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A SELF-ADAPTIVE ITERATIVE METHOD FOR A SPLIT EQUALITY PROBLEM

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Abstract. The purpose of this paper is to introduce an iterative algorithm for the split equality problem with an equilibrium problem, a variational inequality problem, and a fixed point problem of nonexpansive semigroups. We establish a strong convergence theorem of common solutions by the uniformly continuity rather than the Lipchitz continuity of the mappings in real Hilbert spaces. The proposed algorithm only requires one projection each per iteration onto the feasible sets. We also propose a self-adaptive technique that generates non-monotonic sequence of step sizes. Finally, we present a numerical example to illustrate the significance and efficient performance of our algorithm. Our results develop and unify several optimization results in the literature.

Keywords. Equilibrium problem; Split equality problems; Variational inequality; Nonexpansive semigroup.

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1. Introduction

Let C be a nonempty, convex, and closed subset of a real Hilbert space H. Let $\Phi : C\times C \to \mathbb{R}$ be a bifunction. Recall that the equilibrium problem is to find a point $x\in C$ such that

$$\Phi(x,y)\geq 0, \quad \forall y\in C. \tag{1.1}$$

The solution set of (1.1) is denoted by $EP(\Phi)$. Equilibrium problems is quite general. Indeed, it can act as a mathematical modelling for a wide class of problems arising in finance, economics, network analysis, transportation, and elasticity. The equilibrium problem has witnessed an explosive growth in theoretical advances and applications across recently; see, e.g., [2, 4, 5, 6, 7, 8, 19, 22, 26, 25].

If $\Phi(u,v) = \langle v-u, F(u)\rangle$, where $F : C\to H$ is a nonlinear operator, then problem (1.1) reduces to the classical variational inequality, which is to find a point $u^*\in C$ such that

$$\langle v-u^*, F(u)\rangle \geq 0, \quad \forall v\in C. \tag{1.2}$$

Recently, Panyanak et al. [21] proposed a forward-backward explicit iterative algorithms with inertial factors to solve (1.2). Their algorithm reads as follows: for a given $u_0,u_1\in C, \chi\in (0,1), \theta\in(0,1)$, a sequence $\{\tau_n\}$ satisfying $\sum_{n=1}^{\infty}\tau_n<\infty$, and $\{\vartheta_n\}\subset(0,1)$ satisfies the following conditions: $\lim_{n\to\infty}\vartheta_n = 0$ and $\sum_{n=1}^{\infty}\vartheta_n=\infty$. Compute $w_n = (1-\vartheta_n)(u_n+\theta_n(u_n-u_{n-1}))$,

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where θ_n is chosen such that

$$0 \le \theta_n \le \hat{\theta}_n \quad \text{and} \quad \hat{\theta}_n := \begin{cases} \min\{\frac{\theta}{2}, \frac{\|\varepsilon_n\|}{\|u_n - u_{n-1}\|}\} & \text{if } u_n \ne u_{n-1}, \\ \frac{\theta}{2} & \text{otherwise.} \end{cases}$$

where ε_n satisfies the condition $\lim_{n\to\infty} \frac{\varepsilon_n}{\vartheta_n} = 0$. Compute $v_n = P_C[w_n - \lambda_n F(w_n)]$, $u_{n+1} = u_n - \lambda_n(F(v_n) - F(w_n))$, and

$$\lambda_{n+1} := \begin{cases} \min\{\frac{\chi\|w_n - u_n\|}{\|F(w_n) - F(u_n)\|}, \lambda_n + \tau_n\} & \text{if } F(w_n) \ne F(u_n), \\ \lambda_n + \tau_n & \text{otherwise.} \end{cases}$$

Recall that a family $\Gamma_a := \{T(s) : s \ge 0\}$ of mappings from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)u = u$ for all $u \in C$;
(ii) $T(s_1 + s_2) = T(s_1)T(s_1)$ for all $s_1, s_1 \ge 0$;
(iii) $\|T(s)u - T(s)v\| \le \|u - v\|$ for all $u, v \in C$ and $s \ge 0$;
(iv) for all $u \in C$ and $s \ge 0$, $s \mapsto T(s)u$ is continuous.

We denote the set of fixed points of a family Γ_a by $\text{Fix}(\Gamma_a)$, *i.e.*, $\text{Fix}(\Gamma_a) := \{u \in C : T(s)u = u, s \ge 0\}$. A nonexpansive semigroup Γ_a on C is said to be uniformly asymptotically regular (u.a.r) on C if, for all $h > 0$ and any bounded subset E of C, $\lim_{t\to\infty} \sup_{u \in E} \|T(h)(T(t)u) - T(t)u\| = 0$.

Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C, Q be nonempty, convex, and closed subsets of H_1 and H_2, respectively. Let $A_1 : H_1 \to H_3$ and $A_2 : H_2 \to H_3$ two bounded linear operators. Moudafi [16] introduced the following split equality point problem (SEP): find $u \in C$ and $v \in Q$ such that $A_1 u = A_2 v$, which can be seen a generalization of the split feasibility problem introduced by Censor and Elfving [10]. It has been extensively studied recently by many authors; see, e.g., [13, 28, 29] and the references therein.

Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ two bounded linear operators such that $\text{Fix}(S) \ne \emptyset$ and $\text{Fix}(T) \ne \emptyset$. In 2014, Moudafi [17] introduced and studied the following split equality fixed point problem (SEFP):

$$\text{find} \quad u \in \text{Fix}(S), \text{ and } v \in \text{Fix}(T) \quad \text{such that} \quad A_1 u = A_2 v. \tag{1.3}$$

If $H_2 = H_3$ and $A_2 = I$, then split equality fixed point problem (1.3) reduces to the split common fixed point problem (SCFP), originally introduced by Censor and Segal [11]: find $u \in \text{Fix}(S)$ such that $A_1 u \in \text{Fix}(T)$. Moudafi and Al-Shemas [18] proposed the following method for solving (1.3)

$$\begin{cases} u_{n+1} = S(u_n - \gamma_n A_1^*(A_1 u_n - A_2 v_n)), \\ v_{n+1} = T(v_n + \gamma_n A_2^*(A_1 u_n - A_2 v_n)) \end{cases}$$

where S and T are firmly quasi-nonexpansive mappings and $\gamma_n \in (\varepsilon, \frac{2}{\lambda_{A_1} + \lambda_{A_2}} - \varepsilon)$ with λ_{A_1} and λ_{A_2} being the spectral radius of $A_1^* A_1$ and $A_2^* A_2$, respectively. The main advantage of this method is that the step-size γ_n depends on the operator norms $\|A_1\|$ and $\|A_2\|$, which are difficult to compute in some situations. To avoid the knowledge of the operator norms in algorithms, various methods were suggested; see, e.g., [1, 12, 15, 24] and the references therein.

Let $\Gamma_1 := \{T(t) : t \ge 0\}$ and $\Upsilon_1 := \{S(t) : t \ge 0\}$ be two u.a.r nonexpansive semigroups on H_1 and H_2, respectively. Recently, Latif and Eslamian [30] introduced the following iterative scheme for their split equality problem with equilibrium problems, variational inequality problems, and fixed point problems:

$$\begin{cases} z_n = x_n - \gamma_n A_1^*(A_1x_n - A_2y_n)), \\ u_n = P_C[T^{\Phi}_{k_{n,1}} z_n - \lambda_n F(T^{\Phi}_{k_{n,1}} z_n)], \\ v_n = P_C[T^{\Phi}_{k_{n,1}} z_n - \lambda_n F(u_n)], \\ x_{n+1} = \alpha_n \vartheta + \beta_n v_n + \delta_n T(r_n) v_n \\ w_n = y_n + \gamma_n A_2^*(A_1x_n - A_2y_n)), \\ s_n = P_Q[T^{\Psi}_{k_{n,2}} w_n - \rho_n G(T^{\Psi}_{k_{n,2}} w_n)], \\ t_n = P_Q[T^{\Psi}_{k_{n,2}} w_n - \rho_n G(s_n)], \\ x_{n+1} = \alpha_n \zeta + \beta_n t_n + \delta_n S(\iota_n) t_n, \forall n \geq 0, \end{cases}$$

where $F : H_1 \to H_1$ is a monotone and L-Lipschitz continuous operator on C and $G : H_2 \to H_2$ is a monotone and K-Lipschitz continuous operator on Q, $\Phi : C \times C \to \mathbb{R}$ and $\Psi : Q \times Q \to \mathbb{R}$ are functions satisfying Assumption 2.1, $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \delta_n = 1$, and the step-size γ_n is chosen such that, for small enough $\varepsilon > 0$,

$$\gamma_n \in (\varepsilon, \frac{2\|A_1x_n - A_2y_n\|^2}{\|A_2^*(A_1x_n - A_2y_n)\|^2 + \|A_1^*(A_1x_n - A_2y_n)\|^2} - \varepsilon), \quad \text{if} \quad A_1x_n \neq A_2y_n.$$

They proved that $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Omega = \{x \in \text{Fix}(\Gamma_1) \bigcap VI(C,F) \bigcap EP(\Phi)$, $y \in \text{Fix}(\Upsilon_1) \bigcap VI(Q,G) \bigcap EP(\Psi)\}$. We notice that the convergence of this method was established under the assumption that $F(x)$ and $G(y)$ are Lipschitz continuous. However, in many applications, $F(x)$ and $G(y)$ may not be Lipschitz continuous (or it could be difficult to verify their Lipschitz continuity condition). Motivated by the results of Latif and Eslamian [30], Panyanak et al. [21], and the ongoing research in this direction, in this paper, we introduce an iterative algorithm for the split equality problem with an equilibrium problem, a variational inequality problem, and a fixed point problem of nonexpansive semigroups. We establish a strong convergence theorem of solutions by the uniformly continuity rather than the Lipchitz continuity of these mappings. The proposed method is self-adaptive, which does not require any line search technique used in the literature. We also present a numerical example to illustrate the significance and efficient performance of our method.

2. Preliminaries

In this section, we give some useful preliminary results which are used in establishing the convergence of our method in the sequel.

Assumption 2.1. [3] Let $\Phi_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(i) $\Phi_1(u,u) = 0$ for all $u \in C$;
(ii) $\Phi_1(u,v) + \Phi_1(v,u) \leq 0$ for all $u, v \in C$, that is, Φ_1 is monotone;
(iii) $\limsup_{t \to 0} \Phi_1(tw + (1-t)u, v) \leq \Phi_1(u,v)$ for each $u, v, w \in C$;
(iv) $v \to \Phi_1(u,v)$ is lower semicontinuous and convex for each u in C.

Lemma 2.1. [9] *Let $\Phi_1 : C \times C \to \mathbb{R}$ satisfy Assumption 2.1. Define a mapping $T_r^{\Phi_1} : H \to C$, for some $r > 0$ and for all $u \in H$, by $T_r^{\Phi_1} u = \{w \in C : \Phi_1(w,v) + \frac{1}{r}\langle v - w, w - u\rangle \geq 0,\ \forall v \in C\}$. Then the following hold:*

(i) $T_r^{\Phi_1}$ *is single-valued;*
(ii) $\|T_r^{\Phi_1}u - T_r^{\Phi_1}v\|^2 \leq \langle T_r^{\Phi_1}u - T_r^{\Phi_1}v, u-v\rangle$ *for all* $u,v \in H$*, that is,* $T_r^{\Phi_1}$ *is firmly nonexpansive;*
(iii) $\mathrm{Fix}(T_r^{\Phi_1}) = \mathrm{EP}(\Phi_1)$ *is convex and closed.*

Lemma 2.2. [20] *Each Hilbert space* H *satisfies the Opial conditions, i.e., for any sequence* $\{u_n\}$ *with* $u_n \rightharpoonup u$ *the inequality* $\liminf_{n\to\infty}\|u_n - u\| < \liminf_{n\to\infty}\|u_n - v\|$ *holds for every* $v \in H$ *with* $v \neq u$.

Lemma 2.3. [27] *A function* F_1 *defined on a convex domain is uniformly continuous, i.e., for every* $\varepsilon_1 > 0$*, there exists a* $\delta_1 > 0$ *such that* $\|F_1(u) - F_1(v)\| < \varepsilon_1$ *whenever* $\|u - v\| < \varepsilon_1$*, if and only if, for every* $\varepsilon_1 > 0$*, there exists a* $K_1 < \infty$ *such that* $\|F_1(u) - F_1(v)\| \leq K_1\|u - v\| + \varepsilon_1$.

Lemma 2.4. [14] *If* $F : H_1 \to H_1$ *is a monotone and L-Lipschitz continuous operator on* C *and* $G : H_2 \to H_2$ *is a monotone and M-Lipschitz continuous operator on* Q*, then the sequences* $\{\lambda_n\}$ *and* $\{\rho_n\}$ *defined by (3.1) and (3.2), respectively are convergent to* λ *and* ρ*, receptively with* $\min\{\frac{\chi_1}{L}, \lambda_0\} \leq \lambda_n \leq \lambda_0 + P$*, where* $P = \sum_{n=0}^{\infty}\tau_n$ *and* $\min\{\frac{\chi_2}{M}, \rho_0\} \leq \rho_n \leq \rho_0 + M$*, where* $M = \sum_{n=0}^{\infty}\mu_n$.

Lemma 2.5. [23] *Let* $\{a_n\}$ *be a real positive sequence and let* $\{\kappa_n\}$ *be a real sequence in* $(0,1)$ *such that* $\sum_{n=1}^{\infty}\kappa_n = \infty$ *with* $a_{n+1} \leq (1-\kappa_n)a_n + \kappa_n\varphi_n$*, where* φ_n *is a real sequence with* $\limsup_{k\to\infty}\varphi_{n_k} \leq 0$ *for all subsequences* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying* $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$*. Then,* $\lim_{n\to\infty}a_n = 0$.

Finally, we also need the following trivial inequalities and equalities, which hold in Hilbert spaces

(i) $\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v\rangle$ for all $u,v \in H$.
(ii) $2\langle u,v\rangle = \|u\|^2 + \|v\|^2 - \|u-v\|^2 = \|u+v\|^2 - \|u\|^2 - \|v\|^2$ for all $u,v \in H$.
(iii) $\|\eta_1u_1 + \cdots + \eta_m u_m\|^2 = \sum_{i=1}^m \eta_i\|u_i\|^2 - \sum_{1\leq i\leq j\leq m}\eta_i\eta_j\|u_i - u_j\|^2$, where $u_1,\cdots,u_m \in H$ and $\eta_1,\cdots,\eta_m \in [0,1]$ with $\sum_{i=1}^m \eta_i = 1$.

3. The Proposed Method

In this section, we propose and investigate our method in Hilbert spaces. Let $A_1 : H_1 \to H_3$ and $A_2 : H_2 \to H_3$ two bounded and linear operators. Let $\Gamma := \{T_1(t) : t \geq 0\}$ and $\Upsilon := \{T_2(t) : t \geq 0\}$ be two u.a.r nonexpansive semigroups on H_1 and H_2, respectively. Let $F : H_1 \to H_1$ is a monotone and uniformly continuous operator on C and $G : H_2 \to H_2$ be a monotone and uniformly continuous operator on Q. Let $\Phi : C\times C \to \mathbb{R}$ and $\Psi : Q\times Q \to \mathbb{R}$ be functions satisfying Assumption 2.1. Let $\Omega = \{x \in \mathrm{Fix}(\Gamma)\bigcap VI(C,F)\bigcap EP(\Phi),\ y \in \mathrm{Fix}(\Upsilon)\bigcap VI(Q,G)\bigcap EP(\Psi) : A_1x = A_2y\}$ be nonempty. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{r_n\}$, $\{\iota_n\}$, $\{r_{n,1}\}$, $\{r_{n,2}\}$, $\{\tau_n\}$, and $\{\mu_n\}$ be nonnegative sequences satisfying the following conditions:

(a) $\alpha_n + \beta_n + \delta_n = 1$ and $\liminf_{n\to\infty}\beta_n\delta_n > 0$;
(b) $\lim_{n\to\infty} r_n = \infty$ and $\lim_{n\to\infty}\iota_n = \infty$;
(c) $\alpha_n \in (0,\frac{1}{2})$, $\lim_{n\to\infty}\alpha_n = 0$, and $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(d) $\liminf_{n\to\infty} r_{n,1} > 0$ and $\liminf_{n\to\infty} r_{n,2} > 0$;
(e) $\sum_{n=0}^{\infty}\tau_n < \infty$ and $\sum_{n=0}^{\infty}\mu_n < \infty$.;

Algorithm 3.1.

Step 0. The initial step: Give $\gamma > 0$, $(x_0, y_0) \in H_1 \times H_2, (\vartheta, \zeta) \in H_1 \times H_2, \chi_1 \in (0,1), \chi_2 \in (0,1), \lambda_0 > 0$, and $\rho_0 > 0$. Set $n = 0$.

Step 1. Compute

$$z_n = x_n - \gamma_n A_1^*(A_1 x_n - A_2 y_n),$$
$$u_n = P_C[b_n - \lambda_n F(b_n)], \text{ where } b_n = T^{\Phi}_{r_{n,1}}(z_n),$$
$$v_n = u_n - \lambda_n(F(u_n) - F(b_n)),$$
$$x_{n+1} = \alpha_n \vartheta + \beta_n v_n + \delta_n T_1(r_n) v_n,$$

and

$$\lambda_{n+1} := \begin{cases} \min\{\frac{\chi_1 \|u_n - b_n\|}{\|F(u_n) - F(b_n)\|}, \lambda_n + \tau_n\} & \text{if } F(u_n) \neq F(b_n), \\ \lambda_n + \tau_n & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute

$$w_n = y_n + \gamma_n A_2^*(A_1 x_n - A_2 y_n),$$
$$s_n = P_Q[c_n - \rho_n G(c_n)], \text{ where } c_n = T^{\Psi}_{r_{n,2}}(w_n),$$
$$t_n = s_n - \rho_n(G(s_n) - G(c_n)),$$
$$y_{n+1} = \alpha_n \zeta + \beta_n t_n + \delta_n T_2(\iota_n) t_n$$

and

$$\rho_{n+1} := \begin{cases} \min\{\frac{\chi_2 \|s_n - c_n\|}{\|G(s_n) - G(c_n)\|}, \rho_n + \mu_n\} & \text{if } G(s_n) \neq G(c_n), \\ \rho_n + \mu_n & \text{otherwise,} \end{cases} \tag{3.2}$$

where γ_n is chosen such that, for small enough $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon, \frac{2\|A_1 x_n - A_2 y_n\|^2}{\|A_2^*(A_1 x_n - A_2 y_n)\|^2 + \|A_1^*(A_1 x_n - A_2 y_n)\|^2} - \varepsilon\right),$$

if $A_1 x_n \neq A_2 y_n$; otherwise, $\gamma_n = \gamma$.

Set $n := n+1$ and go to Step 1.

Remark 3.1. It follows from Lemma 2.4 that

$$\lim_{n\to\infty} \left(1 - \frac{\lambda_n^2 \chi_1^2}{\lambda_{n+1}^2}\right) = 1 - \chi_1^2 > 0. \tag{3.3}$$

Hence, there exists $n_1 > 0$ such that, for all $n > n_1$, $1 - \frac{\lambda_n^2 \chi_1^2}{\lambda_{n+1}^2} > 0$. Similarly, we have

$$\lim_{n\to\infty} \left(1 - \frac{\rho_n^2 \chi_2^2}{\rho_{n+1}^2}\right) = 1 - \chi_2^2 > 0. \tag{3.4}$$

Thus there exists $n_2 > 0$ such that, for all $n > n_2$, $1 - \frac{\rho_n^2 \chi_2^2}{\rho_{n+1}^2} > 0$.

In what follows, we set $n_0 = \max(n_1, n_2)$, $E_1 := \mathrm{Fix}(\Gamma) \bigcap VI(C,F) \bigcap EP(\Phi)$, and $E_2 := \mathrm{Fix}(\Upsilon) \bigcap VI(Q,G) \bigcap EP(\Psi)$. To prove the global convergence for the proposed method, we first prove the following important lemmas.

Lemma 3.1. *Let $\{(x_n, y_n)\}$ be a sequence generated by the Algorithm 3.1. Then,*

(i) $\|z_n - x^*\|^2 + \|w_n - y^*\|^2 \leq \|x_n - x^*\|^2 + \|y_n - y^*\|^2$;

(ii) $\|v_n - x^*\|^2 \le \|z_n - x^*\|^2 - \|z_n - b_n\|^2 - \left(1 - \frac{\lambda_n^2 \chi_1^2}{\lambda_{n+1}^2}\right)\|u_n - b_n\|^2$;

(iii) $\|t_n - x^*\|^2 \le \|w_n - y^*\|^2 - \|w_n - c_n\|^2 - \left(1 - \frac{\rho_n^2 \chi_2^2}{\rho_{n+1}^2}\right)\|s_n - c_n\|^2$,

where $(x^*, y^*) \in \Omega$.

Proof. Observe

$$\begin{aligned} \|z_n - x^*\|^2 &= \|x_n - x^*\|^2 + \gamma_n^2 \|A_1^*(A_1x_n - A_2y_n)\|^2 - 2\gamma_n \langle x_n - x^*, A_1^*(A_1x_n - A_2y_n)\rangle \\ &= \|x_n - x^*\|^2 + \gamma_n^2 \|A_1^*(A_1x_n - A_2y_n)\|^2 - \gamma_n \|A_1x_n - A_1x^*\|^2 - \gamma_n \|A_1x_n - A_2y_n\|^2 \\ &\quad + \gamma_n \|A_2y_n - A_1x^*\|^2. \end{aligned} \tag{3.5}$$

Similarly, we have

$$\begin{aligned} \|w_n - y^*\|^2 &= \|y_n - y^*\|^2 + \gamma_n^2 \|A_2^*(A_1x_n - A_2y_n)\|^2 - \gamma_n \|A_2y_n - A_2y^*\|^2 - \gamma_n \|A_1x_n - A_2y_n\|^2 \\ &\quad + \gamma_n \|A_1x_n - A_2y^*\|^2. \end{aligned} \tag{3.6}$$

Adding (3.5) and (3.6), we arrive at

$$\begin{aligned} &\|z_n - x^*\|^2 + \|w_n - y^*\|^2 \\ &= \|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \gamma_n^2 [\|A_1^*(A_1x_n - A_2y_n)\|^2 + \|A_2^*(A_1x_n - A_2y_n)\|^2] - \gamma_n [\|A_1x_n - A_1x^*\|^2 \\ &\quad + \|A_2y_n - A_2y^*\|^2] - 2\gamma_n \|A_1x_n - A_2y_n\|^2 + \gamma_n [\|A_2y_n - A_1x^*\|^2 + \|A_1x_n - A_2y^*\|^2]. \end{aligned}$$

In view of

$$(\gamma_n + \varepsilon)(\|A_2^*(A_1x_n - A_2y_n)\|^2 + \|A_1^*(A_1x_n - A_2y_n)\|^2) \le 2\|A_1x_n - A_2y_n\|^2,$$

we find that

$$\begin{aligned} &\gamma_n \varepsilon (\|A_2^*(A_1x_n - A_2y_n)\|^2 + \|A_1^*(A_1x_n - A_2y_n)\|^2) \\ &\le \gamma_n [2\|A_1x_n - A_2y_n\|^2 - \gamma_n (\|A_2^*(A_1x_n - A_2y_n)\|^2 + \|A_1^*(A_1x_n - A_2y_n)\|^2)]. \end{aligned}$$

Since $A_1x^* = A_2y^*$, we have

$$\begin{aligned} &\|z_n - x^*\|^2 + \|w_n - y^*\|^2 \\ &\le \|x_n - x^*\|^2 + \|y_n - y^*\|^2 - \gamma_n \varepsilon [\|A_1^*(A_1x_n - A_2y_n)\|^2 + \|A_2^*(A_1x_n - A_2y_n)\|^2] \\ &\le \|x_n - x^*\|^2 + \|y_n - y^*\|^2. \end{aligned} \tag{3.7}$$

Thus assertion (i) is proved.

Next, we prove assertions (ii) and (iii). Since $T_{r_{n,1}}^{\Phi}$ is firmly nonexpansive, it follows from Lemma 2.1 that

$$\begin{aligned} \|b_n - x^*\|^2 &\le \langle T_{r_{n,1}}^{\Phi}(z_n) - T_{r_{n,1}}^{\Phi}(x^*), z_n - x^*\rangle \\ &= \frac{1}{2}\left(\|T_{r_{n,1}}^{\Phi}(z_n) - T_{r_{n,1}}^{\Phi}(x^*)\|^2 + \|z_n - x^*\|^2 - \|T_{r_{n,1}}^{\Phi}(z_n) - z_n\|^2\right) \\ &= \frac{1}{2}\left(\|b_n - x^*\|^2 + \|z_n - x^*\|^2 - \|b_n - z_n\|^2\right), \end{aligned}$$

that is,

$$\|b_n - x^*\|^2 \le \|z_n - x^*\|^2 - \|z_n - b_n\|^2. \tag{3.8}$$

Similarly, $\|c_n - y^*\|^2 \leq \|T^{\Psi}_{r_{n,2}}(w_n) - y^*\|^2 \leq \|w_n - y^*\|^2 - \|w_n - c_n\|^2$. From (3.1), we have

$$\lambda_{n+1} = \min\left\{\frac{\chi_1\|u_n - b_n\|}{\|F(u_n) - F(b_n)\|}, \lambda_n + \tau_n\right\} \leq \frac{\chi_1\|u_n - b_n\|}{\|F(u_n) - F(b_n)\|}$$

which implies that

$$\|F(u_n) - F(b_n)\| \leq \frac{\chi_1}{\lambda_{n+1}}\|u_n - b_n\|. \tag{3.9}$$

Similarly, we have

$$\|G(s_n) - G(c_n)\| \leq \frac{\chi_2}{\rho_{n+1}}\|s_n - c_n\|. \tag{3.10}$$

Observe that

$$\begin{aligned}
&\|v_n - x^*\|^2 \\
&= \|u_n - x^*\|^2 + \lambda_n^2\|F(u_n) - F(b_n)\|^2 - 2\lambda_n\langle F(u_n) - F(b_n), u_n - x^*\rangle \\
&= \|b_n - x^*\|^2 + \|u_n - b_n\|^2 - 2\|u_n - b_n\|^2 + 2\langle u_n - b_n, u_n - x^*\rangle + \lambda_n^2\|F(u_n) - F(b_n)\|^2 \\
&\quad - 2\lambda_n\langle F(u_n) - F(b_n), u_n - x^*\rangle \\
&= \|b_n - x^*\|^2 - \|u_n - b_n\|^2 + 2\langle u_n - b_n, u_n - x^*\rangle + \lambda_n^2\|F(u_n) - F(b_n)\|^2 \\
&\quad - 2\lambda_n\langle F(u_n) - F(b_n), u_n - x^*\rangle.
\end{aligned} \tag{3.11}$$

In view of $u_n = P_C[b_n - \lambda_n F(b_n)]$ and $x^* \in C$, we obtain $\langle u_n - b_n + \lambda_n F(b_n), x^* - u_n\rangle \geq 0$, which implies that $-\lambda_n\langle F(b_n), u_n - x^*\rangle \geq \langle u_n - b_n, u_n - x^*\rangle$. Since $u_n \in C$ and $x^* \in E_1$, we have $\langle F(u_n), u_n - x^*\rangle \geq 0$, which together with (3.8), (3.9), and (3.11) yields

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \|b_n - x^*\|^2 - \|u_n - b_n\|^2 - 2\lambda_n\langle F(b_n), u_n - x^*\rangle + \lambda_n^2\|F(u_n) - F(b_n)\|^2 \\
&\quad - 2\lambda_n\langle F(u_n) - F(b_n), u_n - x^*\rangle \\
&\leq \|b_n - x^*\|^2 - \|u_n - b_n\|^2 + \frac{\lambda_n^2\chi_1^2}{\lambda_{n+1}^2}\|u_n - b_n\|^2 \\
&\leq \|z_n - x^*\|^2 - \|z_n - b_n\|^2 - \left(1 - \frac{\lambda_n^2\chi_1^2}{\lambda_{n+1}^2}\right)\|u_n - b_n\|^2.
\end{aligned}$$

Similarly, we have $\|t_n - x^*\|^2 \leq \|w_n - y^*\|^2 - \|w_n - c_n\|^2 - \left(1 - \frac{\rho_n^2\chi_2^2}{\rho_{n+1}^2}\right)\|s_n - c_n\|^2$, which completes the proof. □

Lemma 3.2. *Let $\{(x_n, y_n)\}$ be a sequence generated by the Algorithm 3.1. Then $\{(x_n, y_n)\}$ is bounded.*

Proof. Fix $(x^*, y^*) \in \Omega$. It follows from Lemma 3.1 that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \alpha_n\|\vartheta - x^*\|^2 + \beta_n\|v_n - x^*\|^2 + \delta_n\|T_1(r_n)v_n - x^*\|^2 - \beta_n\delta_n\|T_1(r_n)v_n - v_n\|^2\\ &\leq \alpha_n\|\vartheta - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 - \beta_n\delta_n\|T_1(r_n)v_n - v_n\|^2\\ &\leq \alpha_n\|\vartheta - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2 - (1-\alpha_n)\|z_n - b_n\|^2\\ &\quad -(1-\alpha_n)\left(1-\frac{\lambda_n^2\chi_1^2}{\lambda_{n+1}^2}\right)\|u_n - b_n\|^2 - \beta_n\delta_n\|T_1(r_n)v_n - v_n\|^2\\ &\leq \alpha_n\|\vartheta - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2.\end{aligned} \tag{3.12}$$

Similarly, we have

$$\begin{aligned}\|y_{n+1}-y^*\|^2 &\leq \alpha_n\|\zeta - y^*\|^2 + (1-\alpha_n)\|w_n - y^*\|^2 - (1-\alpha_n)\|w_n - c_n\|^2\\ &\quad -(1-\alpha_n)\left(1-\frac{\rho_n^2\chi_2^2}{\rho_{n+1}^2}\right)\|s_n - c_n\|^2 - \beta_n\delta_n\|T_2(\iota_n)t_n - t_n\|^2\\ &\leq \alpha_n\|\zeta - y^*\|^2 + (1-\alpha_n)\|w_n - y^*\|^2,\end{aligned} \tag{3.13}$$

which together with (3.7) and (3.12) implies

$$\|x_{n+1}-x^*\|^2 + \|y_{n+1}-y^*\|^2 \leq \max\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2, \|\vartheta - x^*\|^2 + \|\zeta - y^*\|^2\}.$$

By induction on n, we obtain that $\{(x_n, y_n)\}$ is bounded. Consequently, we deduce that $\{z_n\}$, $\{b_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{c_n\}$, $\{s_n\}$, and $\{t_n\}$ are bounded. □

4. Convergence Analysis

In this section, we prove the strong convergence of the proposed method. Note that the proof of strong convergence result does not need the two cases approach used in [30].

Theorem 4.1. *Let $\{(x_n, y_n)\}$ be a sequence generated by Algorithm 3.1. Then, the sequence $\{(x_n, y_n)\}$ converges strongly to $(\tilde{x}, \tilde{y}) \in \Omega$, where $\tilde{x} = P_{E_1}[\vartheta]$ with $E_1 := \mathrm{Fix}(\Gamma)\bigcap VI(C,F)\bigcap EP(\Phi)$ and $\tilde{y} = P_{E_2}[\zeta]$ with $E_2 := \mathrm{Fix}(\Upsilon)\bigcap VI(Q,G)\bigcap EP(\Psi)$.*

Proof. Let $(\tilde{x},\tilde{y})\in\Omega$, where $\tilde{x}=P_{E_1}[\vartheta]$ and $\tilde{y}=P_{E_2}[\zeta]$. From (3.12), (3.13), and (3.7), we have

$$\begin{aligned}
&\|x_{n+1}-\tilde{x}\|^2+\|y_{n+1}-\tilde{y}\|^2\\
&\leq(1-\alpha_n)(\|z_n-\tilde{x}\|^2+\|w_n-\tilde{y}\|^2)+\alpha_n(\|\vartheta-\tilde{x}\|^2+\|\zeta-\tilde{y}\|^2)\\
&-(1-\alpha_n)(\|z_n-b_n\|^2+\|w_n-c_n\|^2)\\
&-(1-\alpha_n)\left(1-\frac{\lambda_n^2\chi_1^2}{\lambda_{n+1}^2}\right)\|u_n-b_n\|^2-\beta_n\delta_n\|T_1(r_n)v_n-v_n\|^2\\
&-(1-\alpha_n)\left(1-\frac{\rho_n^2\chi_2^2}{\rho_{n+1}^2}\right)\|s_n-c_n\|^2-\beta_n\delta_n\|T_2(\iota_n)t_n-t_n\|^2\\
&\leq(1-\alpha_n)(\|x_n-\tilde{x}\|^2+\|y_n-\tilde{y}\|^2)+\alpha_n(\|\vartheta-\tilde{x}\|^2+\|\zeta-\tilde{y}\|^2)\\
&-(1-\alpha_n)\gamma_n\varepsilon[\|A_1^*(A_1x_n-A_2y_n)\|^2+\|A_2^*(A_1x_n-A_2y_n)\|^2]\\
&-(1-\alpha_n)(\|z_n-b_n\|^2+\|w_n-c_n\|^2)\\
&-(1-\alpha_n)\left(1-\frac{\lambda_n^2\chi_1^2}{\lambda_{n+1}^2}\right)\|u_n-b_n\|^2-\beta_n\delta_n\|T_1(r_n)v_n-v_n\|^2\\
&-(1-\alpha_n)\left(1-\frac{\rho_n^2\chi_2^2}{\rho_{n+1}^2}\right)\|s_n-c_n\|^2-\beta_n\delta_n\|T_2(\iota_n)t_n-t_n\|^2
\end{aligned}\tag{4.1}$$

Suppose that $\{\|x_{n_k}-\tilde{x}\|^2+\|y_{n_k}-\tilde{y}\|^2\}$ is a subsequence of $\{\|x_n-\tilde{x}\|^2+\|y_n-\tilde{y}\|^2\}$ satisfying

$$\lim_{k\to\infty}\inf((\|x_{n_{k+1}}-\tilde{x}\|^2+\|y_{n_{+1}}-\tilde{y}\|^2)-(\|x_{n_k}-\tilde{x}\|^2+\|y_{n_k}-\tilde{y}\|^2))\geq 0.\tag{4.2}$$

From (4.1), we obtain

$$\begin{aligned}
&(1-\alpha_{n_k})\gamma_{n_k}\varepsilon[\|A_1^*(A_1x_{n_k}-A_2y_{n_k})\|^2+\|A_2^*(A_1x_{n_k}-A_2y_{n_k})\|^2]\\
&+(1-\alpha_{n_k})(\|z_{n_k}-b_{n_k}\|^2+\|w_{n_k}-c_{n_k}\|^2)+(1-\alpha_{n_k})\left(1-\frac{\lambda_{n_k}^2\chi_1^2}{\lambda_{n+1}^2}\right)\|u_{n_k}-b_{n_k}\|^2\\
&+\beta_{n_k}\delta_{n_k}\|T_1(r_{n_k})v_{n_k}-v_{n_k}\|^2+(1-\alpha_{n_k})\left(1-\frac{\rho_{n_k}^2\chi_2^2}{\rho_{n+1}^2}\right)\|s_{n_k}-c_{n_k}\|^2+\beta_{n_k}\delta_{n_k}\|T_2(\iota_{n_k})t_{n_k}-t_{n_k}\|^2\\
&\leq(1-\alpha_{n_k})(\|x_{n_k}-\tilde{x}\|^2+\|y_{n_k}-\tilde{y}\|^2)-(\|x_{n_{k+1}}-\tilde{x}\|^2+\|y_{n_{+1}}-\tilde{y}\|^2)+\alpha_{n_k}(\|\vartheta-\tilde{x}\|^2+\|\zeta-\tilde{y}\|^2)
\end{aligned}$$

From (4.2) and $\lim_{n\to\infty}\alpha_n=0$, we have that

$$\begin{aligned}
&\lim_{k\to\infty}[\gamma_{n_k}\varepsilon[\|A_1^*(A_1x_{n_k}-A_2y_{n_k})\|^2+\|A_2^*(A_1x_{n_k}-A_2y_{n_k})\|^2]+\|z_{n_k}-b_{n_k}\|^2+\|w_{n_k}-c_{n_k}\|^2\\
&+\left(1-\frac{\lambda_{n_k}^2\chi_1^2}{\lambda_{n+1}^2}\right)\|u_{n_k}-b_{n_k}\|^2+\beta_{n_k}\delta_{n_k}\|T_1(r_{n_k})v_{n_k}-v_{n_k}\|^2+\left(1-\frac{\rho_{n_k}^2\chi_2^2}{\rho_{n+1}^2}\right)\|s_{n_k}-c_{n_k}\|^2\\
&+\beta_{n_k}\delta_{n_k}\|T_2(\iota_{n_k})t_{n_k}-t_{n_k}\|^2]\leq 0.
\end{aligned}$$

Recalling (3.3), (3.4) and condition (a), we have

$$\lim_{k\to\infty}\|z_{n_k}-b_{n_k}\|=0,\ \lim_{k\to\infty}\|w_{n_k}-c_{n_k}\|=0,\ \lim_{k\to\infty}\|u_{n_k}-b_{n_k}\|=0,$$
$$\lim_{k\to\infty}\|T_1(r_{n_k})v_{n_k}-v_{n_k}\|=0,\ \lim_{k\to\infty}\|s_{n_k}-c_{n_k}\|=0, \tag{4.3}$$
$$\lim_{k\to\infty}\|T_2(\iota_{n_k})t_{n_k}-t_{n_k}\|=0,$$

and

$$\lim_{k\to\infty}\|A_1^*(A_1x_{n_k}-A_2y_{n_k})\|^2+\|A_2^*(A_1x_{n_k}-A_2y_{n_k})\|^2=0.$$

Thus we obtain that $\lim_{k\to\infty}\|A_1^*(A_1x_{n_k}-A_2y_{n_k})\|=\lim_{k\to\infty}\|A_2^*(A_1x_{n_k}-A_2y_{n_k})\|=0$, which implies that $\lim_{k\to\infty}\|A_1x_{n_k}-A_2y_{n_k}\|=0$. Since $\|z_{n_k}-x_{n_k}\|=\gamma_{n_k}\|A_1^*(A_1x_{n_k}-A_2y_{n_k})\|$, $\|w_{n_k}-y_{n_k}\|=\gamma_{n_k}\|A_2^*(A_1x_{n_k}-A_2y_{n_k})\|$, and γ_{n_k} is bounded, we have

$$\lim_{k\to\infty}\|z_{n_k}-x_{n_k}\|=\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=0. \tag{4.4}$$

Since

$$\|v_{n_k}-u_{n_k}\|=\lambda_{n_k}\|F(u_{n_k})-F(b_{n_k})\|\le\frac{\lambda_{n_k}\chi_1}{\lambda_{n_{k+1}}}\|u_{n_k}-b_{n_k}\|$$

and

$$\|t_{n_k}-s_{n_k}\|=\rho_{n_k}\|G(u_{s_k})-G(c_{n_k})\|\le\frac{\rho_{n_k}\chi_2}{\rho_{n_{k+1}}}\|s_{n_k}-c_{n_k}\|.$$

It follows from (4.3) that $\lim_{k\to\infty}\|v_{n_k}-u_{n_k}\|=\lim_{k\to\infty}\|t_{n_k}-s_{n_k}\|=0$, which together with (4.3) and (4.4) yields $\|v_{n_k}-x_{n_k}\|\le\|v_{n_k}-u_{n_k}\|+\|u_{n_k}-b_{n_k}\|+\|b_{n_k}-z_{n_k}\|+\|z_{n_k}-x_{n_k}\|\to 0$ as $k\to\infty$. Similarly, we have $\|t_{n_k}-y_{n_k}\|\le\|t_{n_k}-s_{n_k}\|+\|s_{n_k}-c_{n_k}\|+\|c_{n_k}-w_{n_k}\|+\|w_{n_k}-y_{n_k}\|\to 0$ as $k\to\infty$. Further, we see that

$$\begin{aligned}\|v_{n_k}-T_1(h)v_{n_k}\| &\le \|v_{n_k}-T_1(r_{n_k})v_{n_k}\|+\|T_1(r_{n_k})v_{n_k}-T_1(h)T_1(r_{n_k})v_{n_k}\|\\ &\quad+\|T_1(h)T_1(r_{n_k})v_{n_k}-T_1(h)v_{n_k}\|\\ &\le 2\|v_{n_k}-T_1(r_{n_k})v_{n_k}\|+\|T_1(r_{n_k})v_{n_k}-T_1(h)T_1(r_{n_k})v_{n_k}\|.\end{aligned}$$

Using (4.3) and the fact that $T_1(h)$ is u.a.r. nonexpansive semigroup, we have $\lim_{k\to\infty}\|v_{n_k}-T_1(h)v_{n_k}\|=0$. Similarly, we have $\lim_{k\to\infty}\|t_{n_k}-T_2(h)t_{n_k}\|=0$. It follows from (4.3) and (4.4) that

$$\lim_{k\to\infty}\|x_{n_k}-b_{n_k}\|=\lim_{k\to\infty}\|y_{n_k}-c_{n_k}\|=0. \tag{4.5}$$

From the definition of $x_{n_{k+1}}$, we have

$$\|x_{n_{k+1}}-x_{n_k}\|\le\alpha_{n_k}\|\vartheta-x_{n_k}\|+\beta_{n_k}\|v_{n_k}-x_{n_k}\|+\delta_{n_k}\|T_1(r_{n_k})v_{n_k}-x_{n_k}\|.$$

It follows that $\lim_{k\to\infty}\|x_{n_{k+1}}-x_{n_k}\|=0$. Similarly, we have $\lim_{k\to\infty}\|y_{n_{k+1}}-y_{n_k}\|=0$.

Now, we prove that $(\omega_w(x_n),\omega_w(y_n))\subset\Omega$, where

$$\omega_w(x_n)=\{x\in H_1: x_{n_i}\rightharpoonup x \text{ for some subsequences } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, we have that $\omega_w(x_n)$ and $\omega_w(x_n)$ are nonempty. Let $\tilde{x}\in\omega_w(x_n)$ and $\tilde{y}\in\omega_w(y_n)$. Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\tilde{x}$ as

$k \to \infty$. Using (4.5), we have $b_{n_k} \rightharpoonup \tilde{x}$ as $k \to \infty$. From $u_{n_k} = P_C[b_{n_k} - \lambda_{n_k} F(b_{n_k})]$, we have $\langle u_{n_k} - x, b_{n_k} - \lambda_{n_k} F(b_{n_k}) - u_{n_k}\rangle \geq 0$ for all $x \in C$. Since F is monotone, we have

$$
\begin{aligned}
&\langle \lambda_{n_k} F(x), b_{n_k} - x\rangle \\
&\leq \langle \lambda_{n_k} F(b_{n_k}), b_{n_k} - x\rangle \\
&= \langle \lambda_{n_k} F(b_{n_k}), b_{n_k} - u_{n_k}\rangle + \langle \lambda_{n_k} F(b_{n_k}) - b_{n_k} + u_{n_k}, u_{n_k} - x\rangle + \langle b_{n_k} - u_{n_k}, u_{n_k} - x\rangle \\
&\leq \langle \lambda_{n_k} F(b_{n_k}), b_{n_k} - u_{n_k}\rangle + \langle b_{n_k} - u_{n_k}, u_{n_k} - x\rangle \\
&\leq \lambda_{n_k} \|F(b_{n_k})\| \|b_{n_k} - u_{n_k}\| + \|b_{n_k} - u_{n_k}\| \|u_{n_k} - x\|.
\end{aligned}
$$

Thus $\langle F(x), b_{n_k} - x\rangle \leq \|F(b_{n_k})\| \|b_{n_k} - u_{n_k}\| + \frac{1}{\lambda_{n_k}} \|b_{n_k} - u_{n_k}\| \|u_{n_k} - x\|$. Since $F(b_{n_k})$ is bounded, $\lim_{k\to\infty} \|u_{n_k} - b_{n_k}\| = 0$, $\lim_{k\to\infty} \lambda_{n_k} = \lambda > 0$, and $b_{n_k} \rightharpoonup \tilde{x}$, we obtain

$$\langle F(x), \tilde{x} - x\rangle = \lim_{k\to\infty} \langle F(x), b_{n_k} - x\rangle \leq 0, \quad \forall x \in C,$$

which implies that $\tilde{x} \in VI(C,F)$. Similarly, we can obtain that $\tilde{y} \in VI(Q,G)$.

Next, we show that $\tilde{x} \in \mathrm{Fix}(\Gamma)$ and $\tilde{y} \in \mathrm{Fix}(\Upsilon)$. Since $\lim_{k\to\infty} \|v_{n_k} - x_{n_k}\| = 0$, we have $v_{n_k} \rightharpoonup \tilde{x}$ as $k \to \infty$. Observe that

$$\|v_{n_k} - T_1(r)\tilde{x}\| \leq \|v_{n_k} - T_1(r)v_{n_k}\| + \|T_1(r)v_{n_k} - T_1(r)\tilde{x}\| \leq \|v_{n_k} - T_1(r)v_{n_k}\| + \|v_{n_k} - \tilde{x}\|.$$

It follows that $\liminf_{k\to\infty} \|v_{n_k} - T_1(r)\tilde{x}\| \leq \liminf_{k\to\infty} \|v_{n_k} - \tilde{x}\|$. By the Opial property ((Lemma 2.2)), we obtain that $T_1(r)\tilde{x} = \tilde{x}$ for all $r \geq 0$, which implies that $\tilde{x} \in \mathrm{Fix}(\Gamma)$. Similarly, we obtain that $\tilde{y} \in \mathrm{Fix}(\Upsilon)$. From (4.3), we have $\lim_{k\to\infty} \|z_{n_k} - b_{n_k}\| = \|z_{n_k} - T^{\Phi}_{r_{n,1}} z_{n_k}\| = 0$. It follows from the demiclosed property of nonexpansive mappings that $\tilde{x} \in EP(\Phi)$. Similarly, we have that $\tilde{y} \in EP(\Psi)$. Since $A_1\tilde{x} - A_2\tilde{y} \in \omega_w(A_1x_n - A_2y_n)$, it follows from the weakly lower semi-continuity of the norm that $\|A_1\tilde{x} - A_2\tilde{y}\| \leq \liminf_{n\to\infty} \|A_1x_n - A_2y_n\| = 0$. Hence, $(\tilde{x}, \tilde{y}) \in \Omega$. Since $\tilde{x} \in \omega_w(x_n)$ and $\tilde{x} \in \omega_w(y_n)$, it follows that $(\omega_w(x_n), \omega_w(y_n)) \subset \Omega$.

Next, we show that

$$\limsup_{k\to\infty} \left(\langle \vartheta - \tilde{x}, x_{n_{k+1}} - \tilde{x}\rangle + \langle \zeta - \tilde{y}, y_{n_{k+1}} - \tilde{y}\rangle\right) \leq 0.$$

Let a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\lim_{j\to\infty} \langle \vartheta - \tilde{x}, x_{n_{k_j}} - \tilde{x}\rangle = \limsup_{k\to\infty} \langle \vartheta - \tilde{x}, x_{n_k} - \tilde{x}\rangle$. Since $\{x_{n_{k_j}}\}$ converges weakly to $\hat{x} \in E_1$ and $\tilde{x} = P_{E_1}[\vartheta]$, it follows that

$$\limsup_{k\to\infty} \langle \vartheta - \tilde{x}, x_{n_{k+1}} - \tilde{x}\rangle = \limsup_{k\to\infty} \langle \vartheta - \tilde{x}, x_{n_k} - \tilde{x}\rangle = \langle \vartheta - \tilde{x}, \hat{x} - \tilde{x}\rangle \leq 0. \tag{4.6}$$

By similar argument, we can prove that $\{y_{n_{k_j}}\}$ converges weakly to $\hat{y} \in E_2$ and $\tilde{y} = P_{E_2}[\zeta]$. It follows that

$$\limsup_{k\to\infty} \langle \zeta - \tilde{y}, y_{n_{k+1}} - \tilde{y}\rangle = \limsup_{k\to\infty} \langle \zeta - \tilde{y}, y_{n_k} - \tilde{y}\rangle = \langle \zeta - \tilde{y}, \hat{y} - \tilde{y}\rangle \leq 0. \tag{4.7}$$

Adding (4.6) and (4.7), we obtain $\limsup_{k\to\infty}\left(\langle\vartheta-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle+\langle\zeta-\tilde{y},y_{n_{k+1}}-\tilde{y}\rangle\right)\leq 0$. On the other hand, we have

$$\begin{aligned}\|x_{n_{k+1}}-\tilde{x}\|^2 &\leq \|\beta_{n_k}v_{n_k}+\delta_{n_k}T(r_{n_k})v_{n_k}-(1-\alpha_{n_k})\tilde{x}\|^2+2\alpha_{n_k}\langle\vartheta-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle\\ &= (1-\alpha_{n_k})^2\|\frac{\beta_{n_k}}{(1-\alpha_{n_k})}(v_{n_k}-\tilde{x})+\frac{\delta_{n_k}}{(1-\alpha_{n_k})}(T(r_{n_k})v_{n_k}-\tilde{x})\|^2\\ &\quad +2\alpha_{n_k}\langle\vartheta-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle\\ &\leq (1-\alpha_{n_k})\beta_{n_k}\|v_{n_k}-\tilde{x}\|^2+\delta_{n_k}(1-\alpha_{n_k})\|T(r_{n_k})v_{n_k}-\tilde{x}\|^2\\ &\quad +2\alpha_{n_k}\langle\vartheta-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle\\ &\leq (1-\alpha_{n_k})^2\|v_{n_k}-\tilde{x}\|^2+2\alpha_{n_k}\langle\vartheta-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle. \end{aligned} \tag{4.8}$$

Similarly, we obtain that

$$\|y_{n_{k+1}}-\tilde{y}\|^2\leq(1-\alpha_{n_k})^2\|t_{n_k}-\tilde{y}\|^2+2\alpha_{n_k}\langle\zeta-\tilde{y},y_{n_{k+1}}-\tilde{y}\rangle. \tag{4.9}$$

By adding (4.8) and (4.9), we conclude from Lemma 3.1 that

$$\begin{aligned}&\|x_{n_{k+1}}-\tilde{x}\|^2+\|y_{n_{k+1}}-\tilde{y}\|^2\\ &\leq(1-\alpha_{n_k})^2\left(\|v_{n_k}-\tilde{x}\|^2+\|t_{n_k}-\tilde{y}\|^2\right)+2\alpha_{n_k}(\langle\vartheta-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle+\langle\zeta-\tilde{y},y_{n_{k+1}}-\tilde{y}\rangle)\\ &\leq(1-\alpha_{n_k})^2\left(\|x_{n_k}-\tilde{x}\|^2+\|y_{n_k}-\tilde{y}\|^2\right)+2\alpha_{n_k}(\langle\vartheta-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle+\langle\zeta-\tilde{y},y_{n_{k+1}}-\tilde{y}\rangle)\\ &\leq(1-2\alpha_{n_k})\left(\|x_{n_k}-\tilde{x}\|^2+\|y_{n_k}-\tilde{y}\|^2\right)+2\alpha_{n_k}(\frac{\alpha_{n_k}M}{2}+\langle\vartheta-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle+\langle\zeta-\tilde{y},y_{n_{k+1}}-\tilde{y}\rangle)\\ &=(1-\kappa_{n_k})\left(\|x_{n_k}-\tilde{x}\|^2+\|y_{n_k}-\tilde{y}\|^2\right)+\kappa_{n_k}(\frac{\kappa_{n_k}M}{4}+\langle\vartheta-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle+\langle\zeta-\tilde{y},y_{n_{k+1}}-\tilde{y}\rangle)\end{aligned}$$

where $\kappa_{n_k}=2\alpha_{n_k}$ and $M=\sup\{\|x_{n_k}-\tilde{x}\|^2+\|y_{n_k}-\tilde{y}\|^2:n_k\geq 0\}$. Let

$$\varphi_{n_k}=\frac{\kappa_{n_k}M}{4}+\langle\vartheta-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle+\langle\zeta-\tilde{y},y_{n_{k+1}}-\tilde{y}\rangle.$$

Note that $\sum_{n_k=1}^{\infty}\kappa_{n_k}=\infty$ and $\limsup_{k\to\infty}\varphi_{n_k}\leq 0$. Thus from (4.2) all the conditions of Lemma 2.5 are satisfied. Hence $\lim_{n\to\infty}(\|x_n-\tilde{x}\|^2+\|y_n-\tilde{y}\|^2)=0$. Consequently, $\lim_{n\to\infty}\|x_n-\tilde{x}\|=\lim_{n\to\infty}\|y_n-\tilde{y}\|=0$. Therefore, (x_n,y_n) converges strongly to $(\tilde{x},\tilde{y})$. This completes the proof. □

5. Numerical Example

In this section, we provide a numerical example to illustrate the efficiency of our algorithm.

Example 5.1. Let $H_1=H_2=H_3$ be the set of all real numbers. For $r_{n,i}>0, i=1,2$, consider $C=[-10,10]$ and $Q=[0,20]$. Define the bifunction $\Phi:C\times C\to\mathbb{R}$ and $\Psi:Q\times Q\to\mathbb{R}$ by $\Phi(x,y)=\frac{xy-x^2}{3}$ and $\Psi(x,y)=\frac{xy-x^2}{2}$. It can easily deduced that Φ and Ψ satisfy all conditions of Assumption 2.1. By some simple calculations, it is easy to check that

$$T^{\Phi}_{r_{n,1}}(x)=\frac{3x}{3r_{n,1}+1},\forall x\in C \quad\text{and}\quad T^{\Psi}_{r_{n,2}}(y)=\frac{2y}{2r_{n,2}+1},\forall y\in Q.$$

Let $A_1x=2x$ and $A_2x=5x$. Next, we define $F:H_1\to H_1$ as $F(x)=2x$ and $G:H_2\to H_2$ as $G(x)=3x$. We define the mappings $T_1(r):\mathbb{R}\to\mathbb{R}$ and $T_2(s):\mathbb{R}\to\mathbb{R}$ as follows; $T_1(r)x=10^{-r}x$ and $T_2(s)y=10^{-2s}y$. Clearly, we observe that $T_1(r)$ and $T_2(s)$ are nonexpansive semigroups.

In all test we take, we choose $\lambda_0 = 1.3, \rho_0 = 1.6, \chi_1 = 0.75, \chi_2 = 0.8, \tau_n = \mu_n = \frac{10}{(n+1)^2}, r_{n,1} = 2.3, r_{n,2} = 3.2, s = u = 1, \alpha = \frac{1}{2(n+1)}, \beta_n = \delta_n = \frac{1-\alpha_n}{2}, \vartheta = x_0, \zeta = y_0$, and $\gamma = 0.001$. The algorithm stops if $\|x_{n+1} - x_n\| < 10^{-4}$. We consider the following cases for this numerical experiment.

Case 1: Take $x_0 = -12.9$ and $y_0 = -60.8$.
Case 2: Take $x_0 = 11.5$ and $y_0 = 79.2$.
Case 3: Take $x_0 = 4.8$ and $y_0 = 24.3$.
Case 4: Take $x_0 = 9.7$ and $y_0 = -12.5$.

The result of this experiment is reported in the Table 1 and Figures 1-4 with a comparison of the proposed method to the method in [30].

Table 1: Numerical Results for Example 5.1

	Case 1		Case 2		Case 3		Case 4	
	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
Method [30]	91	0.1198	90	0.1016	80	0.0817	88	0.0885
Our method	19	0.0034	18	0.0031	17	0.0032	18	0.0026

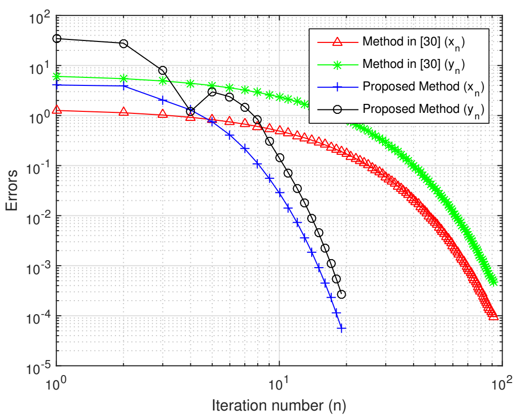

Figure 1:Case 1

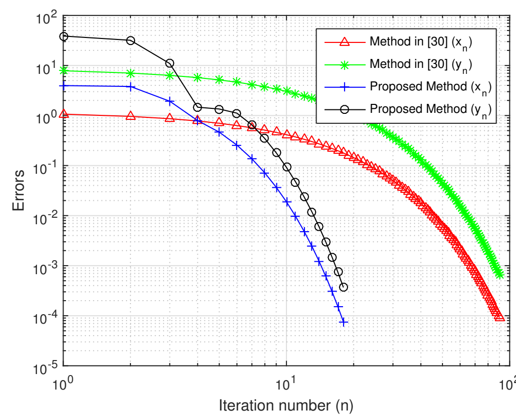

Figure 2: Case 2.

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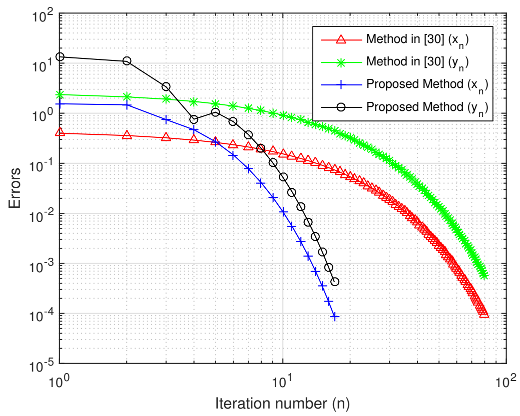

Figure 3:Case 3

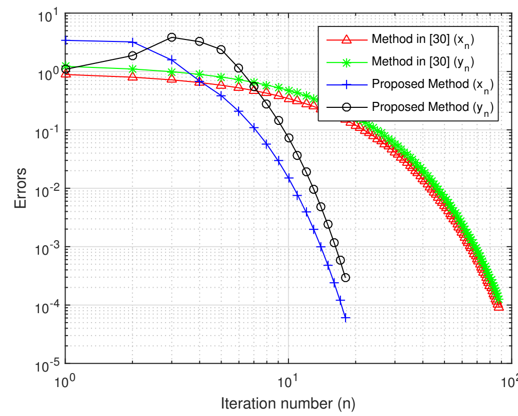

Figure 4: Case 4.

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