to be a *contraction mapping* if $L \in [0, 1)$;

(7) *sequentially weakly continuous* if, for each sequence $\{x_n\}$, $x_n \rightharpoonup x \Rightarrow Ax_n \rightharpoonup Ax$.

It is clear from the definitions above that $(1) \Rightarrow (3) \Rightarrow (5)$ and $(1) \Rightarrow (4) \Rightarrow (5)$. However, the converse is not generally true.

The simplest celebrated projection method for solving *VIP* is the *gradient method* (GM), which involves a single projection onto feasible set C per iteration. Sibony [1] proposed the classical gradient projection algorithm as follows:

$$x_{n+1} = P_C(x_n - \lambda Ax_n). \tag{1.2}$$

Algorithm (1.2) is also called the *projected-gradient method* (PGM), where P_C represents the metric projection onto C, A is L-Lipschitz continuous and η- strongly monotone, and step size $\lambda \in (0, \frac{2\eta}{L^2})$. If *VI(C, A)* is nonempty, then the iterative sequence $\{x_n\}$ generated by (1.2) converges to a solution of the (VIP). The sequence generated by algorithm (1.2) only converges weakly when the operator is either strongly monotone or inverse-strongly monotone but fails to converge when the operator is only monotone.

Recently, Malitsky [2] proposed a *projected reflected gradient method* (PRGM), which is an improvement over the PGM. The (PRGM) is defined as follows:

$$x_{n+1} = P_C(x_n - \gamma A(2x_n - x_{n-1})), \quad \forall n \geq 1.$$

Malitsky proved that the sequence generated by (1) converges to an element in $VI(C,A)$ when mapping A is monotone.

Korpelevich [3] and Antipin [4] relaxed the conditions in Algorithm (1.2) and proposed the extragradient method (EGM) for solving the VIP (1.1). Initially, the algorithm proposed by Korpelevich was used to solve saddle point problems but later it was extended to solve *VIPs* in both Euclidean and infinite dimensional Hilbert spaces. The (EGM) method is proposed as follows:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases}$$

where $\lambda \in (0, \frac{1}{L})$, A is monotone and L-Lipschitz continuous, and P_C is the metric projection from H onto C. The algorithm only converges weakly to an element in *VI(C,A)* if set *VI(C,A)* is nonempty.

Over the years, (EGM) has been extensively used and extended by several researchers to approximate the solution of *(VIP)* in infinite dimensional spaces. Recently, this iterative scheme was extended by Vuong [5] to solve pseudomonotone variational inequalities in Hilbert spaces. However, due to the extensive amount of time required in executing the EGM method, as a result of calculating two projections onto closed and convex set C in each iteration, Censor et al. [6] proposed the subgradient extragradient method (SEGM) in which the second projection onto C was replaced by a projection onto a half space, which is given as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ T_n = \{w \in H : \langle x_n - \lambda Ax_n - y_n, w - y_n\rangle \leq 0\}, \\ x_{n+1} = PT_n(x_n - \lambda Ay_n) \quad \forall n \geq 0, \end{cases} \tag{1.3}$$

where $\lambda \in (0, \frac{2}{L})$. The weak convergence result of algorithm (1.3) motivated the authors in [6] to introduce a hybrid subgradient extragradient method in [7], which generates a strong convergence sequence. In the same vein, Tseng [8] improved the (EGM) method, by introducing the Tseng's extragradient method (TEGM) which only requires one projection per iteration. The scheme is as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = y_n + \lambda(Ax_n - Ay_n), \forall n \geq 0, \end{cases} \tag{1.4}$$

where A is monotone, L-Lipschitz continuous, and $\lambda \in (0, \frac{2}{L})$. TEGM (1.4) converges to a solution of the *VIP* with the assumption that *VI(C, A)* is nonempty.

In this work, we consider the inertial technique which is a two-step iteration process for accelerating the speed of convergence of iterative schemes. This technique was derived by Polyak [9] from a dynamic system called the heavy ball with friction. Over the years, several authors have incorporated this technique in their methods for solving various optimization problems; see, e.g., [10–15] and the references therein.

Recently, Tan and Qin in [16] proposed an inertial Tseng's extragradient method (ITEM) for approximating the solution of VIP (1.1) in Hilbert spaces. Their proposed algorithm is as follows:

$$\begin{cases} s_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = P_C(s_n - \gamma_n As_n), \\ z_n = y_n - \gamma_n(Ay_n - As_n), \\ x_{n+1} = \alpha_n f(z_n) + (I - \varphi_n)z_n, \end{cases}$$

where

$$\delta_n = \begin{cases} \min\left\{\frac{\varepsilon_n}{\|x_n - x_{n-1}}, \delta\right\}, & \text{if } x_n \neq x_{n-1}, \\ \delta, & \text{otherwise} \end{cases}$$

and

$$\gamma_{n+1} = \begin{cases} \min\left\{\frac{\phi\|r_n - y_n\|}{\|Ar_n - Ay_n\|}, \psi_n\right\}, & \text{if } Ar_n - Ay_n \neq 0, \\ \psi_n, & \text{otherwise.} \end{cases}$$

where f is a contraction and A is a pseudomonotone Lipschitz continuous and sequentially weakly continuous mapping.

Another problem of interest in this study is the fixed point problem. Let $S : H \to H$ be a nonlinear map. The *fixed point problem* (FPP) is defined as finding a point $p \in H$ (called the *fixed point of* S) such that $Sp = p$. We denote by $F(S)$, the set of all fixed points of S, i.e., $F(S) = \{p \in H : Sp = p\}$. If S is a multivalued mapping, i.e., $S : H \to 2^H$, then $p \in H$ is called a fixed point of S if $p \in Sp$. Our interest in this study is to approximate a common solution of VIP (1.1) and FPP $p \in Sp$.. That is, to find a point $x^* \in H$ such that $x^* \in VI(C,A) \cap F(S)$, where S is a multivalued mapping.

Recently, the common solution problem has received great research attention and several iterative methods have been proposed for finding its solution; see, e.g., [11, 17–22] and the references therein. The motivation for studying such a problem lies in its potential application to mathematical models whose constraints can be expressed as VIP and FPP. This arises in areas like image recovery, signal processing, and network resource allocation. A particular example of this is in network bandwidth allocation problem for two services in a heterogeneous wireless access networks in which the bandwidth of the services are related mathematically; see, e.g., [23, 24].

Very recently, Cai et al. [25] introduced the following inertial Tseng's extragradient method for finding the common solution of pseudomonotone variational inequality problem and fixed point problem for nonexpansive mappings in real Hilbert spaces:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \psi A w_n), \\ z_n = y_n - \psi(Ay_n - Aw_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)[\beta_n T z_n + (1-\beta_n) z_n], \end{cases} \tag{1.5}$$

where f is a contraction, T is a nonexpansive mapping, A is pseudomonotone, L-Lipschitz and sequentially weakly continuous, and $\psi \in (0, \frac{1}{L})$. They established strong convergence result for the proposed method under certain conditions. One of the major drawbacks of the above algorithms for solving pseudomonotone variational inequality problems and several other existing methods in the literature is that the pseudomonotone operator is required to be sequentially weakly continuous, which is a stringent condition. Moreover, several of the existing methods, such as Algorithm (1.5), require knowledge of the Lipschitz constant of the pseudomonotone operator for their implementation. However, the Lipschitz constant is unknown or difficult to calculate or even estimate, which makes the implementation of these algorithms infeasible.

Motivated by the above results and the ongoing research activities in this direction, in this paper, we propose a new generalized viscosity method, which combines the inertial Tseng's extragradient method with self-adaptive step size for approximating a common solution of pseudomonotone variational inequality problem and common fixed point of finite family of demicontractive multivalued mappings in Hilbert spaces. We prove strong convergence theorems without the sequentially weakly continuity condition of the pseudomonotone operator and without the knowledge of Lipschitz constants. More precisely, our proposed method has the following features:

(i) Our algorithm converges to the common solution of pseudomonotone variational inequality problem and common fixed points of a finite family of multivalued demicontractive mappings.

(ii) The proposed method only requires one projection per iteration onto the feasible set, which improves the efficiency of the algorithm and minimizes computational cost.

(iii) The viscosity method and the inertial technique, which are two of the efficient techniques for accelerating rate of convergence of iterative methods are employed in our proposed algorithm.
(iv) In addition, the proposed method does not require the sequentially weakly continuity condition of the pseudomonotone operator neither does it require knowledge of the Lipschitz constant of the operator for its implementation.
(v) We establish strong convergence results without following the conventional "*two-cases*" approach, which was often employed; see, eg., [25–27].

Moreover, we apply our result to convex minimization problems and present several numerical experiments to demonstrate the efficiency of our proposed method in comparison with existing methods in the literature. Our result in this paper complements several of the recently announced results in this direction. The organization of our paper is built as follows. In Section 2, we give relevant definitions and lemmas needed in the subsequent sections. In Section 3, we present the proposed algorithm while in Section 4, we analyze its convergence. We give applications of our proposed algorithm in Section 5. In Section 6 we present some numerical experiments. Finally, in Section 7, we give a concluding remark.

2. PRELIMINARIES

Let C be a convex, closed, and nonempty set in a real Hilbert space H. The weak convergence and strong convergence of $\{x_n\}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$ respectively.

We recall that a mapping $P_C : H \to C$ is called the metric projection from H onto C if, for all $x \in H$, there is a unique nearest point in C represented by $P_C(x)$ such that $P_C(x) := argmin\{\|x - y\|, y \in C\}$. The following statements are true and useful

(i) $\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall y \in C$;
(ii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall y \in H$;
(iii) $\|x - P_C x\|^2 \leq \|x - y\|^2 - \|y - P_C x\|^2, \quad \forall y \in C$.

Recall that a bounded linear operator G on H is said to be *strongly positive* if there exists a constant $\hat{\gamma} > 0$ such that $\langle Gx, x\rangle \geq \hat{\gamma}\|x\|^2$ for all $x \in H$. Let ξ be a positive real number and let $0 < \rho \leq ||G||^{-1}$. From [28], one has $||I - \rho G|| \leq 1 - \rho\xi$. Recall that a subset K of H is called proximal if, for each $x \in H$, there exists $y \in K$ such that $\|x - y\| = d(x, K) = \inf\{\|x - z\| : z \in K\}$. In this study, we denote the families of all nonempty closed bounded subsets, nonempty closed convex subsets, nonempty compact subsets, and nonempty proximal bounded subsets of C by $CB(C)$, $CC(C)$, $KC(C)$, and $P(C)$, respectively. The *Pompeiu-Hausdorff* metric on $CB(C)$ is defined by:

$$H(A, B) := \max\left\{\sup_{x \in A} d(x, B),\ \sup_{y \in B}(y, A)\right\}.$$

for all $A, B \in CB(C)$, where $d(x, B) = \inf_{b \in B}\|x - b\|$. Let $S : C \to 2^C$ be a multivalued mapping. S is said to satisfy the *end point* condition if $Sp = \{p\}$ for all $p \in F(S)$. For multivalued mappings, $S_i : C \to 2^C$ $(i \in \mathbb{N})$ with $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, then S_i is said to satisfy the *common endpoint* condition if $S_i(p) = \{p\}$ *for all* $i \in \mathbb{N},\ p \in \bigcap_{i=1}^{\infty} Fix(S_i)$.

Recall that a multivalued mapping $S : C \to CB(C)$ is said to be:

(i) *nonexpansive* if $H(Sa, Sb) \leq \|a - b\|$ for all $a, b \in C$;
(ii) *quasi-nonexpansive* if $F(S) \neq \emptyset$ and $H(Sa, Sp) \leq \|a - p\|$ for all $a \in C$ and $p \in F(S)$;

(iii) *nonspreading* if $2H(Sa,Sb)^2 \leq d(b,Sa)^2 + d(a,Sb)^2$ for all $a,b \in C$;

(iv) *k-hybrid* if there exists $k \in \mathbb{R}$, such that

$$(1+k)H(Sa,Sb)^2 \leq (1-k)\|a-b\|^2 + kd(b,Sa)^2 + kd(a,Sb)^2, \quad \forall\, a,b \in C;$$

(v) λ-*demicontractive* for $0 \leq \lambda < 1$ if $F(S) \neq \emptyset$ and

$$H(Sa,Sp)^2 \leq \|a-p\|^2 + \lambda d(a,Sa)^2, \quad \forall\, a \in C, p \in F(S).$$

Remark 2.1. It can easily be observed from the above definitions that the class of λ-demicontractive maps is more general than all other classes of maps listed above.

Let C be a convex ad closed subset of a real Hilbert space H and let $S : C \to CB(C)$ be a multivalued mapping. The mapping $I - S$ is said to be *demiclosed at zero* if, for any sequence $\{x_n\}$ in C, the conditions $x_n \rightharpoonup p$ and $\lim_{n\to\infty} d(x_n, Sx_n) = 0$, imply $p \in F(S)$.

The following Lemmas are needed to establish our results.

Lemma 2.1. *[29] For each $x,y \in H$, and $\delta \in \mathbb{R}$, we have the following results:*

(i) $||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle$;

(ii) $||x+y||^2 = ||x||^2 + 2\langle x,y\rangle + ||y||^2$;

(iii) $||\delta x + (1-\delta)y||^2 = \delta||x||^2 + (1-\delta)||y||^2 - \delta(1-\delta)||x-y||^2$.

Lemma 2.2. *[29] For each $x_1,\dots,x_m \in H$ and $\alpha_1,\dots,\alpha_m \in [0,1]$ with $\sum_{i=1}^m \alpha_i = 1$, the equality holds*

$$||\alpha_1 x_1 + \ldots + \alpha_m x_m||^2 = \sum_{i=1}^m \alpha_i ||x_i||^2 - \sum_{1\leq i<j\leq m} \alpha_i\alpha_j ||x_i - x_j||^2.$$

Lemma 2.3. *[30] Let $\{a_n\}$ be a non-negative real sequence and let $\{b_n\}$ be a real sequence. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ with $\sum_{n=1}^\infty \alpha_n = \infty$. Assume that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n$ for all $n \geq 1$. If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.4. *[31] Consider the VIP (1.1) with C being a closed and convex subset of a real Hilbert space H and $A : C \to H$ being pseudomonotone and continuous. Then p is a solution to VIP (1.1) if and only if $\langle Ax, x-p\rangle \geq 0$ for all $x \in C$.*

3. THE ALGORITHM

In this section, we present our algorithm and highlight some of its important features. Let $S_i : H \longrightarrow CB(H)$ be a finite family of multivalued demicontractive mappings with constant k_i such that each $I - S_i$ is demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^m F(S_i)$, and $k = \max\{k_i\}$. Let $G : H \longrightarrow H$ be a strongly positive and bounded linear operator with coefficient $\hat{\gamma} > 0$ and let $f : H \longrightarrow H$ be a contraction mapping with coefficient $\rho \in (0,1)$ such that $0 < \gamma < \frac{\hat{\gamma}}{\rho}$. We establish the strong convergence result of our algorithm under the following assumptions:

(A1) solution set $\Omega = \bigcap_{i=1}^m F(S_i) \cap VI(C,A)$ is non-empty;

(A2) mapping $A : H \longrightarrow H$ is pseudomonotone and L-Lipschitz continuous (however, the knowledge of the Lipschitz constant is not required);

(A3) $A : H \to H$ satisfies the following property. Whenever $\{x_n\} \subset C$, $x_n \rightharpoonup z$, one has $||Az|| \leq \liminf_{n\to\infty} ||Ax_n||$.

(A4) $\{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ and the positive sequence ε_n satisfies $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$;

(A5) $\beta_{n,0} \subset (0,1), \sum_{i=0}^{m} \beta_{n,i} = 1, \liminf_n (\beta_{n,0} - k)\beta_{n,i} > 0$ for each $1 \leq i \leq m$;

(A6) Let $\{\phi_n\}$ be a nonnegative sequence such that $\sum_{n=1}^{\infty} \phi_n < +\infty$.

Now, the algorithm is presented as follows:

Algorithm 1

Initialization: Give $\delta > 0, \quad \gamma_1 > 0$, and $\phi \in (0,1)$. Let $x_0, x_1 \in H$ be two initial points.
Iterative Steps: Calculate the next iteration point x_{n+1} as follows:

$$\begin{cases} w_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \gamma_n A w_n), \\ z_n = y_n - \gamma_n(Ay_n - Aw_n), \\ u_n = \beta_{n,0} z_n + \sum_{i=1}^{m} \beta_{n,i} v_{n,i}, \quad v_{n,i} \in S_i z_n, \\ x_{n+1} = \alpha_n \gamma f(w_n) + (I - \alpha_n G) u_n, \end{cases} \tag{3.1}$$

where δ_n and γ_n are updated by (3.2) and (3.3), respectively

$$\delta_n = \begin{cases} \min\left\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \ \delta\right\}, & \text{if } x_n \neq x_{n-1}, \\ \delta, & \text{otherwise} \end{cases} \tag{3.2}$$

and

$$\gamma_{n+1} = \begin{cases} \min\left\{\frac{\phi\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \ \gamma_n + \phi_n\right\}, & \text{if } Aw_n - Ay_n \neq 0 \\ \gamma_n + \phi_n, & \text{otherwise.} \end{cases} \tag{3.3}$$

Remark 3.1.

- If A is sequentially weakly continuous, then A satisfies condition (A3), but the converse is false. Thus, condition (A3) is strictly weaker than the sequentially weakly continuity condition commonly employed in the literature (see, e.g., [32]).
- Inertial technique is employed to accelerate the convergence speed of our proposed algorithm. Observe that (3.2) updates the inertial factor is easily implemented since the value of $\|x_n - x_{n-1}\|$ is known prior to choosing δ_n.

Remark 3.2. From (3.2) and condition (A5), it follows that $\lim_{n\to\infty} \frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$. It is clearly seen that $\delta_n\|x_n - x_{n-1}\| \leq \varepsilon_n$ for all $n \in \mathbb{N}$, which together with $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$ gives $\lim_{n\to\infty} \frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$.

4. Convergence Analysis

First, we present some lemmas, that are needed to prove our strong convergence theorem for the proposed algorithm.

Lemma 4.1. *Suppose that $\{\gamma_n\}$ is the sequence generated by (3.3). Then $\lim_{n\to\infty} \gamma_n = \gamma$, where $\gamma \in [\min\{\frac{\phi}{L}, \gamma_1\}, \gamma_1 + \Phi]$ and $\Phi = \sum_{n=1}^{\infty} \phi_n$.*

Proof. The method of proof is similar to the result in [33]. Hence, the proof is omitted here. □

Lemma 4.2. *Let $\{w_n\}$ and $\{y_n\}$ be two sequences generated by Algorithm 1 such that conditions (A1)-(A3) hold. If there exists a subsequence $\{w_{n_k}\}$ which is weakly convergent to $u \in H$ and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$, then $u \in VI(C,A)$.*

Proof. From (3.1) and the characterization of the projection, we have $\langle w_{n_k} - \gamma_{n_k} Aw_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0$ for all $x \in C$, which implies that $\frac{1}{\gamma_{n_k}}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq \langle Aw_{n_k}, x - y_{n_k}\rangle$ for all $x \in C$. Thus

$$\frac{1}{\gamma_{n_k}}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle Aw_{n_k}, y_{n_k} - w_{n_k}\rangle \leq \langle Aw_{n_k}, x - w_{n_k}\rangle, \quad \forall x \in C. \tag{4.1}$$

Since the subsequence $\{w_{n_k}\}$ is weakly convergent to $u \in H$, then $\{w_{n_k}\}$ is a bounded subsequence. By the Lipschitz continuity of A and $\|w_{n_k} - y_{n_k}\| \to 0$, we assert that $\{Aw_{n_k}\}$ and $\{y_{n_k}\}$ are bounded as well. Observe that $\gamma_{n_k} \geq \min\{\gamma_1, \frac{\phi}{L}\}$. Bby applying (4.1), we have

$$\liminf_{n\to\infty}\langle Aw_{n_k}, x - w_{n_k}\rangle \geq 0, \forall x \in C. \tag{4.2}$$

We also have

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k} - Aw_{n_k}, x - w_{n_k}\rangle + \langle Aw_{n_k}, x - w_{n_k}\rangle + \langle Ay_{n_k}, w_{n_k} - y_{n_k}\rangle. \tag{4.3}$$

Observe $\|w_{n_k} - y_{n_k}\| \to 0$ as $k \to \infty$. By the Lipschitz continuity of A, we have $\lim_{k\to\infty} \|Aw_{n_k} - Ay_{n_k}\| = 0$, which together with (4.2) and (4.3) gives $\liminf_{n\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \geq 0$. Now, let $\{\Phi_k\}$ be a decreasing sequence of positive numbers such that $\Phi_k \to 0$ as $k \to \infty$. Let N_k represent the smallest positive integer for any k such that

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \Phi_k \geq 0, \quad \forall j \geq N_k. \tag{4.4}$$

Clearly, the sequence $\{N_k\}$ is increasing since $\{\Phi_k\}$ is decreasing. From $\{y_{N_k}\} \subset C$, for any k, suppose $Ay_{N_k} \neq 0$ (otherwise y_{N_k} is a solution) and let $u_{N_k} = \frac{Ay_{N_k}}{\|Ay_{N_k}\|^2}$. Thus $\langle Ay_{N_k}, u_{N_k}\rangle = 1$ for each k. From (4.4), we obtain $\langle Ay_{N_k}, x + \Phi_k u_{N_k} - y_{N_k}\rangle \geq 0$ for all k. By the pseudomonotonicity of A, we have $\langle A(x + \Phi_k u_{N_k}), x + \Phi_k u_{N_k} - y_{N_k}\rangle \geq 0$, which gives

$$\langle Ax, x - y_{N_k}\rangle \geq \langle Ax - A(x + \Phi_k u_{N_k}), x + \Phi_k u_{N_k} - y_{N_k}\rangle - \Phi_k\langle Ax, u_{N_k}\rangle. \tag{4.5}$$

We now prove that $\lim_{k\to\infty} \Phi_k u_{N_k} = 0$. Since $w_{n_k} \rightharpoonup u$ and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$, we have $y_{N_k} \rightharpoonup u$, so $u \in C$. Since A satisfies condition (A3), we have $0 < \|Au\| \leq \liminf_{k\to\infty}\|Ay_{n_k}\|$. Using $\{y_{N_k}\} \subset \{y_{n_k}\}$ and $\Phi_k \to 0$ as $k \to \infty$, we have

$$0 \leq \limsup_{k\to\infty}\|\Phi_k u_{N_k}\| = \limsup_{k\to\infty}\left(\frac{\Phi_k}{\|Ay_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty} \Phi_k}{\liminf_{k\to\infty}\|Ay_{n_k}\|} = 0,$$

which implies that $\limsup_{k\to\infty} \Phi_k u_{N_k} = 0$. From the facts that A is Lipschitz continuous, $\{y_{N_k}\}$ and $\{u_{N_k}\}$ are bounded, and $\lim_{k\to\infty} \Phi_k u_{N_k} = 0$, it follows from (4.5) that $\liminf_{k\to\infty}\langle Ax, x - y_{N_k}\rangle \geq 0$. Hence, we obtain $\langle Ax, x - u\rangle = \lim_{k\to\infty}\langle Ax, x - y_{N_k}\rangle = \liminf_{k\to\infty}\langle Ax, x - y_{N_k}\rangle \geq 0$ for all $x \in C$. By invoking Lemma 2.4, it follows that $u \in VI(C,A)$ as required. □

Lemma 4.3. *Suppose that $\{y_n\}$ and $\{z_n\}$ are two sequences generated by Algorithm 1 such that conditions (A1)- (A3) hold. Then, we have the following inequalities*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - (1 - \phi^2 \frac{\gamma_n^2}{\gamma_{n+1}^2})\|w_n - y_n\|^2, \quad \forall p \in \Omega \tag{4.6}$$

and

$$\|z_n - y_n\| \leq \phi \frac{\gamma_n}{\gamma_{n+1}} \|w_n - y_n\|. \tag{4.7}$$

Proof. By the definition of γ_{n+1}, we have

$$\|Aw_n - Ay_n\| \leq \frac{\phi}{\gamma_{n+1}} \|w_n - y_n\|, \quad \forall n \in \mathbb{N}. \tag{4.8}$$

Observe that (4.8) holds both when $Aw_n = Ay_n$ and $Aw_n \neq Ay_n$. Using the definition of z_n and applying Lemma 2.1, we obtain

$$\begin{aligned} \|z_n - p\|^2 &= \|y_n - p\|^2 + \gamma_n^2 \|Ay_n - Aw_n\|^2 - 2\gamma_n \langle y_n - p, Ay_n - Aw_n \rangle \\ &= \|w_n - p\|^2 + \|y_n - w_n\|^2 + 2\langle y_n - w_n, w_n - p \rangle \\ &\quad + \gamma_n^2 \|Ay_n - Aw_n\|^2 - 2\gamma_n \langle y_n - p, Ay_n - Aw_n \rangle \\ &= \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2\langle y_n - w_n, y_n - p \rangle + \gamma_n^2 \|Ay_n - Aw_n\|^2 \\ &\quad - 2\gamma_n \langle y_n - p, Ay_n - Aw_n \rangle. \end{aligned} \tag{4.9}$$

The characterization of the projection yields that $\langle y_n - w_n + \gamma_n Aw_n, y_n - p \rangle \leq 0$, which is equivalent to $\langle y_n - w_n, y_n - p \rangle \leq -\gamma_n \langle Aw_n, y_n - p \rangle$. From (4.8) and (4.9), we obtain

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \phi^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\right) \|w_n - y_n\|^2 - 2\gamma_n \langle y_n - p, Ay_n \rangle.$$

Since $p \in VI(C, A)$ and $y_n \in C$, we have $\langle Ap, y_n - p \rangle \geq 0$. By the pseudomonotonicity of A, it follows that $\langle Ay_n, y_n - p \rangle \geq 0$. Thus

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \phi^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\right) \|w_n - y_n\|^2.$$

Also, from the definition of z_n and (4.8), we see that $\|z_n - y_n\| \leq \phi \frac{\gamma_n}{\gamma_{n+1}} \|w_n - y_n\|$, which completes the proof. □

Lemma 4.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then $\{x_n\}$ is bounded.*

Proof. First, we recall from Algorithm 1 that $u_n = \beta_{n,0}z_n + \sum_{i=1}^m \beta_{n,i}v_{n,i}$. By applying Lemma 2.2, we have

$$\begin{aligned}
\|u_n - p\|^2 &\le \beta_{n,0}\|z_n - p\|^2 + \sum_{i=1}^m \beta_{n,i}\|v_{n,i} - p\|^2 - \beta_{n,0}\sum_{i=1}^m \beta_{n,i}\|v_{n,i} - z_n\|^2 \\
&\le \beta_{n,0}\|z_n - p\|^2 + \sum_{i=1}^m \beta_{n,i}H(S_iz_n, S_ip)^2 - \beta_{n,0}\sum_{i=1}^m \beta_{n,i}\|v_{n,i} - z_n\|^2 \\
&\le \beta_{n,0}\|z_n - p\|^2 + \sum_{i=1}^m \beta_{n,i}(\|z_n - p\|^2 + k_id(z_n, S_iz_n)^2) - \beta_{n,0}\sum_{i=1}^m \beta_{n,i}\|v_{n,i} - z_n\|^2 \\
&\le \beta_{n,0}\|z_n - p\|^2 + \sum_{i=1}^m \beta_{n,i}(\|z_n - p\|^2 + k_i\|z_n - v_{n,i}\|^2) - \beta_{n,0}\sum_{i=1}^m \beta_{n,i}\|v_{n,i} - z_n\|^2 \\
&= \|z_n - p\|^2 - \sum_{i=1}^m \beta_{n,i}(\beta_{n,0} - k_i)\|z_n - v_{n,i}\|^2 \qquad (4.10) \\
&\le \|z_n - p\|^2.
\end{aligned}$$

Using (3.1) yields $\|w_n - p\| \le \|x_n - p\| + \alpha_n \frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\|$. Thanks to Remark 3.2, we have $\lim_{n\to\infty} \frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$. It follows that there exists a constant $M_1 > 0$ such that $\frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\| \le M_1$ for all $n \ge 1$. It thus follows that

$$\|w_n - p\| \le \|x_n - p\| + \alpha_n M_1. \qquad (4.11)$$

Observe that from Lemma 4.1, we have

$$\lim_{n\to\infty}\left(1 - \phi^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\right) = 1 - \phi^2 > 0,$$

which implies that there exists $n_0 \in \mathbb{N}$ such that $1 - \phi^2 \frac{\gamma_n^2}{\gamma_{n+1}^2} > 0$ for all $n \ge n_0$. Thus, from (4.6), we obtain $\|z_n - p\| \le \|w_n - p\|$ for all $n \ge n_0$. Combining (4.10) and (4.11), we have that, for all $n \ge n_0$,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n\gamma f(w_n) - \alpha_n Gp + (I - \alpha_n G)(u_n - p)\| \\
&\le \alpha_n(\|\gamma f(w_n) - \gamma f(p)\| + \|\gamma f(p) - Gp\|) + (1 - \alpha_n\hat{\gamma})\|u_n - p\| \\
&\le \alpha_n\gamma\rho\|w_n - p\| + \alpha_n\|\gamma f(p) - Gp\| + (1 - \alpha_n\hat{\gamma})(\|x_n - p\| + \alpha_n M_1) \\
&\le \alpha_n\gamma\rho(\|x_n - p\| + \alpha_n M_1) + \alpha_n\|\gamma f(p) - Gp\| + (1 - \alpha_n\hat{\gamma})(\|x_n - p\| + \alpha_n M_1) \\
&\le (1 - \alpha_n(\hat{\gamma} - \gamma p))\|x_n - p\| + \alpha_n(\hat{\gamma} - \gamma p)\left\{\frac{\|\gamma f(p) - Gp\|}{\hat{\gamma} - \gamma p} + \frac{M_1}{\hat{\gamma} - \gamma p}\right\} \\
&\le \max\left\{\|x_n - p\|, \frac{\|\gamma f(p) - Gp\|}{\hat{\gamma} - \gamma p} + \frac{M_1}{\hat{\gamma} - \gamma p}\right\} \\
&\dots \\
&\le \max\left\{\|x_{n_0} - p\|, \frac{\|\gamma f(p) - Gp\|}{\hat{\gamma} - \gamma p} + \frac{M_1}{\hat{\gamma} - \gamma p}\right\},
\end{aligned}$$

which presents the boundedness of $\{\|x_n - p\|\}$. Consequently, $\{x_n\}$ is bounded. Moreover, $\{w_n\}, \{y_n\}, \{z_n\}$, and $\{u_n\}$ are all bounded. $\square$

Lemma 4.5. *The following inequality holds for all $p \in \Omega$ and $n \in \mathbb{N}$:*

$$\begin{aligned}||x_{n+1}-p||^2 \leq &\Big(1-\frac{2\alpha_n(\hat{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big)||x_n-p||^2+\frac{2\alpha_n(\hat{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big\{\frac{\alpha_n\hat{\gamma}^2}{2(\hat{\gamma}-\gamma\rho)}M_3\\ &+\frac{3M_2(1-\alpha_n\hat{\gamma})^2+\alpha_n\gamma\rho}{2(\hat{\gamma}-\gamma\rho)}\frac{\delta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{1}{(\hat{\gamma}-\gamma\rho)}\langle\gamma f(p)-Gp,x_{n+1}-p\rangle\Big\}\\ &-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\gamma\rho)}\Big\{\Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)||w_n-y_n||^2+\sum_{i=1}^m\beta_{n,i}(\beta_{n,0}-k_i)||z_n-v_{n,i}||^2\Big\}.\end{aligned}$$

Proof. Using Lemma 2.1, we have

$$\begin{aligned}\|w_n-p\|^2 &= \|x_n-p\|^2+\delta_n^2\|x_n-x_{n-1}\|^2+2\delta_n(x_n-p,x_n-x_{n-1})\\ &\leq \|x_n-p\|^2+\delta_n\|x_n-x_{n-1}\|(\delta_n\|x_n-x_{n-1}\|+2\|x_n-p\|)\\ &\leq \|x_n-p\|^2+3M_2\alpha_n\frac{\delta_n}{\alpha_n}\|x_n-x_{n-1}\|, \end{aligned} \tag{4.12}$$

where $M_2 := \sup\{\|x_n - p\|, \delta_n\|x_n - x_{n-1}\|\} > 0$.

Next, by applying Lemma 2.1, we obtain from (4.6), (4.10), and (4.12) that

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\leq (1-\alpha_n\hat{\gamma})^2\|u_n-p\|^2+2\alpha_n\langle\gamma f(w_n)-Gp,x_{n+1}-p\rangle\\ &\leq (1-\alpha_n\hat{\gamma})^2(\|z_n-p\|^2-\sum_{i=1}^m\beta_{n,i}(\beta_{n,0}-k_i)\|z_n-v_{n,i}\|^2)\\ &\quad +2\alpha_n\gamma\langle f(w_n)-f(p),x_{n+1}-p\rangle+2\alpha_n\langle\gamma f(p)-Gp,x_{n+1}-p\rangle\\ &\leq (1-\alpha_n\hat{\gamma})^2(\|z_n-p\|^2-\sum_{i=1}^m\beta_{n,i}(\beta_{n,0}-k_i)\|z_n-v_{n,i}\|^2)\\ &\quad +\alpha_n\gamma\rho(\|w_n-p\|^2+\|x_{n+1}-p\|^2)+2\alpha_n\langle\gamma f(p)-Gp,x_{n+1}-p\rangle\\ &\leq (1-\alpha_n\hat{\gamma})^2\Big(\|w_n-p\|^2-(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2})\|w_n-y_n\|^2-\sum_{i=1}^m\beta_{n,i}(\beta_{n,0}-k_i)\|z_n-v_{n,i}\|^2\Big)\\ &\quad +\alpha_n\gamma\rho(\|w_n-p\|^2+\|x_{n+1}-p\|^2)+2\alpha_n\langle\gamma f(p)-Gp,x_{n+1}-p\rangle\\ &\leq (1-\alpha_n\hat{\gamma})^2\Big(\|x_n-p\|^2+3M_2\alpha_n\frac{\delta_n}{\alpha_n}\|x_n-x_{n-1}\|\\ &\quad -(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2})\|w_n-y_n\|^2-\sum_{i=1}^m\beta_{n,i}(\beta_{n,0}-k_i)\|z_n-v_{n,i}\|^2\Big)\\ &\quad +\alpha_n\gamma\rho\Big(\|x_n-p\|^2+3M_2\alpha_n\frac{\delta_n}{\alpha_n}\|x_n-x_{n-1}\|+\|x_{n+1}-p\|^2\Big)+2\alpha_n\langle\gamma f(p)-Gp,x_{n+1}-p\rangle.\end{aligned}$$

This consequently leads to

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\leq \frac{(1-2\alpha_n\hat{\gamma}+(\alpha_n\hat{\gamma})^2+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}\|x_n-p\|^2+\frac{3M_2((1-\alpha_n\hat{\gamma})^2+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}\alpha_n\frac{\delta_n}{\alpha_n}\|x_n-x_{n-1}\|\\
&+\frac{2\alpha_n}{(1-\alpha_n\gamma\rho)}\langle\gamma f(p)-Gp,x_{n+1}-p\rangle-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\gamma\rho)}\Big\{\Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|w_n-y_n\|^2\\
&+\sum_{i=1}^{m}\beta_{n,i}(\beta_{n,0}-k_i)\|z_n-v_{n,i}\|^2\Big\}\\
&=\frac{(1-2\alpha_n\hat{\gamma}+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}\|x_n-p\|^2+\frac{(\alpha_n\hat{\gamma})^2}{(1-\alpha_n\gamma\rho)}\|x_n-p\|^2\\
&+\frac{3M_2((1-\alpha_n\hat{\gamma})^2+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}\alpha_n\frac{\delta_n}{\alpha_n}\|x_n-x_{n-1}\|+\frac{2\alpha_n}{(1-\alpha_n\gamma\rho)}\langle\gamma f(p)-Gp,x_{n+1}-p\rangle\\
&-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\gamma\rho)}\Big\{\Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|w_n-y_n\|^2+\sum_{i=1}^{m}\beta_{n,i}(\beta_{n,0}-k_i)\|z_n-v_{n,i}\|^2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq\Big(1-\frac{2\alpha_n(\hat{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big)||x_n-p||^2+\frac{2\alpha_n(\hat{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big\{\frac{\alpha_n\hat{\gamma}^2}{2(\hat{\gamma}-\gamma\rho)}M_3\\
&+\frac{3M_2(1-\alpha_n\hat{\gamma})^2+\alpha_n\gamma\rho}{2(\hat{\gamma}-\gamma\rho)}\frac{\delta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{1}{(\hat{\gamma}-\gamma\rho)}\langle\gamma f(p)-Gp,x_{n+1}-p\rangle\Big\}\\
&-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\gamma\rho)}\Big\{\Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|w_n-y_n\|^2+\sum_{i=1}^{m}\beta_{n,i}(\beta_{n,0}-k_i)||z_n-v_{n,i}||^2\Big\},
\end{aligned}
$$

where $M_3:=\sup\{\|x_n-p\|^2:n\in\mathbb{N}\}$. Thus the required inequality is obtained. □

Theorem 4.1. *Suppose that assumptions $(A1)-(A6)$ hold. Then, the sequence $\{x_n\}$ generated by Algorithm 1 converges to $\hat{x}\in\Omega$ in norm, where $\hat{x}=P_\Omega(I-G+\gamma f)(\hat{x})$ is a solution to the variational inequality $\langle(G-\gamma f)\hat{x},\hat{x}-q\rangle\leq 0$ for all $q\in\Omega$.*

Proof. Let $\hat{x}=P_\Omega(I-G+\gamma f)(\hat{x})$. Then, from Lemma 4.5, we obtain

$$
\begin{aligned}
||x_{n+1}-\hat{x}||^2\leq&\Big(1-\frac{2\alpha_n(\hat{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big)||x_n-\hat{x}||^2+\frac{2\alpha_n(\hat{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big\{\frac{\alpha_n\hat{\gamma}^2}{2(\hat{\gamma}-\gamma\rho)}M_3\\
&+\frac{3M_2(1-\alpha_n\hat{\gamma})^2+\alpha_n\gamma\rho}{2(\hat{\gamma}-\gamma\rho)}\frac{\delta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{1}{(\hat{\gamma}-\gamma\rho)}\langle\gamma f(\hat{x})-G\hat{x},x_{n+1}-\hat{x}\rangle\Big\}.
\end{aligned}
\tag{4.13}
$$

Next, we claim that the sequence $\{\|x_n-\hat{x}\|^2\}$ converges to zero. To establish this claim, in view of Lemma 2.3, Remark 3.2, and the fact that $\lim_{n\to\infty}\alpha_n=0$, it suffices to prove that $\limsup_{k\to\infty}\langle\gamma f(\hat{x})-Gp,x_{n_k+1}-\hat{x}\rangle\leq 0$ for every subsequence $\{\|x_{n_k}-\hat{x}\|\}$ of $\{\|x_n-\hat{x}\|\}$ satisfying $\liminf_{k\to\infty}(\|x_{n_k+1}-\hat{x}\|-\|x_{n_k}-\hat{x}\|)\geq 0$. Suppose that $\{\|x_{n_k}-\hat{x}\|\}$ is a subsequence of $\{\|x_n-\hat{x}\|\}$ such that

$$\liminf_{k\to\infty}(\|x_{n_k+1}-\hat{x}\|-\|x_{n_k}-\hat{x}\|)\geq 0. \tag{4.14}$$

Again, from Lemma 4.5 we have

$$\frac{(1-\alpha_{n_k}\hat{\gamma})^2}{(1-\alpha_{n_k}\gamma\rho)}\Big\{\Big(1-\phi^2\frac{\gamma_{n_k}^2}{\gamma_{n_k+1}^2}\Big)||w_{n_k}-y_{n_k}||^2+\sum_{i=1}^{m}\beta_{n_k,i}(\beta_{n_k,0}-k_i)||z_{n_k}-v_{n_k,i}||^2\Big\}$$
$$\leq\Big(1-\frac{2\alpha_{n_k}(\hat{\gamma}-\gamma\rho)}{(1-\alpha_{n_k}\gamma\rho)}\Big)||x_{n_k}-\hat{x}||^2-||x_{n_k+1}-\hat{x}||^2+\frac{2\alpha_{n_k}(\hat{\gamma}-\gamma\rho)}{(1-\alpha_{n_k}\gamma\rho)}\Big\{\frac{\alpha_{n_k}\hat{\gamma}^2}{2(\hat{\gamma}-\gamma\rho)}M_3$$
$$+\frac{3M_2(1-\alpha_{n_k}\hat{\gamma})^2+\alpha_{n_k}\gamma\rho}{2(\hat{\gamma}-\gamma\rho)}\frac{\delta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}||+\frac{1}{(\hat{\gamma}-\gamma\rho)}\langle\gamma f(\hat{x})-G\hat{x},x_{n_k+1}-\hat{x}\rangle\Big\}.$$

Applying (4.14) and the fact that $\lim_{k\to\infty}\alpha_{n_k}=0$, we have

$$\frac{(1-\alpha_{n_k}\hat{\gamma})^2}{(1-\alpha_{n_k}\gamma\rho)}\Big\{\Big(1-\phi^2\frac{\gamma_{n_k}^2}{\gamma_{n_k+1}^2}\Big)\|w_{n_k}-y_{n_k}\|^2+\sum_{i=1}^{m}\beta_{n_k,i}(\beta_{n_k,0}-k_i)||z_{n_k}-v_{n_k,i}||^2\Big\}\to 0,\quad k\to\infty.$$

By the conditions on the control parameters, we have

$$\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=\lim_{k\to\infty}\|z_{n_k}-v_{n_k,i}\|=0. \tag{4.15}$$

From (4.7) and (4.15), it follows that $\lim_{k\to\infty}\|z_{n_k}-y_{n_k}\|=0$. From the definition of u_n and by applying (4.15), we have

$$\|u_{n_k}-z_{n_k}\|\leq\beta_{n_k,0}\|z_{n_k}-z_{n_k}\|+\sum_{i=1}^{m}\beta_{n_k,i}\|v_{n_k,i}-z_{n_k}\|\to 0,\quad k\to\infty. \tag{4.16}$$

Applying Remark 3.2 and the definition of w_n, we obtain

$$\|x_{n_k}-w_{n_k}\|=\delta_{n_k}\|x_{n_k}-x_{n_k-1}\|\to 0,\quad k\to\infty. \tag{4.17}$$

From (4.15)-(4.17), we obtain

$$\lim_{k\to\infty}\|x_{n_k}-u_{n_k}\|=\lim_{k\to\infty}\|x_{n_k}-z_{n_k}\|=0. \tag{4.18}$$

Now, applying (4.18) and the fact that $\lim_{k\to\infty}\alpha_{n_k}=0$, we see that

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\|&=\|\alpha_{n_k}\gamma f(w_{n_k})-\alpha_{n_k}Gx_{n_k}+(I-\alpha_{n_k}G)(u_{n_k}-x_{n_k})\|\\&\leq\alpha_{n_k}\|\gamma f(w_{n_k})-Gx_{n_k}\|+(1-\alpha_{n_k}\hat{\gamma})\|u_{n_k}-x_{n_k}\|\to 0,\quad k\to\infty.\end{aligned} \tag{4.19}$$

To complete the proof, we need to establish $w_\omega(x_n)\subset\Omega$. Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $x^*\in w_\omega(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$ as $k\to\infty$. From (4.17), we have $w_{n_k}\rightharpoonup x^*$ as $k\to\infty$. Consequently, by Lemma 4.2 and (4.15), we obtain $x^*\in VI(C,A)$. Since $x^*\in w_\omega(x_n)$ is arbitrary, it follows that $w_\omega(x_n)\subset VI(C,A)$. By applying (4.15), we obtain

$$d(z_{n_k},S_iz_{n_k})\leq\|z_{n_k}-v_{n_k,i}\|\to 0,\quad k\to\infty\ \forall i=1,2,\ldots,m. \tag{4.20}$$

From (4.18), we have $z_{n_k}\rightharpoonup x^*$ as $k\to\infty$. Since $I-S_i$ is demiclosed at zero for each $i=1,2,\ldots,m$, then it follows from (4.20) that $x^*\in F(S_i)$ for each $i=1,2,\ldots,m$, which implies that $x^*\in\bigcap_{i=1}^m F(S_i)$. That is, $w_\omega(x_n)\subset\bigcap_{i=1}^m F(S_i)$. Hence, $w_\omega(x_n)\subset\Omega$.

Now, we see from (4.18) that $w_\omega\{x_n\}=w_\omega\{z_n\}$. By the boundedness of $\{x_{n_k}\}$, we further sees that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}}\rightharpoonup x^\dagger$ and

$$\lim_{j\to\infty}\langle\gamma f(\hat{x})-G\hat{x},x_{n_{k_j}}-\hat{x}\rangle=\limsup_{k\to\infty}\langle\gamma f(\hat{x})-G\hat{x},x_{n_k}-\hat{x}\rangle=\limsup_{k\to\infty}\langle\gamma f(\hat{x})-G\hat{x},z_{n_k}-\hat{x}\rangle.$$

Since $\hat{x} = P_\Omega(I - G + \gamma f)(\hat{x})$, it follows that

$$\begin{aligned}\limsup_{k\to\infty}\langle \gamma f(\hat{x}) - G\hat{x}, x_{n_k} - \hat{x}\rangle &= \lim_{j\to\infty}\langle \gamma f(\hat{x}) - G\hat{x}, x_{n_{k_j}} - \hat{x}\rangle \\ &= \langle \gamma f(\hat{x}) - G\hat{x}, x^\dagger - \hat{x}\rangle \le 0.\end{aligned} \tag{4.21}$$

Hence, from (4.19) and (4.21), we have

$$\begin{aligned}&\limsup_{k\to\infty}\langle \gamma f(\hat{x}) - G\hat{x}, x_{n_k+1} - \hat{x}\rangle \\ &= \limsup_{k\to\infty}\langle \gamma f(\hat{x}) - G\hat{x}, x_{n_k+1} - x_{n_k}\rangle + \limsup_{k\to\infty}\langle \gamma f(\hat{x}) - G\hat{x}, x_{n_k} - \hat{x}\rangle \\ &= \langle \gamma f(\hat{x}) - G\hat{x}, x^\dagger - \hat{x}\rangle \le 0.\end{aligned}$$

Applying Lemma 2.3 to (4.13) and using the fact that $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, we deduce that $\lim_{n\to\infty}||x_n - \hat{x}|| = 0$ as desired. □

5. Applications

5.1. **Convex minimization problem with fixed point constraints.** Let C be a nonempty, closed, and convex subset of a real Hilbert space H, the *constrained convex minimization problem* is formulated as follows: Find a point $x^* \in C$ such that

$$\psi(x^*) = \min_{x\in C}\psi(x), \tag{5.1}$$

where ψ is a real-valued convex function. The set of solutions of the constrained convex minimization problem is denoted by $\arg\min_{x\in C}\psi(x)$.

Lemma 5.1. *[34] Let H be a real Hilbert space, and let C be a nonempty, closed, and convex subset of H. Let ψ be a convex function of H into $\mathbb{R}$. If ψ is differentiable. Then z is a solution to problem (5.1) if and only if $z \in VI(C, \nabla\psi)$.*

Applying Theorem 4.1 and Lemma 5.1, we obtain the following result immediately.

Theorem 5.1. *Let H be a real Hilbert space, and let $\psi : H \to \mathbb{R}$ be a differentiable convex function such that $\nabla\psi$ is $\alpha-$ism. Let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 2

Initialization: *Give* $\delta > 0, \quad \gamma_1 > 0, \quad \phi \in (0,1)$. *Let* $x_0, x_1 \in H$ *be two initial points.*
Iterative Steps: *Calculate the next iterate* x_{n+1} *as follows:*

$$\begin{cases} w_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \gamma_n \nabla \psi w_n), \\ z_n = y_n - \gamma_n(\nabla \psi y_n - \nabla \psi w_n), \\ u_n = \beta_{n,0} z_n + \sum_{i=1}^m \beta_{n,i} v_{n,i}, \quad v_{n,i} \in S_i z_n, \\ x_{n+1} = \alpha_n \gamma f(w_n) + (I - \alpha_n G) u_n, \end{cases}$$

where δ_n *and* γ_n *are updated by* (5.2) *and* (5.3)*, respectively,*

$$\delta_n = \begin{cases} \min\left\{ \frac{\varepsilon_n}{||x_n - x_{n-1}||}, \ \delta \right\}, & \text{if } x_n \neq x_{n-1}, \\ \delta, & \text{otherwise} \end{cases} \tag{5.2}$$

and

$$\gamma_{n+1} = \begin{cases} \min\left\{ \frac{\phi ||w_n - y_n||}{||\nabla \psi w_n - \nabla \psi y_n||}, \ \gamma_n + \phi_n \right\}, & \text{if } \nabla \psi w_n - \nabla \psi y_n \neq 0 \\ \gamma_n + \phi_n, & \text{otherwise.} \end{cases} \tag{5.3}$$

If all other conditions of Theorem 4.1 hold, then sequence $\{x_n\}$ *converges to*

$$\hat{x} \in \Gamma = \arg\min_{x \in C} \psi(x) \bigcap \cap_{i=1}^m F(S_i) \neq \emptyset$$

in norm, where $\hat{x} = P_\Gamma(I - G + \gamma f)(\hat{x})$ *is a solution to the variational inequality*

$$\langle (G - \gamma f)\hat{x}, \hat{x} - q \rangle \leq 0, \quad \forall q \in \Gamma.$$

Proof. Since ψ is convex, then $\nabla \psi$ is monotone [34] and thus pseudomonotone. Now, setting $A = \nabla \psi$ in Theorem 4.1 and applying Lemma 5.1, we immediately obtain the desired result. $\square$

6. Numerical Examples

In this section, we give some numerical examples to illustrate the performance of our Algorithm 1 in comparison with Algorithm 1.5 proposed by Gang *et al.* (Gang *et al.* Alg.), Appendix 7.1 proposed by Chen *et al.* (Chen *et al.* Alg.), Appendix 7.2 by Ceng *et al.* (Ceng *et al.* Alg. (1)) and Appendix 7.3 by Ceng *et al.* (Ceng *et al.* Alg. (2)). The parameters are selected as follows:

- Take $f(x) = \frac{1}{3}x$, that is, $\rho = \frac{1}{3}$ is the Lipschitz constant for f. Let $G(x) = \frac{x}{2}$ with constant $\bar{\gamma} = \frac{1}{2}$. Then we take $\gamma = 1$, which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\rho}$. Let $S_i x = \{-\frac{(i+2)}{3}x\}$ for $i = 1,2,\ldots,5$. Choose $\delta = 0.9, \gamma_1 = 0.65, \phi = 0.8, \phi_n = \frac{1}{(n+2)^2}, \alpha_n = \frac{1}{n+5}, \varepsilon_n = \frac{1}{(n+5)^3}, \beta_{n,0} = \frac{n}{n+1}$, and $\beta_{n,i} = \frac{1}{5(n+1)}$ in our Algorithm 1.
- Take $Tx = \frac{x}{3}, \psi = 0.2$, and $\theta_n = \frac{1}{(n+2)^2}$ in Algorithm (1.5).
- Let $Ux = -\frac{3}{2}x, Gx = x - x_1, \gamma_n = \frac{1}{n+2}, \omega = 0.09$, and $\rho_n = \frac{n+1}{2n+1}$ in Appendix 7.1.
- Take $T_n x = -\frac{2}{n \mod 5}x, \lambda = m = \mu = \frac{2}{3}, \sigma_n = \frac{1}{n+2}, \tau_n = \frac{1}{3}, \gamma_n = \frac{1}{6}$, and $\mu_n = \frac{1}{2}$, in Appendix 7.2 and Appendix 7.3.

The graph of errors is plotted against the number of iterations in each case and $\|x_{n+1}-x_n\| \leq 10^{-2}$ is used as the stopping criterion. All numerical computations were carried out by using Matlab 2022(b) and the numerical results are reported in Tables 1 - 2 and Figures 1 - 8.

Example 6.1. Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be a linear operator in the form $A(x) = Sx + q$, where $q \in \mathbb{R}^m$ and $S = NN^T + Q + D, N$ is a $m \times m$ matrix, Q is a $m \times m$ skew-symmetric matrix, and D is a $m \times m$ diagonal matrix with its diagonal entries being nonnegative (thus S is positive symmetric definite). We take feasible set C as $C = \{x \in \mathbb{R}^m : -2 \leq x_i \leq 5, i = 1, \ldots, m\}$. Clearly, A is monotone and Lipschitz continuous with constant $L = \|S\|$. In this experiment, all entries of N, Q are randomly generated in $[-2, 2]$ while D is randomly generated in $[0, 2]$ and $q = 0$. The choices of the initial values x_0 and x_1 are generated randomly in $\mathbb{R}^m$ for $m = 10, 20, 25, 50$.

TABLE 1. Numerical Results for Example 6.1

	$m = 10$		$m = 20$		$m = 25$		$m = 50$	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Gang *et al. Alg.*	26	3.0827	31	3.6278	31	4.0547	39	2.6929
Chen *et al. Alg.*	9	0.8600	10	0.9154	10	1.0204	10	0.5639
Ceng *et al. Alg. (1)*	11	0.8903	12	1.2412	12	1.2354	12	0.6088
Ceng *et al. Alg. (2)*	22	1.7677	23	1.9747	24	2.0761	25	1.2937
Proposed Alg. 1	8	0.7119	10	0.7540	10	0.7718	10	0.5087

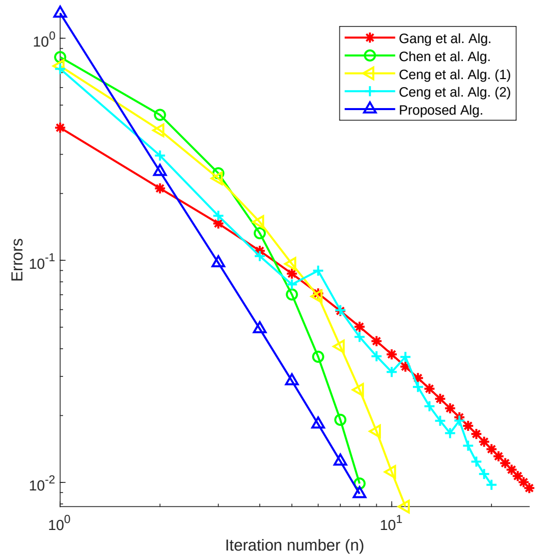

FIGURE 1. Example 6.1 $(m = 10)$

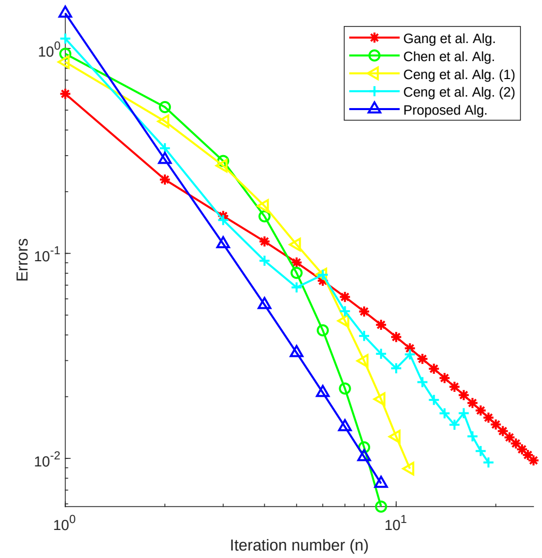

FIGURE 2. Example 6.1 $(m = 20)$

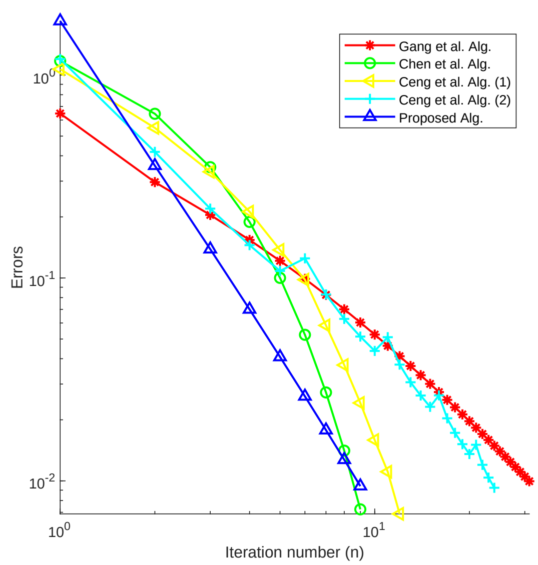

FIGURE 3. Example 6.1 $(m = 25)$

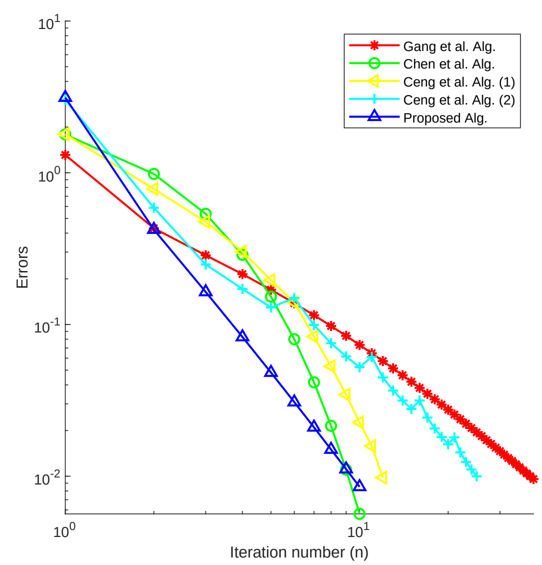

FIGURE 4. Example 6.1 $(m = 50)$

Example 6.2. Let $H = (\ell_2(\mathbb{R}), \|\cdot\|_2)$, where $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^2 < +\infty\}$, $||x||_2 = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$, and $\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j$ for all $x \in \ell_2(\mathbb{R})$. Let $\alpha, \beta \in \mathbb{R}$ be such that $\beta > \alpha > \frac{\beta}{2} > 0$ and $C = \{x \in H : \|x\| \leq \alpha\}$. We define the operator $A : H \to H$ by $A(x) = (\beta - \|x\|)x$, $\forall x \in H$. It can be verified that A is pseudomonotone. For this experiment, we choose $\beta = 3$ and $\alpha = 2$.

We choose different initial values as follows:
Case I: $x_0 = (-\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \cdots)$ and $x_1 = (-1, \frac{1}{3}, -\frac{1}{9}, \cdots)$;
Case II: $x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$ nad $x_1 = (1, \frac{1}{3}, \frac{1}{\cdot}, \cdots)$;
Case III: $x_0 = (-\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \cdots)$ and $x_1 = (1, \frac{1}{2}, \frac{1}{4}, \cdots)$;
Case IV: $x_0 = (-\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \cdots)$ and $x_1 = (-1, \frac{1}{2}, -\frac{1}{4}, \cdots)$.

TABLE 2. Numerical Results for Example 6.2

	Case I		Case II		Case III		Case IV	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Gang *et al. Alg.*	9	0.0022	9	0.0234	10	0.0231	9	0.0253
Chen *et al. Alg.*	6	0.0173	6	0.0179	6	0.0204	6	0.0171
Ceng *et al. Alg. (1)*	6	0.0159	6	0.0132	8	0.0184	10	0.0171
Ceng *et al. Alg. (2)*	4	0.0089	4	0.0080	4	0.0077	4	0.0096
Proposed Alg. 1	4	0.0113	4	0.0112	4	0.0108	4	0.0092

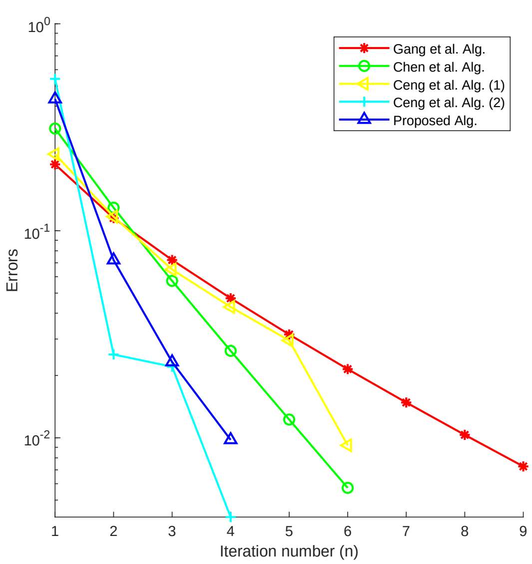

FIGURE 5. Example 6.2 Case I

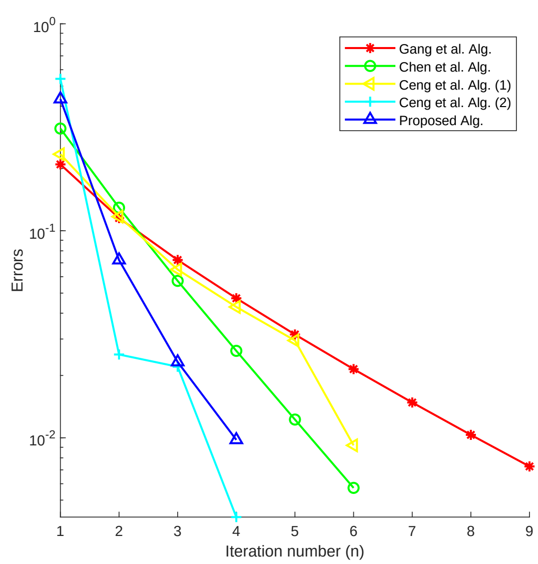

FIGURE 6. Example 6.2 Case II

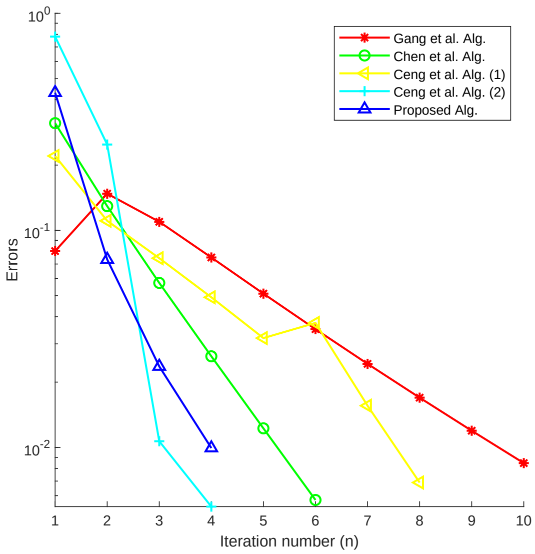

FIGURE 7. Example 6.2 Case III

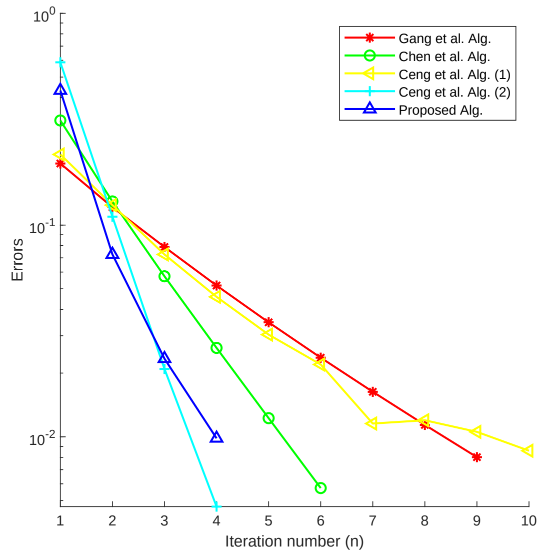

FIGURE 8. Example 6.2 Case IV

7. CONCLUSION

In this paper, we studied the problem of finding the solution of the pseudomonotone variational inequality problem with constraints of fixed points of a finite family of demicontractive multivalued mappings. We proposed a new generalized viscosity inertial Tseng's extragradient method, which uses self-adaptive step sizes. We proved strong convergence results and presented several numerical experiments to demonstrate the efficiency of the proposed method in comparison with other existing methods in the literature.

Acknowledgments
The authors sincerely thank the anonymous referee for his careful reading, constructive comments, and useful suggestions that improved the manuscript. The first author was supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant Number 119903). The research of the third author was wholly supported by the University of KwaZulu-Natal, Durban, South Africa Postdoctoral Fellowship.

REFERENCES

[1] M. Sibony, Methodes iteratives pour les equation set en equations aux derives partielles nonlinearesde type monotone, Calcolo, 7 (1970), 65-183.
[2] Y. Malitsky, Projected reflected gradient methods for monotone variational inequalities, SIAM J. Optim. 25 (2015), 502-520.
[3] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, Ekonom. Mat, Methody 12 (1976), 747-756.
[4] A.S. Antipin, On a method for convex programs using a symmetrical modification of the Lagrange function, Ekonom. i Mathe. Met. 12 (1976), 1164-1173.

[5] P.T. Vuong, On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities, J. Optim. Theory Appl. 176 (2018), 399-409.
[6] Y. Censor, A. Gibali, S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert spaces, J. Optim. Theory Appl. 148 (2011), 318-335.
[7] Y. Censor, A. Gibali, S. Reich, Strong convergence of subgradient extragradient methods for variational inequality problems in Hilbert space, Optim. Meth. Softw. 26 (2011), 827-845.
[8] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, SIAM J. Control Optim. 38 (2000), 431-446.
[9] B.T. Polyak, Some methods of speeding up the convergence of iteration methods, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 4 (1964), 1-17.
[10] J. Fan, L. Liu, X. Qin, A subgradient extragradient algorithm with inertial effects for solving strongly pseudomonotone variational inequalities, Optimization, 69 (2020), 2199-2215.
[11] L. Liu, X. Qin, Strong convergence theorems for solving pseudo-monotone variational inequality problems and applications, Optimization, 71 (2022), 3603-3626.
[12] G.N. Ogwo, C. Izuchukwu, Y. Shehu, O.T. Mewomo, Convergence of relaxed inertial subgradient extragradient methods for quasimonotone variational inequality problems, J. Sci. Comput. (2021). DOI:10.1007/s10915-021-01670-1.
[13] A. Taiwo, L.O. Jolaoso, O.T. Mewomo, Inertial-type algorithm for solving split common fixed point problems in Banach spaces, J. Sci. Comput. 86 (2021), 12.
[14] B. Tan, X. Qin, S.Y. Cho, Revisiting subgradient extragradient methods for solving variational inequalities, Numer. Algo. 90 (2022), 1593-1615.
[15] B. Tan, S.Y. Cho, Strong convergence of inertial forward–backward methods for solving monotone inclusions, Appl. Anal. 101 (2022), 5386-5414
[16] B. Tan, X. Qin, Strong convergence of an inertial Tseng's extragradient algorithm for pseudomonotone variational inequalities with applications to optimal control problems, arXiv preprint arXiv:2007.11761v1 [math. O.C], (2020).
[17] J.S. Jung, A general iterative algorithm for split variational inclusion problems and fixed point problems of a pseudocontractive mapping, J. Nonlinear Funct. Anal. 2022 (2022), 13.
[18] L.C. Ceng, A. Petruse, X. Qin, J.C. Yao, A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems, Fixed Point Theory 21 (2020), 93-108.
[19] F. Ogbuisi, Y. Shehu, J.C. Yao, An alternated inertial method for pseudomonotone variational inequalities in Hilbert spaces, Optim. Eng. 23 (2022), 917-945.
[20] B. Tan, S.Y. Cho, Inertial extragradient methods for solving pseudomonotone variational inequalities with non-Lipschitz mappings and their optimization applications, Appl. Set-Valued Anal. Optim. 3 (2021), 165-192.
[21] E.C. Godwin, T.O. Alakoya, O.T. Mewomo, J.C. Yao, Approximation of solutions of the split minimization problem with multiple output sets and common fixed point problems in real Banach spaces, J. Nonlinear Var. Anal. 6 (2022), 333-358.
[22] V.A. Uzor, T.O. Alakoya, O.T. Mewomo, On split monotone variational inclusion problem with multiple output sets with fixed point constraints, Comput. Meth. Appl. Math. (2022),

DOI: 10.1515/cmam-2022-0199.

[23] H. Iiduka, I. Yamada, A use of conjugate gradient direction for the convex optimization problem over the fixed point set of a nonexpansive mapping, SIAM J. Optim. 19 (2008), 1881-1893.

[24] P.E. Maingé, Projected subgradient techniques and viscosity methods for optimization with variational inequality constraints, Eur. J. Oper. Res. 205 (2010), 501-506.

[25] G. Cai, Q.L. Dong, Y. Peng, Strong convergence theorems for inertial Tseng's extragradient method for solving variational inequality problems and fixed point problems, Optim. Lett. 15 (2021), 1457-1474.

[26] H.A. Abass, C. Izuchukwu, O.T. Mewomo, On split equality generalized equilibrium problems and fixed point problems of Bregman relatively nonexpansive semigroups, J. Appl. Numer. Optim. 4 (2022), 357-380.

[27] D.V. Thong, D.V. Hieu, Some extragradient-viscosity algorithms for solving variational inequality problems and fixed point problems, Numer. Algo. 82 (2019), 761-789.

[28] G. Marino, H.K. Xu, A general iterative method for nonexpansive mapping in Hilbert spaces, J. Math. Anal. Appl. 318 (2006), 43-52.

[29] H. Zegeye, N. Shahzad, Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings, Comput. Math. Appl. 62 (2011), 4007-4014.

[30] S. Saejung, P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces, Nonlinear Anal. 75 (2012), 742-750.

[31] R.W. Cottle, J.C. Yao, Pseudomonotone complementary problems in Hilbert space, J. Optim. Theory Appl. 75 (1992), 281-295.

[32] D.V. Thong, D.V. Hieu, T.M. Rassias, Self adaptive inertial subgradient extragradient algorithms for solving pseudomonotone variational inequality problems, Optim. Lett. 14 (2020), 115-144.

[33] Liu, H., Yang, J., Weak convergence of iterative methods for solving quasimonotone variational inequalities, *Comput. Optim. Appl.* **77** 491-508 (2020).

[34] M. Tian, B.N. Jiang, Weak convergence theorem for a class of split variational inequality problems and applications in a Hilbert space, J. Ineqal. Appl. 2017 (2017), 123.

[35] X. Chen, Z.B. Wang, Z.Y. Chen, A new method for solving variational inequalities and fixed points problems of demi-contractive pappings in Hilbert spaces, J. Sci. Comput. 85 (2020), 19.

[36] L.C. Ceng, A. Petrusel, X. Qin, Y.C. Yao, Two inertial subgradient extragradient algorithms for variational inequalities with fixed-point constraints, Optimization 70 (2021), 1337-1358.

Appendix 7.1. (Algorithm 3.3 in [35])

Take $x_1 \in H$, $\psi_1 > 0$, $\omega \in (0, \frac{1-\beta}{2}]$, and $\phi \in (0,1)$. Choose the sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfying the assumptions made on the control parameters

Step 1. Compute

$$y_n = P_C(x_n - \psi_n A(x_n)).$$

Step 2. Compute

$$z_n = P_{H_n}(x_n - \psi_n A(y_n)),$$

where

$$H_n := \{x \in H : \langle x_n - \psi_n A(x_n) - y_n, x - y_n \rangle \leq 0\},$$

and

$$\psi_{n+1} := \begin{cases} \min\{\frac{\phi ||x_n - y_n||}{||A(x_n) - A(y_n)||}, \psi_n\}, & \text{if } A(x_n) - A(y_n) \neq 0, \\ \psi_n, & \text{otherwise.} \end{cases}$$

Step 3. Compute

$$t_n := (1-\rho_n)x_n + \rho_n z_n.$$

Step 4. Compute

$$v_n = t_n - \gamma_n G(t_n).$$

Step 5. Compute

$$x_{n+1} = [(1-\omega)I + \omega U]v_n.$$

Let $n = n+1$ and return to Step 1.

Appendix 7.2. (Algorithm 1 in [36])

Initial step: Give $x_0, x_1 \in H$ arbitrarily and let $\lambda > 0$, $m \in (0,1), \mu \in (0,1)$

Iteration steps: Compute x_{n+1} below:

Step 1. Put $v_n = x_n - \sigma_n(x_{n-1} - x_n)$ and calculate $u_n = P_C(v_n - l_n A v_n)$, where l_n is picked to be the largest $l \in \{\lambda, \lambda^m, \lambda^{m^2}, \ldots\}$ s.t

$$l\|Av_n - Au_n\| \leq \mu\|v_n - u_n\|.$$

Step 2. Calculate

$$z_n = (1-\alpha_n)P_{C_n}(v_n - l_n A(u_n)) + \alpha_n f(x_n),$$

where

$$C_n := \{v \in H : \langle v_n - l_n A v_n - u_n, u_n - v \rangle \geq 0\}.$$

Step 3. Compute

$$x_{n+1} = \gamma_n P_{C_n}(v_n - l_n A u_n) + \mu_n T_n z_n + \tau_n x_n.$$

Update $n = n+1$ and return to Step 1.

Appendix 7.3. (Algorithm 2 in [36])

Initial step: Give $x_0, x_1 \in H$ arbitrarily, and let $\lambda > 0$, $m \in (0,1), \mu \in (0,1)$

Iteration steps: Compute x_{n+1} below:

Step 1. Put $v_n = x_n - \sigma_n(x_{n-1} - x_n)$ and calculate $u_n = P_c(v_n - l_n A v_n)$, where l_n is picked to be the largest $l \in \{\lambda, \lambda^m, \lambda^{m^2}, \ldots\}$ s.t

$$l\|Av_n - Au_n\| \leq \mu\|v_n - u_n\|$$

Step 2. Calculate

$$z_n = (1-\alpha_n)P_{C_n}(v_n - l_n A(u_n)) + \alpha_n f(x_n),$$

where

$$C_n := \{v \in H : \langle v_n - l_n A v_n - u_n, u_n - v\rangle \geq 0\}.$$

Step 3. Compute

$$x_{n+1} = \gamma_n P_{C_n}(v_n - l_n A u_n) + \mu_n T_n z_n + \tau_n v_n$$

Update $n = n+1$ and return to Step 1.