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OPTIMALITY AND SCALARIZATION OF APPROXIMATE SOLUTIONS FOR VECTOR EQUILIBRIUM PROBLEMS VIA MICHEL-PENOT SUBDIFFERENTIAL

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Abstract. This paper is devoted to the investigation of the optimality and scalarization for approximate solutions to a Constrained Vector Equilibrium Problem (CVEP). The optimality conditions are given in terms of Michel-Penot subdifferentials, and the scalarization theorems are proposed via a strongly monotone cone convex function. We firstly establish a necessary condition for an approximate quasi weakly efficient solution to problem (CVEP). Then, a sufficient condition for approximate quasi Benson proper efficient solutions to problem (CVEP) is examined under the newly introduced generalized convexity assumptions. Finally, by using the properties of Bishop-Phelps cone, we present the scalarization theorems for approximate quasi weakly (Benson proper) efficient solutions.

Keywords. Approximate solution; Michel-Penot subdifferential; Optimality condition; Scalarization; Vector equilibrium.

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1. INTRODUCTION

The Vector Equilibrium Problems (VEP) are natural extensions of vector variational inequalities and complementarity problems. Various kinds of solutions of VEP were considered recently, such as, efficiency, weakly efficiency, Henig proper efficiency, Benson proper efficiency and super efficiency, and so on; see [1]. The optimality conditions and scalarization are of two important topics in VEP. The necessary condition is the condition that the optimal solution must satisfy, and the sufficient condition refers to the condition that a feasible solutions becomes an optimal solution. In general, scalarization means the replacement of an equilibrium problem by a suitable scalar equilibrium problem which is an vector equilibrium problem with a real-valued objective functional. The purpose of this paper is to study the optimality conditions and scalarizations of approximate solutions for a class of Constrained Vector Equilibrium Problems (CVEP). It is worth noting that most mathematical models are usually non-smooth, that is, the objective and constraint functions are nondifferentiable. The subgradient and subdifferential are powerful tools to characterize non-smooth analysis. In recent years, there are abundant works related to optimality conditions for VEP via different subdifferentials. For example, Feng and Qiu [2] and Gong [3] established optimality conditions for weakly efficiency, efficiency, Henig efficiency, and super efficiency of VEP by utilizing the approximate subdifferential; Vau and

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Hang [4, 5] derived optimality theorems for (weakly) efficient solutions of VEP in terms of the Clarke subdifferentials; Zhang and Yu [6] focused on the approximate quasi weakly efficient solution of VEP via the Clarke subdifferentials. Michel-Penot subdifferentials is an important non-smooth analysis tool; see, e.g.,[7, 8]. It is the refinement of Clarke subdifferential and has many attractive properties. Most recently, Luu and Mai [9] and Su and Hang [10] derived optimality conditions for Henig efficiency and super efficiency of VEP in the sense of Michel-Penot subdifferentials. It is one of the main aim of this paper to establish the necessary conditions for approximate quasi weakly efficient solutions to problem CVEP via Michel-Penot subdifferentials. In addition, we propose the notion of approximate pseudoconvex functions in the form of Michel-Penot subdifferentials, and use it to examine the sufficient optimality conditions.

Scalarization turns out to be of great importance for the vector equilibrium theory. Based on Tammer nonlinear function, Qiu and Hao [11] obtained the scalarization theorems for approximate Henig proper and weakly efficient solutions of VEP. Gong [12] established the scalarizations of super efficient solutions to VEP via Baire theorem. Gasimov introduced a scalar function, termed the strongly monotone cone convex function, and applied it to present the characterization for Benson properly efficiency. It should be pointed out that the properties of Bishop-Phelps cones play an vital role in Gasimov's work. In this paper, we establish the scalarization theorms to problem CVEP with respect to approximate quasi weakly (Benson proper) efficient solutions by employing strongly monotone cone convex functions and Bishop-Phelps cones.

The article is organized as follows: Section 2 gives some symbols, concepts, and lemmas which are used in the subsequent sections. Section 3 and Section 4 present the optimality conditions and scalarization theorems for the approximate quasi weakly (Benson proper) efficient solutions to problem CVEP, respectively.

2. Preliminaries

Throughout this paper, let X, Y and Z be Banach spaces, and let their topological dual spaces be denoted by X^*, Y^* and Z^*, respectively. $\mathbb{R}^n$ denotes the n dimension Euclidean space, and

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_i \geq 0, \ i = 1, ..., n\}, \ \ \mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : \ x_i > 0, \ i = 1, ..., n\}.$$

For a set $K \subset X$, we use int$K$, cl$K$, co$K$, and cone$K$ to represent the interior, closure, convex hull, and cone hull of K, respectively. Let $\mathbb{B}(\bar{x}, r)$ stand for the open ball of radius $r > 0$ around $\bar{x} \in X$, and let $\|\cdot\|$ denote the norm of X. The value of linear functional $x^* \in X^*$ at point $x \in X$ is denoted by $\langle x, x^* \rangle$. Let C be a pointed convex cone in Y. The dual cone C^* to C is defined as (see [13]) $C^* = \{y^* \in Y^* : \ \langle y, y^* \rangle \geq 0, \ \forall y \in C\}$.

Lemma 2.1. *(see [13]) Let $C \subset Y$ be pointed, closed, and convex cone, and let $intC \neq \emptyset$. Then*
(i) If $\lambda \in C^* \backslash \{0\}$, $y \in intC$, then $\langle \lambda, y \rangle > 0$;
(ii) If $\lambda \in intC^*$, $y \in C \backslash \{0\}$, then $\langle \lambda, y \rangle > 0$.

Let K be a nonempty subset of X. The Clarke contingent cone and normal cone associated with set K at point $\bar{x} \in K$ are denoted by (see [13])

$$T(\bar{x}, K) = \{v \in X : \ \exists\, t_n \to 0, \ v_n \to v, \ s.t \ \bar{x} + t_n v_n \in K, \ \forall\, n \in \mathbb{N}\}$$

and

$$N_C(\bar{x}; K) = \{\xi \in X^* : \ \langle \xi, v \rangle \leq 0, \ \forall\, v \in T(\bar{x}; K)\}.$$

If K is a convex set, Clarke normal cone associated with set K at point $\bar{x} \in K$ is characterized by (see [13])

$$N_C(\bar{x};K) = \{\xi \in X^* : \langle \xi, x-\bar{x} \rangle \leq 0, \ \forall x \in K\}.$$

The indicator function of the set K is defined by (see [13])

$$\delta_K(x) = \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K. \end{cases}$$

If K is a closed set, then $\delta_K(x)$ is a lower semicontinuous function and

$$N_C(x;K) = \partial \delta_K(x), \tag{2.1}$$

where ∂ is the symbol of Clarke subdifferential.

Let $F: X \to Y$ be a vector valued mapping. F is said to be locally Lipschitz at $\bar{x} \in X$ if there exist constant $L > 0$ and $r > 0$ such that

$$\| F(x_1) - F(x_2) \| \leq L \| x_1 - x_2 \|, \ \forall \, x_1, x_2 \in \mathbb{B}(\bar{x}, r).$$

If, for any $x \in X$, F is locally Lipschitz at x, then F is called a locally Lipschitz mapping.

Let $f: X \to \mathbb{R}$ ($\mathbb{R}$ denotes the real number set) be a real valued local Lipschitz function. The Michel-Penot generalized directional derivative of f at $\bar{x} \in X$ with respect to the direction $v \in X$ is defined as (see [14])

$$f^{MP}(\bar{x};d) = \sup_{y \in X} \limsup_{t \to 0^+} \frac{f(x+ty+td) - f(x+ty)}{t}.$$

The Michel-Penot subdifferential of f at $\bar{x} \in X$ is given by (see [14])

$$\partial^{MP} f(\bar{x}) = \{\xi \in X^* : f^{MP}(\bar{x};d) \geq \langle \xi, d \rangle, \ \forall d \in X\}.$$

We summarize some properties of Michel-Penot subdifferential as follows, which are used in later sections.

Lemma 2.2. *(see [14]) Let $f, g: X \to \mathbb{R}$ be local Lipschitz functions at $x \in X$. Then the following assertions hold.*

(i) $\partial^{MP} f(x)$ is a non-empty, compact, and convex set;
(ii) for all $t \in \mathbb{R}$, $\partial^{MP} f(tx) = t\partial^{MP} f(x)$;
(iii) $\partial^{MP}(f+g)(x) \subset \partial^{MP} f(x) + \partial^{MP} g(x)$;
(iv) $\partial^{MP} f(x) \subset \partial f(x)$;
(v) if x is a local minimum or maximum point of f, then $0 \in \partial^{MP} f(x)$.

Lemma 2.3. *(see [8]) Let F be a local Lipschitz function from X to $\mathbb{R}^n$ at $\bar{x} \in X$, and let f be a local Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}$ at $F(\bar{x})$. Then*

$$\partial^{MP}(f \circ F)(\bar{x}) \subset cl\big(co(\bigcup_{\bar{\Lambda} \in \partial^{MP} f(F(\bar{x}))} \partial^{MP}(\bar{\Lambda} \circ F)(\bar{x}))\big).$$

Tammer's function is an important nonlinear scalarization function, which was proved to be one of the effective tools to study vector optimization problems (see [2]). The concept and some important properties of Tammer's function are given below.

Lemma 2.4. *(see [11, 15]) Let $C \subset Y$ be a pointed, closed, and convex cone, $\hat{e} \in intC \neq \emptyset$. Tammer's function $\Psi_{\hat{e}} : Y \to \mathbb{R}$ is defined as $\Psi_{\hat{e}}(y) = \inf\{r \in \mathbb{R} : y \in r\hat{e} - C\}$, $y \in Y$. Then*

(i) $\Psi_{\hat{e}}$ is a continuous local Lipschitz function;

(ii) $\Psi_{\hat{e}}(y) \leq r \Leftrightarrow y \in r\hat{e} - C$;

(iii) $\Psi_{\hat{e}}(y) \geq r \Leftrightarrow y \notin r\hat{e} - intC$;

(iv) $\partial\Psi_{\hat{e}}(y) = \{\lambda \in C^* : \langle \lambda, y\rangle = \Psi_{\hat{e}}(y)\}$;

(v) $\partial\Psi_{\hat{e}}(y) \subset C^* \backslash \{0\}$.

From now on, we assume that $K \subset X$ is a nonempty and closed set, $C \subset Y$ and $D \subset Z$ are pointed closed and convex cones with $intC \neq \emptyset$ and $intD \neq \emptyset$, $F : K \times K \to Y$ and $G : K \to Z$ are two vector valued mappings. Consider the following Constrained Vector Equilibrium Problem (CVEP):

$$\text{find a } \bar{x} \in K \text{ such that } F(\bar{x}, x) \notin -\text{int}C, \ \forall\, x \in \Omega,$$

where $\Omega := \{x \in K : G(x) \in -D\}$. Let $F_{\bar{x}}(y) = F(\bar{x}, y)$. It is supposed that $F_{\bar{x}}(\bar{x}) = 0$.

Definition 2.1. (see [6]) Let $\varepsilon \geq 0$, $e \in \text{int}C$,and $\bar{x} \in \Omega$. $\bar{x}$ is called to be an εe-quasi weakly efficient solution to (CVEP) if $F_{\bar{x}}(x) + \varepsilon\|x - \bar{x}\|e \notin -\text{int}C$, $\forall\, x \in \Omega$.

Based on the concept of Benson proper efficiency (see [16]), we propose the following definition of approximate quasi Benson proper efficient solutions to problem (CVEP).

Definition 2.2. Let $\varepsilon \geq 0$, $e \in \text{int}C$ and $\bar{x} \in \Omega$. $\bar{x}$ is called to be an εe-quasi Benson proper efficient solution with respect to cone C to problem (CVEP) if

$$clcone(F_{\bar{x}}(x) + C + \varepsilon\|x - \bar{x}\|e) \cap (-C) = \{0\},\ \forall\, x \in \Omega.$$

Here is an example of approximate quasi weakly (Benson proper) efficient solution for problem (CVEP).

Example 2.1. In problem (CVEP), let $K = \mathbb{R}_+$, $\bar{x} = 0$, $C = D = \mathbb{R}^2_+$, $e \in intC = \mathbb{R}^2_{++}$, $F :\ K \times K \to \mathbb{R}^2$, and $G :\ K \to \mathbb{R}^2$ be defined as

$$F_{\bar{x}}(x) = (e^x - \frac{|x - \bar{x}|}{2} - 1,\ x^2 - \frac{|x - \bar{x}|}{3})$$

and $G(x) = (-|\sin x|,\ -|\cos x|)$. Obviously, $\Omega = \mathbb{R}_+$. Taking $\varepsilon = 1, e = (\frac{1}{2},\ \frac{1}{3}) \in \mathbb{R}^2_{++}$, for any $x \in \Omega$, we have

$$\begin{aligned} F_{\bar{x}}(x) + \varepsilon\|x - \bar{x}\|e &= (e^x - \frac{|x - \bar{x}\,|}{2} - 1,\ x^2 - \frac{|x - \bar{x}|}{3}) + (\frac{1}{2},\ \frac{1}{3})|x - \bar{x}| \\ &= (e^x - \frac{|x - \bar{x}|}{2} + \frac{|x - \bar{x}|}{2} - 1,\ x^2 - \frac{|x - \bar{x}|}{3} + \frac{|x - \bar{x}|}{3}) \\ &= (e^x - 1,\ x^2) \notin -\mathbb{R}^2_{++}. \end{aligned}$$

Hence, for all $x \in \Omega$,

$$F_{\bar{x}}(x) + C + \varepsilon\|x - \bar{x}\|e = (e^x - 1,\ x^2) + C = (e^x - 1,\ x^2) + \mathbb{R}^2_+ = \mathbb{R}^2_+,$$

and cl cone$(F_{\bar{x}}(x) + C + \varepsilon\|x - \bar{x}\|e) =$ cl cone$\mathbb{R}^2_+ = \mathbb{R}^2_+ = C$. Therefore, $\bar{x} = 0$ is not only a $1 \cdot (\frac{1}{2},\ \frac{1}{3})$-quasi weakly efficient solution, but also a $1 \cdot (\frac{1}{2},\ \frac{1}{3})$-quasi Benson proper efficient solution to problem (CVEP).

3. Optimality Conditions

In this section, we firstly present a necessary optimality condition for approximate quasi weakly efficient solutions for problem (CVEP). Then, the concepts of quasiconvex and approximate pseudoconvex functions in the form of Michel-Penot subdifferential are introduced. Under their assumptions, we propose a sufficient optimality condition for approximate quasi Benson proper efficient solutions to problem (CVEP).

Theorem 3.1. *In problem (CVEP), let $\bar{x} \in \Omega$, $\varepsilon \geq 0$, $(e,\ \bar{e}) \in intC \times intD$, and $F_{\bar{x}}$ and G be locally Lipschitz at $\bar{x}$. If $\bar{x}$ is an εe-quasi weakly efficient solution to (CVEP), then there exists $(\lambda,\ \mu) \in (C^* \times D^*)\backslash\{(0,\ 0)\}$ such that*

$$0 \in \partial^{MP}(\lambda \circ F_{\bar{x}})(\bar{x}) + \partial^{MP}(\mu \circ G)(\bar{x}) + \langle \lambda, e\rangle \varepsilon \mathbb{B}_{X^*} + N_C(\bar{x};K), \tag{3.1}$$

$$\langle \mu, G(\bar{x})\rangle = 0, \tag{3.2}$$

where $\mathbb{B}_{X^*}$ be closed unit ball in X^*.

Proof. Since $\bar{x}$ is an εe-quasi weakly efficient solution to (CVEP), then $F_{\bar{x}}(x) + \varepsilon\|x - \bar{x}\|e \notin -intC$ for all $x \in \Omega$, which means that

$$(F_{\bar{x}}(x) + \varepsilon\|x - \bar{x}\|e,\ G(x)) \notin -\text{int}(C \times D),\ \forall\, x \in K.$$

From Lemma 2.4 (iii), we have $\Psi_{(e\times\bar{e})}(F_{\bar{x}}(x) + \varepsilon\|x - \bar{x}\|e,\ G(x)) \geq 0$ for all $x \in K$. Let $H(x) = (F_{\bar{x}}(x) + \varepsilon\|x - \bar{x}\|e,\ G(x))$. It follows that

$$\Psi_{(e\times\bar{e})}(H(x)) \geq 0,\ \forall\, x \in K. \tag{3.3}$$

It is clear that

$$\Psi_{(e\times\bar{e})}(H(\bar{x})) \geq 0. \tag{3.4}$$

Noting that $F_{\bar{x}}(\bar{x}) = 0$ and $(0,\ G(\bar{x})) \in -(C \times D)$, we see from Lemma 2.4 (ii) that

$$\Psi_{(e\times\bar{e})}(H(\bar{x})) = \Psi_{(e\times\bar{e})}(0,\ G(\bar{x})) \leq 0.$$

From (3.4), we arrive at $\Psi_{(e\times\bar{e})}(H(\bar{x})) = 0$. In view if (3.3), we have

$$\Psi_{(e\times\bar{e})}(H(x)) \geq \Psi_{(e\times\bar{e})}(H(\bar{x})) = 0,\ \forall\, x \in K,$$

which means that $\bar{x}$ is the minimum point to function $\Psi_{(e\times\bar{e})} \circ H$ on K. Furthermore, this leads to that $\bar{x}$ is a minimal solution to function $\Psi_{(e\times\bar{e})} \circ H(\cdot) + \delta_K(\cdot)$ on X. By Lemma 2.2 (iii) and (v), we have

$$0 \in \partial^{MP}(\Psi_{(e\times\bar{e})} \circ H + \delta_K)(\bar{x}) \subset \partial^{MP}(\Psi_{(e\times\bar{e})} \circ H)(\bar{x}) + \partial^{MP}\delta_K(\bar{x}). \tag{3.5}$$

Noting that K is a closed set, we obtain from Lemma 2.2 (iv) and (2.1) that

$$\partial^{MP}\delta_K(\bar{x}) \subset \partial\delta_K(\bar{x}) = N_C(\bar{x};K).$$

From (3.5), we have $0 \in \partial^{MP}(\Psi_{(e\times\bar{e})} \circ H)(\bar{x}) + N_C(\bar{x};K)$. Since $F_{\bar{x}},\ G,\ \|\cdot - \bar{x}\|e$, and Tammer's function $\Psi_{(e\times\bar{e})}$ are locally Lipschitz, we conclude from Lemma 2.3 that H is also locally Lipschitz and

$$\partial^{MP}\big(\Psi_{(e\times\bar{e})} \circ H)(\bar{x})\big) \subset \text{cl}\ \big(\text{co} \bigcup_{\bar{\Lambda} \in \partial^{MP}\Psi_{(e\times\bar{e})}(H(\bar{x}))} \partial^{MP}(\bar{\Lambda} \circ H)(\bar{x})\big).$$

Therefore, there exists $\Lambda = (\lambda,\ \mu) \in \partial^{MP}\Psi_{(e\times\bar{e})}(H(\bar{x}))$ such that

$$0 \in \text{cl}\left(\text{co}\partial^{MP}((\lambda,\ \mu)\circ H)(\bar{x}))\right) + N_C(\bar{x};K). \tag{3.6}$$

From Lemma 2.2 (iv) and Lemma 2.4 (v), we can deduce that

$$\partial^{MP}\Psi_{(e\times\bar{e})}(H(\bar{x})) \quad\subset\quad \partial\Psi_{(e\times\bar{e})}(H(\bar{x})) \subset (C\times D)^*\backslash\{(0,0)\} = (C^*\times D^*)\backslash\{(0,0)\}.$$

Hence, $(\lambda,\ \mu) \in (C^*\times D^*)\backslash\{(0,0)\}$. Since $H(x) = (F_{\bar{x}}(x) + \varepsilon\|x-\bar{x}\|e,\ G(x))$, we derive from (3.6) that

$$\begin{aligned} 0 &\in \text{cl}\left(\text{co}\partial^{MP}((\lambda,\ \mu)\circ H)(\bar{x}))\right) + N_C(\bar{x};K) \\ &= \text{cl}\left(\text{co}\partial^{MP}((\lambda,\ \mu)\circ(F_{\bar{x}} + \varepsilon\|\cdot-\bar{x}\|e,\ G))(\bar{x})\right) + N_C(\bar{x};K) \\ &= \text{cl}\left(\text{co}(\partial^{MP}(\lambda\circ(F_{\bar{x}} + \varepsilon\|\cdot-\bar{x}\|e))(\bar{x}) + \partial^{MP}(\mu\circ G)(\bar{x}))\right) + N_C(\bar{x};K). \end{aligned}$$

By Lemma 2.2 (iii) and (iv), we obtain

$$\begin{aligned} 0 &\in \text{cl}\left(\text{co}(\partial^{MP}(\lambda\circ(F_{\bar{x}} + \varepsilon\|\cdot-\bar{x}\|e))(\bar{x}) + \partial^{MP}(\mu\circ G)(\bar{x}))\right) + N_C(\bar{x};K) \\ &\subset \text{cl}\left(\text{co}(\partial^{MP}(\lambda\circ F_{\bar{x}})(\bar{x})) + \partial^{MP}(\mu\circ G)(\bar{x}) + \langle\lambda, e\rangle\varepsilon\partial^{MP}\|\cdot-\bar{x}\|)\right) + N_C(\bar{x};K) \\ &\subset \text{cl}\left(\text{co}(\partial^{MP}(\lambda\circ F_{\bar{x}})(\bar{x}) + \partial^{MP}(\mu\circ G)(\bar{x}) + \langle\lambda, e\rangle\varepsilon\partial^{C}\|\cdot-\bar{x}\|)\right) + N_C(\bar{x};K) \\ &= \text{cl}\left(\text{co}(\partial^{MP}(\lambda\circ F_{\bar{x}})(\bar{x}) + \partial^{MP}(\mu\circ G)(\bar{x}) + \langle\lambda, e\rangle\varepsilon\mathbb{B}_{X^*})\right) + N_C(\bar{x};K). \end{aligned}$$

From Lemma 2.2 (i), we have

$$0 \in \partial^{MP}(\lambda\circ F_{\bar{x}})(\bar{x}) + \partial^{MP}(\mu\circ G)(\bar{x}) + \langle\lambda, e\rangle\varepsilon\mathbb{B}_{X^*} + N_C(\bar{x};K).$$

Now, we prove that (3.2) holds. Observe that $(\lambda,\ \mu) \in \partial\Psi_{(e\times\bar{e})}(0, G(\bar{x}))$. By Lemma 2.4 (iv), we have $\langle(\lambda,\ \mu),\ (0, G(\bar{x}))\rangle = \Psi_{(e\times\bar{e})}(0, G(\bar{x})) = \Psi_{(e\times\bar{e})}(H(\bar{x})) = 0$, that is, $\langle\mu, G(\bar{x})\rangle = 0$. □

The following example verifies the conclusion of Theorem 3.1.

Example 3.1. In problem (CVEP), let $\varepsilon \geq 0,\ K = \mathbb{R}^2,\ C = D = \mathbb{R}^2_+,\ e \in \text{int}C = \mathbb{R}^2_{++}$, and $F : K\times K \to \mathbb{R}^2,\ G : K \to \mathbb{R}^2$ be defined as

$$F_{\bar{x}}(x) = (\max\{|x_1|, -x_2)\}(1 + \bar{x}_1\ln(1+\bar{x}_2)),\ |x_1|(1 - \bar{x}_1\cos(\bar{x}_2 + \frac{\pi}{2}))),$$

and $G(x) = (-|x_1|, -|x_2|)$. Taking $\bar{x} = (0,\ 0)$, one has

$$\begin{aligned} F_{\bar{x}}(x) &= (\max\{|x_1|, -x_2)\}(1 + \bar{x}_1\ln(1+\bar{x}_2)),\ |x_1|(1 - \bar{x}_1\cos(\bar{x}_2 + \frac{\pi}{2}))) \\ &= (\max\{|x_1|, -x_2)\},\ |x_1|). \end{aligned}$$

Obviously, $\Omega = \mathbb{R}^2,\ N_C(\bar{x};K) = (0,\ 0)$. Taking $\varepsilon = 1,\ e = (1,\ 1) \in \mathbb{R}^2_{++}$, for any $x \in \Omega$, we have

$$\begin{aligned} F_{\bar{x}}(x) + \varepsilon\|x-\bar{x}\|e &= (F_{1,\bar{x}}(x),\ F_{2,\bar{x}}(x)) + \varepsilon\|x-\bar{x}\|e \\ &= (\max\{|x_1|, -x_2\},\ |x_1|) + (1,\ 1)\|x-\bar{x}\| \\ &= (\max\{|x_1|, -x_2\} + \|x\|,\ |x_1| + \|x\|) \\ &\notin -\mathbb{R}^2_{++}. \end{aligned}$$

Hence, $\bar{x}$ is $1\cdot(1,\ 1)$ approximate quasi efficient weakly solution.

Taking $(\lambda,\ \mu)=((0,\ 1),\ (1,\ 0))\in C^*\times D^*\setminus\{((0,0),(0,0))\}$, we derive

$$(\lambda\circ F_{\bar{x}})(x)=\langle(0,\ 1),\ (\max\{|x_1|,-x_2\}\},\ |x_1|)\rangle=|x_1|.$$
$$(\mu\circ G)(x)=\langle(1,\ 0),\ (-|x_1|,-|x_1|)\rangle=-|x_1|.$$
$$\langle\mu,\ G(\bar{x})\rangle=\langle(1,\ 0),\ (0,\ 0)\rangle=0.$$

Then $\partial^{MP}(\lambda\circ F_{\bar{x}})(\bar{x})=\{(\xi_1,0):\xi_1\in[-1,\ 1]\}$ and $\partial^{MP}(\mu\circ G_{\bar{x}})(\bar{x})=\{(\nu_1,0):\nu_1\in[-1,\ 1]\}$. Hence

$$(0,\ 0)\in\partial^{MP}(\lambda\circ F_{\bar{x}})(\bar{x})+\partial^{MP}(\mu\circ G)(\bar{x})+\langle\lambda,e\rangle\varepsilon\mathbb{B}_{\mathbb{R}^2}+N_C(\bar{x};K).$$

In [9], the following concept of quasiconvex functions in the form of Michel-Penot subdifferential was presented.

Definition 3.1. Let $K\subset X$ be a nonempty set, and let $f:K\to\mathbb{R}$ be a locally Lipschtiz function in $\bar{x}\in K$. f is said to be quasiconvex at $\bar{x}\in K$ if there exists $\xi\in\partial^{MP}f(\bar{x})$ such that

$$f(x)-f(\bar{x})\leq 0\Rightarrow\langle\xi,x-\bar{x}\rangle\leq 0,\ \forall x\in K.$$

Motivated by the quasiconvexity above, we propose the following definition of approximate pseudoconvex functions, which contribute to the sufficient optimality conditions for approximate quasi Benson proper efficient solutions to problem (CVEP).

Definition 3.2. Let $K\subset X$ be a nonempty set, $\varepsilon\geq 0$, and $f:\ K\to\mathbb{R}$ be locally Lipschitz at $\bar{x}\in K$. f is said to be ε-pseudoconvex at $\bar{x}$ if there exists $\xi\in\partial^{MP}f(\bar{x})$ satisfying

$$\langle\xi,x-\bar{x}\rangle+\varepsilon\|x-\bar{x}\|\geq 0\Rightarrow f(x)-f(\bar{x})+\varepsilon\|x-\bar{x}\|\geq 0,\ \forall\, x\in K.$$

Remark 3.1. It is worth noting that if $\varepsilon=0$, then the ε-pseudoconvexity degenerates to the pseudoconvex function defined in [10] (See Definition 2.1). If f is pseudoconvex at $\bar{x}$, then f is also ε-pseudoconvex at $\bar{x}$. However, the opposite conclusion does not necessarily hold.

Example 3.2. Let $K=\mathbb{R}_+,\ \varepsilon\geq 0$. The function $f:\ K\to\mathbb{R}$ is defined by

$$f(x)=\begin{cases}\ln(x+1)+x, & x\in[0,3],\\ -2x+3, & x\in(3,+\infty).\end{cases}$$

Obviously, $\partial^{MP}f(0)=\{2\}$, Taking $\bar{x}=0$, $\varepsilon=2$, and $\xi=2$, one has

$$\langle 2,x-0\rangle+2\parallel x-0\parallel=2x+|x|=3x\geq 0,\ \forall x\in\mathbb{R}_+.$$

Hence

$$f(x)-f(0)+2\parallel x-0\parallel=\begin{cases}\ln(x+1)+3x, & x\in[0,3],\\ 3, & x\in(3,+\infty).\end{cases}$$

So, f is 2-pseudoconvex at 0. For $\xi=2\in\partial^{MP}f(0)$, we have $\langle 2,x-0\rangle\geq 0$ for all $x\in\mathbb{R}_+$ and $f(x)-f(0)=-2x+3<0$ for all $x\in(3,+\infty)$. Therefore, f is not pseudoconvex at $\bar{x}=0$.

Theorem 3.2. *In problem (CVEP), let $\bar{x}\in\Omega$, $\varepsilon\geq 0$, $(e,\ \bar{e})\in intC\times intD$, and $F_{\bar{x}}$ and G be locally Lipschitz at $\bar{x}$. Assume that there exist $\lambda\in C^*\backslash\{0\}$ and $\mu\in D^*\backslash\{0\}$ such that (3.1) and (3.2) hold. If $(\lambda\circ F_{\bar{x}})$ and $(\mu\circ G)$ are ε-pseudoconvex and quasiconvex at $\bar{x}$, respectively, then $\bar{x}$ is an εe-quasi Bensen proper efficient solution to problem (CVEP).*

Proof. It follows from (3.1) that there exist $\alpha \in \partial^{MP}(\lambda \circ F_{\bar{x}})(\bar{x})$, $\beta \in \partial^{MP}(\mu \circ G)(\bar{x})$, $\eta \in N_C(\bar{x};K)$, and $\rho \in \mathbb{B}_{X^*}$ such that $\alpha+\beta+\eta+\langle\lambda,e\rangle\varepsilon\rho = 0$. Thus, for any $x \in K$,

$$\langle\alpha,x-\bar{x}\rangle+\langle\beta,x-\bar{x}\rangle+\langle\eta,x-\bar{x}\rangle+\langle\lambda,e\rangle\varepsilon\langle\rho,x-\bar{x}\rangle = 0,\ \forall x \in K. \tag{3.7}$$

It follows from the fact that K is convex that $\langle\eta,x-\bar{x}\rangle \leq 0$ for all $x \in K$. From (3.7), we see that

$$\langle\alpha,x-\bar{x}\rangle+\langle\beta,x-\bar{x}\rangle+\langle\lambda,e\rangle\varepsilon\langle\rho,x-\bar{x}\rangle \geq 0. \tag{3.8}$$

Since $\rho \in \mathbb{B}_{X^*}$, then $\|\rho\| \leq 1$, and $\langle\rho,x-\bar{x}\rangle \leq \|x-\bar{x}\|$ for all $x \in K$. Again, it follows from (3.8) that

$$\langle\alpha,x-\bar{x}\rangle+\langle\beta,x-\bar{x}\rangle+\langle\lambda,e\rangle\varepsilon\|x-\bar{x}\| \geq 0,\ \forall\, x \in K. \tag{3.9}$$

Assume that $\bar{x}$ is not an εe-quasi Benson proper efficient solution to (CVEP). There exists $\hat{x} \in \Omega$ such that

$$\text{cl cone}(F_{\bar{x}}(\hat{x})+C+\varepsilon\|\hat{x}-\bar{x}\|e)\cap(-C\setminus\{0\}) \neq \emptyset.$$

Thus, there is a $c \in C$ such that $F_{\bar{x}}(\hat{x})+c+\varepsilon\|\hat{x}-\bar{x}\|e \in -C\setminus\{0\}$. Since $\lambda \in C^*\setminus\{0\}$, we obtain

$$\langle\lambda, F_{\bar{x}}(\hat{x})+c+\varepsilon\|\hat{x}-\bar{x}\|e\rangle < 0,$$

that is, $\langle\lambda,F_{\bar{x}}(\hat{x})\rangle+\langle\lambda,c\rangle+\langle\lambda,e\rangle\varepsilon\|\hat{x}-\bar{x}\| < 0$. Because $F_{\bar{x}}(\bar{x}) = 0$ and $c \in C$, we have

$$\langle\lambda,F_{\bar{x}}(\hat{x})\rangle-\langle\lambda,F_{\bar{x}}(\bar{x})\rangle+\langle\lambda,e\rangle\varepsilon\|\hat{x}-\bar{x}\| < 0.$$

Combing with $G(\hat{x}) \in -D$, $\mu \in D^*\setminus\{0\}$ and (3.2), we arrive at $\langle\mu,G(\hat{x})-G(\bar{x})\rangle \leq 0$. It follows from the ε-pseudoconvexity of $(\lambda \circ F_{\bar{x}})$ and quasiconvexity of $(\mu \circ G)$ at $\bar{x}$ that $\langle\alpha,\hat{x}-\bar{x}\rangle+\langle\lambda,e\rangle\varepsilon\|\hat{x}-\bar{x}\| < 0$ and $\langle\beta,\hat{x}-\bar{x}\rangle \leq 0$, which leads to $\langle\alpha,\hat{x}-\bar{x}\rangle+\langle\beta,\hat{x}-\bar{x}\rangle+\langle\lambda,e\rangle\varepsilon\|\hat{x}-\bar{x}\| < 0$. This contradicts (3.9). Thus, $\bar{x}$ is an εe-quasi Benson efficient solution to (CVEP). □

4. Scalarization

In order to obtain the scalarization theorems with respect to the approximate quasi weakly (Benson proper) efficient solutions to problem (CVEP), we start with introducing the next monotonicity concepts.

Definition 4.1. (see [17]) Let $C \subset Y$ be closed and convex cone with $intC \neq \emptyset$, and let $\varphi : Y \to \mathbb{R}$ be a given function.

(i) φ is called monotonically increasing if, for all $y,\ \bar{y} \in Y$ such that $y-\bar{y} \in C \Rightarrow \varphi(\bar{y}) \leq \varphi(y)$.

(ii) φ is called strongly monotonically increasing if, for any $y,\ \bar{y} \in Y$, $y-\bar{y} \in \text{int}C \Rightarrow \varphi(\bar{y}) < \varphi(y)$.

(iii) φ is called strictly monotonically increasing, if for any $y,\ \bar{y} \in Y$, $y-\bar{y} \in C\setminus\{0\} \Rightarrow \varphi(\bar{y}) < \varphi(y)$.

The following set $\mathbb{U}$ and Bishop-Phelps cone were defined in [18] and [19]:

$$\mathbb{U} := \{(\alpha,y^*) \in \mathbb{R}_+\times C^* : \alpha\|y\|-\langle y,y^*\rangle < 0,\ y \in C\setminus\{0\}\},$$

$$C(\alpha,y^*) := \{(\alpha,y^*) \in (0,\ 1]\times Y^* : \alpha\|y\|-\langle y,y^*\rangle \leq 0,\ y \in C\setminus\{0\}, \|y^*\| = 1\}.$$

From now on, we suppose that the interior of Bishop-Phelps cone is nonempty, i.e., $intC(\alpha,y^*) \neq \emptyset$.

Lemma 4.1. *(see [18]) Let $(\alpha,y^*) \in \mathbb{U}$, and let function $\psi : Y \to \mathbb{R}$ be defined as $\psi(y) := \alpha\|y\|+\langle y,y^*\rangle$ for all $y \in Y$. Then, ψ is strongly monotonically increasing on Y.*

Let $\bar{x} \in \Omega$, $\varepsilon \geq 0$. Consider the following scalar optimization problem (P_ψ) associated with (CVEP)

$$(P_\psi) \qquad \min \psi(F_{\bar{x}}(x)), \;\; \text{s.t. } x \in \Omega.$$

It is said that $\bar{x}$ is a quasi ε solution to problem (P_ψ) (see [6]) if

$$\psi(F_{\bar{x}}(x)) - \psi(F_{\bar{x}}(\bar{x})) + \varepsilon\|x - \bar{x}\| \geq 0, \; \forall\, x \in \Omega.$$

Theorem 4.1. *Let $\bar{x} \in \Omega$, $\varepsilon \geq 0$, and $e \in intC$. If, for any $\bar{\alpha} \in [0,1)$, $\bar{x} \in \Omega$ is a quasi $\bar{\alpha}\varepsilon\|e\|$ solution to problem (P_ψ), then $\bar{x}$ is an εe-quasi weakly efficient solution to problem (CVEP).*

Proof. Since $\bar{x} \in \Omega$ is a quasi $\bar{\alpha}\varepsilon\|e\|$ solution of problem (P_ψ), we have

$$\psi(F_{\bar{x}}(x)) - \psi(F_{\bar{x}}(\bar{x})) + \bar{\alpha}\varepsilon\|e\|\|x - \bar{x}\| \geq 0, \;\; \bar{\alpha} \in [0,1), \, x \in \Omega. \tag{4.1}$$

Assume that $\bar{x}$ is not an εe-quasi weakly efficient solution of problem (CVEP). There exists $\hat{x} \in \Omega$ such that $F_{\bar{x}}(\hat{x}) + \varepsilon\|\hat{x} - \bar{x}\|e \in -\text{int}C$. In view of $F_{\bar{x}}(\bar{x}) = 0$, one has

$$F_{\bar{x}}(\hat{x}) - (F_{\bar{x}}(\bar{x}) - \varepsilon\|\hat{x} - \bar{x}\|e) \in -\text{int}C.$$

Since ψ is strongly monotonically function on Y, then $\psi(F_{\bar{x}}(\hat{x})) < \psi(F_{\bar{x}}(\bar{x}) - \varepsilon\|\hat{x} - \bar{x}\|e)$. Based upon sublinear of ψ, we obtain $\psi(F_{\bar{x}}(\hat{x})) < \psi(F_{\bar{x}}(\bar{x})) + \psi(-\varepsilon\|\hat{x} - \bar{x}\|e)$. According to the definition of function ψ, for any $(\alpha, y^*) \in \mathbb{U}$, we arrive at

$$\begin{aligned} \psi(F_{\bar{x}}(\hat{x})) &< \psi(F_{\bar{x}}(\bar{x})) + \varepsilon\alpha\|e\|\|\hat{x} - \bar{x}\| - \varepsilon\|\hat{x} - \bar{x}\|\langle e,\, y^*\rangle \\ &= \psi(F_{\bar{x}}(\bar{x})) + \varepsilon\|\hat{x} - \bar{x}\|(\alpha\|e\| - \langle e,\, y^*\rangle). \end{aligned} \tag{4.2}$$

From the definition of $\mathbb{U}$, we get that $\alpha\|e\| < \langle e,\, y^*\rangle$. Hence, there exists $\bar{\alpha}' \in [0,\ 1)$ such that $(\alpha + \bar{\alpha}')\|e\| \leq \langle e,\, y^*\rangle$. Furthermore, it follows from (4.2) that

$$\begin{aligned} \psi(F_{\bar{x}}(\hat{x})) &< \psi(F_{\bar{x}}(\bar{x})) + \varepsilon\|\hat{x} - \bar{x}\|(\alpha\|e\| - \langle e,\, y^*\rangle) \\ &\leq \psi(F_{\bar{x}}(\bar{x})) + \varepsilon\|\hat{x} - \bar{x}\|(\alpha\|e\| - (\alpha + \bar{\alpha}')\|e\|) \\ &= \psi(F_{\bar{x}}(\bar{x})) - \varepsilon\bar{\alpha}'\|\hat{x} - \bar{x}\|\|e\|. \end{aligned}$$

This is a contradiction to (4.1). Therefore, $\bar{x}$ is an εe-quasi weakly efficient solution to problem (CVEP). □

Example 4.1 is to illustrate the conclusion of Theorem 4.1.

Example 4.1. Let $K = \mathbb{R}_+$, $\varepsilon \geq 0$, $C = D = \mathbb{R}^2_+$, $e \in \mathbb{R}^2_{++}$, and let $F_{\bar{x}} : K \times K \to \mathbb{R}^2$ and $G(x) : K \to \mathbb{R}^2$ be defined by $F_{\bar{x}}(x) = (x + \sin\bar{x},\ x + \cos(\bar{x} - \frac{\pi}{2}))$ and $G(x) = (-|\sin x|,\ -|\cos x|)$. Let

$$\mathbb{U} = \{(\alpha, y^*) \in \mathbb{R}_+ \times \mathbb{R}^2_+ : \alpha\|y\| - \langle y,\, y^*\rangle < 0, \;\; y \in \mathbb{R}^2_+ \setminus \{(0,\ 0)\}.$$

Taking $\bar{x} = 0$, one has $F_{\bar{x}}(x) = (x,\ x)$. It is obvious that $\Omega = K = \mathbb{R}_+$ and $\psi(F_{\bar{x}}(\bar{x})) = \psi(0) = 0$. For any $\varepsilon \geq 0$, $\bar{\alpha} \in [0,1)$, $(\alpha,\ y^*) \in \mathbb{U}$, $x \in \Omega$, we derive

$$\begin{aligned} \psi(F_{\bar{x}}(x)) - \psi(F_{\bar{x}}(\bar{x})) + \bar{\alpha}\varepsilon\|x - \bar{x}\|\|e\| &= \alpha\|F_{\bar{x}}(x)\| + \langle F_{\bar{x}}(x),\, y^*\rangle + \bar{\alpha}\varepsilon\|x\|\|e\| \\ &= \alpha\|(x,\ x)\| + \langle (x,\ x), (y_1^*,\ y_2^*)\rangle + \bar{\alpha}\varepsilon x\|e\| \\ &= \alpha\|(x,\ x)\| + xy_1^* + xy_2^* + \bar{\alpha}\varepsilon x\|e\| \\ &\geq 0, \end{aligned}$$

which leads to that $\bar{x}$ is a quasi $\bar{\alpha}\varepsilon\|e\|$ solution to problem (P_ψ).

Furthermore, let us verify that $\bar{x}$ is an εe-quasi weakly efficient solution to problem (CVEP). Indeed, for $e=(e_1,\ e_2)\in\mathbb{R}^2_{++}$ and $x\geq 0$, one has

$$\begin{aligned}F_{\bar{x}}(x)+\varepsilon\|x-\bar{x}\|e &= (x,\ x)+\varepsilon\|x\|(e_1,\ e_2)\\ &= (x+\varepsilon xe_1,\ x+\varepsilon xe_2)\\ &= \big((1+\varepsilon e_1)x,\ (1+\varepsilon e_2)x\big)\\ &\notin -\mathbb{R}^2_{++}.\end{aligned}$$

Therefore, $\bar{x}$ is an εe-quasi weakly efficient solution to problem (CVEP).

The following theorem proves that an εe-quasi Benson proper efficient solution to problem (CVEP) with respect to Bishop-Phelps cone $C(\alpha,y^*)$ is a quasi approximate solution to problem (P_ψ). From now on, we suppose that the interior of Bishop-Phelps cones is nonempty.

Theorem 4.2. *Let $\bar{x}\in\Omega$, $\varepsilon\geq 0$, and $e\in intC(\alpha,y^*)$. If $\bar{x}$ is an εe-quasi Benson proper efficient solution to (CVEP) with respect to $C(\alpha,y^*)$, then $\bar{x}$ is a quasi $\varepsilon'\|e\|$ $(\varepsilon<\varepsilon'\leq 2\varepsilon)$ solution to problem (P_ψ).*

Proof. Since $\bar{x}\in\Omega$ is an εe-quasi Benson proper efficient solution to problem (CVEP), we obtain

$$\mathrm{cl}\ \big(\mathrm{cone}(F_{\bar{x}}(x)+C(\alpha,y^*)+\varepsilon\|x-\bar{x}\|e\big)\cap\big(-C(\alpha,y^*)\big)=\{0\},\ \forall x\in\Omega.$$

Hence, we have

$$(F_{\bar{x}}(x)+C(\alpha,y^*)+\varepsilon\|x-\bar{x}\|e)\cap\big(-C(\alpha,y^*)\big)=\{0\}. \tag{4.3}$$

Noticing that $-C(\alpha,y^*)=\big\{y\in Y:\alpha\|y\|+\langle y,\ y^*\rangle\leq 0\big\}$ (See [18] pp. 193), we obtain from (4.3) that, for all $x\in\Omega$, $c\in C(\alpha,y^*)$, and $(\alpha,y^*)\in\mathbb{U}$,

$$\alpha\|F_{\bar{x}}(x)+c+\varepsilon\|x-\bar{x}\|e\|+\langle F_{\bar{x}}(x)+c+\varepsilon\|x-\bar{x}\|e,\ y^*\rangle\geq 0.$$

So, one has

$$\begin{aligned}0&\leq\alpha\|F_{\bar{x}}(x)+c+\varepsilon\|x-\bar{x}\|e\|+\langle F_{\bar{x}}(x)+c+\varepsilon\|x-\bar{x}\|e,\ y^*\rangle\\ &\leq\alpha\|F_{\bar{x}}(x)\|+\alpha\|c\|+\alpha\varepsilon\|e\|\|x-\bar{x}\|+\langle c,\ y^*\rangle+\langle F_{\bar{x}}(x),\ y^*\rangle+\varepsilon\|x-\bar{x}\|\langle e,\ y^*\rangle\\ &=\alpha\|F_{\bar{x}}(x)\|+\alpha\varepsilon\|e\|\|x-\bar{x}\|+\langle F_{\bar{x}}(x),y^*\rangle+\varepsilon\|x-\bar{x}\|\|e\|+\alpha\|c\|+\langle c,\ y^*\rangle\\ &=\|F_{\bar{x}}(x)\|+\langle F_{\bar{x}}(x),\ y^*\rangle+(1+\alpha)\varepsilon\|x-\bar{x}\|\|e\|+\alpha\|c\|+\langle c,\ y^*\rangle.\end{aligned}$$

Because above equation holds for any $c\in C(\alpha,y^*)$, it follows that

$$\alpha\|F_{\bar{x}}(x)\|+\langle F_{\bar{x}}(x),\ y^*\rangle+\varepsilon'\|x-\bar{x}\|\|e\|\geq 0,$$

where $\varepsilon'=(1+\alpha)\varepsilon$. Since $\alpha\in(0,1]$, then $\varepsilon<\varepsilon'\leq 2\varepsilon$. Noting that $F_{\bar{x}}(\bar{x})=0$ and $\psi(0)=0$, we derive $\psi(F_{\bar{x}}(x))-\psi(F_{\bar{x}}(\bar{x}))+\varepsilon'\|e\|\|x-\bar{x}\|\geq 0$. Therefore, $\bar{x}$ is a quasi $\varepsilon'\|e\|$ solution to problem (P_ψ). □

5. Conclusions

A necessary optimality condition in the form of Michel-Penot subdifferential was proved for approximate quasi weakly efficient solutions to problem (CVEP). A sufficient optimality condition with respect to the approximate quasi Benson prpper efficiency was presented. Finally, two scalarization theorems were established for the approximate quasi weakly (Benson proper) efficiency to problem (CVEP) by utilizing a strongly monotone function and Bishop-Phelps cone. The definitions and main conclusions presented in this paper were also verified by specific examples.

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