Appl. Set-Valued Anal. Optim. 6 (2024), No. 2, pp. 245-260
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.6.2024.2.08

# OPTIMALITY CONDITIONS FOR NONCONVEX MATHEMATICAL PROGRAMMING PROBLEMS USING WEAK SUBDIFFERENTIALS AND AUGMENTED NORMAL CONES

TRAN VAN SU*, CHU VAN TIEP

*Faculty of Mathematics, The University of Danang - University of Science and Education, Da Nang 550000, Vietnam*

**Abstract.** In this paper, we study some characterizations of the class of weakly subdifferentiable functions and formulate optimality conditions for nonconvex mathematical programming problems described by the class of weakly subdifferentiable functions in real normed spaces. The necessary and sufficient optimality conditions for a nonconvex scalar function with a global minimum/or a global maximum at a given vector via the weak subdifferentials and augmented normal cones are established. Additionally, the necessary and sufficient optimality conditions for a nonconvex vector function with a weakly efficient solution/or an efficient solution at a given vector via the augmented weak subdifferentials and normal cones are presented too. Finally, our optimality conditions are used to derive the necessary optimality conditions for nonsmooth nonconvex mathematical programming problems with set, inequality, and equality constraints.

**Keywords.** Augmented normal cones; Efficient solutions; Nonsmooth and nonconvex problems; Optimality conditions; Weak subdifferentials.

**2020 Mathematics Subject Classification.** 90C26, 90C30, 90C46.

## 1. Introduction

The tool of weak subdifferentials and augmented normal cones plays a crucial role from viewpoint of developing optimality conditions in nonsmooth and nonconvex optimization; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8] and the references therein. Applying the notion of weak subdifferentials and augmented normal cones, the necessary and sufficient optimality conditions given in the kind of variational inequality of nonsmooth convex optimization are generalized to the nonconvex case, e.g., in Kasimbeyli and Mammadov [9]. Up to our knowledge, for the problem of minimizing/maximizing a convex function $l$ over a closed convex feasible set $C$ in a $n$-dimensional Euclidean space, the necessary and sufficient optimality conditions for the global optimality were formulated by Rockafellar [6, 7, 8] and some other related authors, using the tool of subdifferentials and normal cones in the case of convex analysis. However, the classic subdifferentials and normal cones are not particularly useful tools in deriving optimality conditions for any nonsmooth nonconvex mathematical programming problem, which

*Corresponding author.
E-mail address: tvsu@ued.udn.vn (T.V. Su), cvtiep@ued.udn.vn (C.V. Tiep).
Received 18 May 2023; Accepted 17 December 2023; Published online 27 March 2024.

makes it very interesting to look for concepts of weak subdifferentials and augmented normal cones for any nonsmooth nonconvex real-valued function. The theory of weak subdifferentials and augmented normal cones was naturally introduced and independently developed by Azimov and Gasimov [10], in which they used the class of supperlinear functions defined as an augmented norm term (with degree equal to one) with a linear part added. The fundamental characterization of weak subdifferentials and augmented normal cones with a class of lower Lipschitz functions in real normed spaces was investigated in [10, 11, 12, 13, 14, 15, 16]. To the best of our knowledge, the weakly subdifferentiable nonconvex functions on the feasible set at the point under consideration was investigated in [10] and also [13, 14]. The question of whether a nonconvex extended-real-valued function attains a global minimum (or a global maximum) on a set is weakly subdifferentiable on that set is still an open problem. Additionally, a similar question whether a nonconvex vector function that attains a weakly efficient solution (or an efficient solution) on a set is augmented weakly subdifferentiable on that set remains is also an open problem. This is the motivation for our current work. To answer the previous open problems, let us give a mapping $l : X \to Y$ and a nonempty subset $C \subset X$, where $X$ is the normed space with norm $\|.\|$ and $(Y, \|\,.\,\|)$ is the other normed space, which is partially ordered by a convex cone $Q$. Let $\bar{x} \in C$. We pointed out here that if $l : X \to \mathbb{R}$ attains, either a global minimum on $C$ at $\bar{x}$, or, a global maximum on $C$ at $\bar{x}$ with $\sup\limits_{x \in C \setminus \{\bar{x}\}} \frac{|l(x) - l(\bar{x})|}{\|x - \bar{x}\|} < +\infty$, then $l$ is weakly subdifferentiable on $C$ at $\bar{x}$. Additionally, if $l : X \to Y$ attains, either a weakly efficient solution on $C$ at $\bar{x}$ with $\mathrm{int}Q \neq \emptyset$, or an efficient solution on $C$ at $\bar{x}$ in which there exists a pointed convex cone $H$ in $Y$ with $\emptyset \neq Q \setminus \{0\} \subset \mathrm{int}H$, then $l$ is augmented weakly subdifferentiable on $C$ at $\bar{x}$ (see Theorems 3.1, 3.7, and 3.8). These results are applied directly to the problem of minimizing/maximizing function $l : C \to \mathbb{R}$ over the subset $C \subset X$ as well as the nonconvex mathematical programming problems in normed spaces, which makes a huge contribution to the nonconvex mathematical programs. It is important to mention that these results are still very few; see, e.g., [9, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] and the references therein.

The current paper is the continuation of the investigation initialed in [11] and is related to the new nonconvex mathematical programming theory described by the class of weakly subdifferentiable functions in normed spaces. This paper is organized as follows. In Section 2, we briefly review the definitions of weak subdifferentials, augmented weak subdifferentials, augmented normal cones and provide some preliminaries results. Section 3 provides the necessary and sufficient optimality conditions for a nonconvex real-valued function with a global minimum (resp., maximum)/and for a nonconvex vector function with a weakly efficient solution (resp., efficient solution) at a given point via the weak subdifferentials and the augmented normal cones in real normed spaces. Section 4 derives the necessary optimality conditions for nonconvex mathematical programming problems by means of the weak subdifferentials and augmented normed cones in real normed spaces without convexity assumptions. Finally, Section 5 presents some conclusions.

## 2. PRELIMINARIES

We consider the real normed space $(X, \|\,.\,\|)$ and the real normed space $(Y, \|\,.\,\|)$, where $Y$ is partially ordered by a convex cone $Q$; $X^*, Y^*$ the topological duals of $X, Y$ respectively, and the Euclidean space $\mathbb{R}^n$ with norm $\|x\| = \sqrt{\langle x, x \rangle}$, where $\langle .,. \rangle$ is the scalar product. In the product space $X \times \mathbb{R}$, a norm is taken as $\|(x,r)\| = \|x\| + |r|$ for every $(x,r) \in X \times \mathbb{R}$. For simplicity, one writes 0 instead of the origin of any real normed space $X$; $\mathscr{P}$ means the set of all continuous positively homogeneous subadditive convex functions on $Y$; int$C$, cl$C$, bd$C$, co$C$, cone$C$, and span$C$ stand for the interior, the closure, the boundary, the convex hull, the generated cone, and the linear hull of a nonempty subset $C \subset X$, respectively, where cone$C := \{tc \,|\, t \in \mathbb{R}_+,\ c \in C\}$ in which $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$, $\mathbb{R}^n_+ = (\mathbb{R}_+)^n$ and $\mathbb{R}^n_{++} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 > 0, \ldots, x_n > 0\}$.

We propose for the case $Q \setminus \{0\} \neq \emptyset$ the following set

$$\mathscr{P}_w := \{P \in \mathscr{P} | y_2 - y_1 \in Q \setminus \{0\} \Longrightarrow \langle P, y_1 \rangle < \langle P, y_2 \rangle \ \ (\forall y_1, y_2 \in Y)\}.$$

For the case int$Q \neq \emptyset$, we also propose the following set

$$\mathscr{P}^w := \{P \in \mathscr{P} | y_2 - y_1 \in \mathrm{int}Q \Longrightarrow \langle P, y_1 \rangle < \langle P, y_2 \rangle \ \ (\forall y_1, y_2 \in Y)\}.$$

**Proposition 2.1.** *We have the following assertions:*

(i) *If int$Q \neq \emptyset$, then $\mathscr{P}^w$ is nonempty.*

(ii) *If there exists a pointed convex cone $H$ with $\emptyset \neq Q \setminus \{0\} \subset \mathrm{int}H$, then $\mathscr{P}_w$ is nonempty.*

*Proof.* (i): By the hypotheses int$Q \neq \emptyset$, we see that there exists $e \in \mathrm{int}Q$. Defining the Gerstewitz mapping $P_1 : Y \to \mathbb{R}$ by $P_1(y) = \inf\{a \in \mathbb{R} | y \in ae - Q\}$ for all $y \in Y$, we have that $P_1 \in \mathscr{P}$ satisfying $y_2 - y_1 \in \mathrm{int}Q$, implies $\langle P_1, y_1 \rangle < \langle P_1, y_2 \rangle$ for every $y_1, y_2 \in Y$. By the notion of $\mathscr{P}^w$, one can obtain $P_1 \in \mathscr{P}^w$.

(ii): Let $H$ be a pointed convex cone such that $\emptyset \neq Q \setminus \{0\} \subset \mathrm{int}H$. We take $\hat{e} \in \mathrm{int}H$, and then define the Gerstewitz mapping $P_2 : Y \to \mathbb{R}$ by $P_2(y) = \inf\{a \in \mathbb{R} | y \in a\hat{e} - H\}$ for all $y \in Y$. Thus $P_2 \in \mathscr{P}$ satisfying $y_2 - y_1 \in \mathrm{int}H$ implies $\langle P_2, y_1 \rangle < \langle P_2, y_2 \rangle$ for all $y_1, y_2 \in Y$. Since $Q \setminus \{0\} \subset \mathrm{int}H$, one can achieve that if $y_2 - y_1 \in Q \setminus \{0\}$, then $\langle P_2, y_1 \rangle < \langle P_2, y_2 \rangle$ for any $y_1, y_2 \in Y$, that is, $P_2 \in \mathscr{P}_w$, as we have to show. □

The notion of the weak subgradient, which is a generalization of the classic subgradient in the case of convex analysis, was proposed by Azimov and Gasimov in [10, 14], where the authors used a class of supperlinear conic functions defined as an augmented norm with a linear part added, instead of (only) linear part used in convex analysis. For the illustration, let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a real-valued function and $\bar{x} \in X$ be a given vector, where $f(\bar{x})$ is finite. Then, we recall that a pair $(\xi, r) \in X^* \times \mathbb{R}_+$ is said to be the weak subgradient of $f$ at $\bar{x}$ iff

$$f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \ \text{for every}\ \ x \in X. \tag{2.1}$$

**Definition 2.1.** ([10, 14]) Let $f : X \to \mathbb{R}$ be a real-valued function and $\bar{x} \in X$ be a given vector, where $f(\bar{x})$ is finite. The set $\partial^w f(\bar{x}) := \{(\xi, r) \in X^* \times \mathbb{R}_+ |\ (2.1) \text{ is fulfilled}\}$ is said to be the weak subdifferential of $f$ at $\bar{x}$. If $\partial^w f(\bar{x}) \neq \emptyset$, then $f$ is said to be weakly subdifferentiable at $\bar{x}$. If (2.1) is satisfied only $x \in C$, where $C \subset X$, then we say that $f$ is weakly subdifferentiable at $\bar{x}$ on $C$. One writes $\partial^w_C f(\bar{x})$ stands for the weak subdifferential of $f$ at $\bar{x}$ on $C$. It can be verified that $\partial^w f(\bar{x}) \subset \partial^w_C f(\bar{x})$.

It should be mentioned that if $f$ is weakly subdifferentiable on $C$ at $\bar{x}$, then $\partial^w_C f(\bar{x})$ is a closed and convex set. By extending the weak subgradient notion in a natural way, we arrive at the augmented weak subgradient notion, which can be illustrated as follows: let $f : X \to Y$ be a vector-valued mapping and $\bar{x} \in X$ be a given vector. Then, a triple $(\xi, P, r) \in X^* \times (\mathscr{P} \setminus \{0\}) \times \mathbb{R}_+$ is said to be the augmented weak subgradient of $f$ on a subset $C \subset X$ at $\bar{x} \in C$ iff

$$\langle P, f(x) - f(\bar{x}) \rangle \geq \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \ \text{for every}\ \ x \in C. \tag{2.2}$$

**Definition 2.2.** Let $f : X \to Y$ be a vector-valued mapping, $C \subset X$, and $\bar{x} \in C$ be a given vector. Then $\partial^w_{a,C} f(\bar{x}) := \Big\{(\xi, P, r) \in X^* \times (\mathscr{P} \setminus \{0\}) \times \mathbb{R}_+ \,|\, (2.2) \text{ is fulfilled}\Big\}$ is said to be the augmented weak subdifferential of $f$ on a subset $C$ at $\bar{x}$. If $\partial^w_{a,C} f(\bar{x}) \neq \emptyset$, then we say that $f$ is augmented weakly subdifferentiable on a subset $C$ at $\bar{x}$. In the case $C = X$, one writes $\partial^w_a f(\bar{x})$ instead of $\partial^w_{a,X} f(\bar{x})$. It can be verified that $\partial^w_a f(\bar{x}) \subset \partial^w_{a,C} f(\bar{x})$.

**Remark 2.1.** Given a function $P \in \mathscr{P} \setminus \{0\}$. It is evident that $(\xi, P, r) \in X^* \times (\mathscr{P} \setminus \{0\}) \times \mathbb{R}_+$ is an augmented weak subgradient of $f$ on a subset $C \subset X$ at $\bar{x} \in C$ if and only if $(\xi, r) \in X^* \times \mathbb{R}_+$ is a weak subgradient of $P_0 f$ on a subset $C \subset X$ at $\bar{x} \in C$.

The next definition, which generalizes the notion of normal cone in the case of convex analysis, was introduced by Kasimbeyli and Mammadov in [9].

**Definition 2.3.** Let $C \subset X$ and $\bar{x} \in C$ be a given vector. Then

$$N^a_C(\bar{x}) = \{(\xi, r) \in X^* \times \mathbb{R}_+ \,|\, \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \leq 0 \ (\forall x \in C)\}$$

is said to be an augmented normal cone to $C$ at $\bar{x}$.

**Remark 2.2.** If $(\xi, r) \in X^* \times \mathbb{R}_+$ such that $\|\xi\| \leq r$, then, for any $x \in C$, it is evident that $\langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \leq 0$, that is, $(\xi, r) \in N^a_C(\bar{x})$. An augmented normal cone consisting of only such elements is called trivial and expressed as $N^{triv}_C(\bar{x})$. It is an obviousness from the definitions that $N^{triv}_C(\bar{x}) \subset N^a_C(\bar{x})$. Especially, for the case that $C = X$, one can achieve that

$$N^{triv}_C(\bar{x}) := \{(\xi, r) \in X^* \times \mathbb{R}_+ \,|\, \|\xi\| \leq r\} = N^a_C(\bar{x}).$$

**Proposition 2.2.** *Let $C \subset X$ and a vector $\bar{x} \in C$. Then $N^a_C(\bar{x})$ is a nonempty, closed, and convex cone.*

*Proof.* In view of $(0, 0) \in N^a_C(\bar{x})$, it is obvious that $N^a_C(\bar{x})$ is $\emptyset$. It is not hard to check that $N^a_C(\bar{x})$ is a cone. For two pairs $(\xi, r), (\eta, s) \in N^a_C(\bar{x})$ arbitrarily taken and for every $t \in [0, 1]$, we have

$$\begin{aligned} &\langle t\xi + (1-t)\eta, x - \bar{x} \rangle - (tr + (1-t)s)\|x - \bar{x}\| \\ &= t\big(\langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\|\big) + (1-t)\big(\langle \eta, x - \bar{x} \rangle - s\|x - \bar{x}\|\big) \leq 0, \end{aligned}$$

that is, $t(\xi, r) + (1-t)(\eta, s) \in N^a_C(\bar{x})$. Thus $N^a_C(\bar{x})$ is a convex cone. Finally, for any sequence $((\xi_n, r_n))_{n \geq 1} \subset N^a_C(\bar{x})$ with $(\xi_n, r_n) \longrightarrow (\xi, r)$, it results that

$$0 \geq \langle \xi_n, x - \bar{x} \rangle - r_n\|x - \bar{x}\| \longrightarrow \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\|.$$

Consequently, $\langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \leq 0$. It follows that $(\xi, r) \in N^a_C(\bar{x})$ is fulfilled. So, $N^a_C(\bar{x})$ is also closed and convex. $\square$

We mention that the closedness and convexity of $\partial^w_C f(\bar{x})$ are similarly argumented as in the proof of Proposition 2.2. Also, we recall that $l : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous at $\bar{x} \in X$, where $l(\bar{x})$ is finite if, for every $\varepsilon > 0$, there exists an open neighborhood $U$ of $\bar{x}$ such that $l(x) \geq l(\bar{x}) - \varepsilon$ for all $x \in U$. Since $X$ is a normed space, the condition above can be rewritten as $\liminf_{x \to \bar{x}} l(x) \geq l(\bar{x})$.

## 3. Optimality Conditions for the Problem of Minimizing/Maximizing of Nonconvex Functions

By applying the weak subdifferentials and augmented normal cones, we derive some necessary and sufficient optimality conditions for the problem of minimizing/maximizing of nonconvex functions in real normed spaces. To begin with, the problem of minimizing/maximizing function $l : X \longrightarrow \mathbb{R}$ over the nonempty subset $C \subset X$ is considered.

**Definition 3.1.** ([1, 18]) Let $C$ be nonempty subset of $X$, the real-valued function $l : X \to \mathbb{R}$, and $\bar{x} \in X$. It is said that

(i) $l$ attains a global minimum at $\bar{x}$ if $l(x) \geq l(\bar{x})$ for all $x \in X$;
(ii) $l$ attains a global maximum at $\bar{x}$ if $l(x) \leq l(\bar{x})$ for all $x \in X$;
(iii) $l$ attains a global minimum at $\bar{x} \in C$ on $C$ if $l(x) \geq l(\bar{x})$ for all $x \in C$;
(iv) $l$ attains a global maximum at $\bar{x} \in C$ on $C$ if $l(x) \leq l(\bar{x})$ for all $x \in C$.

The following theorems characterize the class of weakly subdifferentiable functions in normed spaces.

**Theorem 3.1.** *Let $l : X \to \mathbb{R}$, $C$ be subset of $X$, and $\bar{x} \in C$. Then the following assertions hold:*

(i) *If $l$ attains a global minimum on $C$ at $\bar{x}$, then $l$ is weakly subdifferentiable on $C$ at $\bar{x}$ and*
$$N_C^a(\bar{x}) \subset \partial_C^w l(\bar{x}). \tag{3.1}$$
(ii) *If $l$ attains a global maximum on $C$ at $\bar{x}$ with $\sup\limits_{x \in C \setminus \{\bar{x}\}} \frac{|l(\bar{x}) - l(x)|}{\|x - \bar{x}\|} < +\infty$, then $l$ is weakly subdifferentiable on $C$ at $\bar{x}$.*

*Proof.* (i) According to Proposition 2.2, $N_C^a(\bar{x})$ is not null. By taking $(\xi, r) \in N_C^a(\bar{x})$, we obtain, for every $x \in C$, the $\langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \leq 0$, which combined with $l$ attains a global minimum on $C$ at $\bar{x}$. This guarantees that $\langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \leq l(x) - l(\bar{x})$ for all $x \in C$. Therefore, $(\xi, r) \in \partial_C^w l(\bar{x})$, and then (i) is fulfilled.

(ii) We set
$$r := \sup_{x \in C \setminus \{\bar{x}\}} \frac{|l(\bar{x}) - l(x)|}{\|x - \bar{x}\|}.$$
Then, $(0, r) \in X^* \times \mathbb{R}_+$. Since $l$ attains a global maximum on $C$ at $\bar{x}$, it yields that
$$r \geq \frac{|l(\bar{x}) - l(x)|}{\|x - \bar{x}\|} = \frac{l(\bar{x}) - l(x)}{\|x - \bar{x}\|} \text{ for all } x \in C \setminus \{\bar{x}\},$$
that is, $\langle 0, x - \bar{x} \rangle - r\|x - \bar{x}\| \leq l(x) - l(\bar{x})$ for all $x \in C$. Thus $(0, r) \in \partial_C^w l(\bar{x})$, which completes the proof. □

**Remark 3.1.** It should be noted that if $\sup\limits_{x \in C \setminus \{\bar{x}\}} \frac{|l(\bar{x}) - l(x)|}{\|x - \bar{x}\|} = +\infty$, then the obtained result in Theorem 3.1 (ii) is not true. To illustrate this, let us take $C = [0, 1]$, $\bar{x} = 0 \in C$, and let $l : X \longrightarrow \mathbb{R}$ be given by
$$l(x) = \begin{cases} -\sqrt{x} & \text{if } x \in C, \\ 0 & \text{otherwise.} \end{cases}$$

By directly calculating, one can obtain that

$$\sup_{x\in[0,1]\setminus\{0\}} \frac{\left|l(0)-l(x)\right|}{\left|x-0\right|} = \sup_{x\in]0,1]} \frac{1}{\sqrt{x}} = +\infty,$$

while the set of the weak subdifferential $\partial^w_{[0,1]} l(0)$ is empty, as we need to check.

**Theorem 3.2.** *Let $l : X \to \mathbb{R}$ and $\bar{x} \in X$. If $l$ attains, either a global minimum at $\bar{x}$, or a global maximum at $\bar{x}$ with $\sup_{x\in X\setminus\{\bar{x}\}} \frac{\left|l(\bar{x})-l(x)\right|}{\|x-\bar{x}\|} < +\infty$, then $l$ is lower semicontinuous at $\bar{x}$.*

*Proof.* By taking $C = X$, one sees from Theorem 3.1 that $l$ is weakly subdifferentiable at $\bar{x}$, that is, there exists a subgradient pair $(\xi, r) \in \partial^w l(\bar{x})$ such that $l(x) - l(\bar{x}) \geq \langle \xi, x-\bar{x}\rangle - r\|x-\bar{x}\|$ for all $x \in X$. By passing the limit inferior of the both sides of the inequality above when $x \to \bar{x}$, one obtains $\liminf_{x\to\bar{x}} l(x) \geq l(\bar{x})$, which proves the lower semicontinuity of $l$ at $\bar{x}$. □

**Remark 3.2.** We mention here that the converse of Theorem 3.2 may fail. To illustrate this, we can let $l : \mathbb{R} \to \mathbb{R}$ be defined by $l(x) = -x^2$ for all $x \in \mathbb{R}$, which can be verified that $\bar{x} = 0$ is not a global minimum of $l$, where $\bar{x} = 0$ is a global maximum of $l$. We further calculate that

$$\sup_{x\in\mathbb{R}\setminus\{0\}} \frac{\left|l(0)-l(x)\right|}{\left|x-0\right|} = \sup_{x\in\mathbb{R}\setminus\{0\}} |x| = +\infty,$$

while $l$ is a continuous function.

**Corollary 3.1.** *Let $l : X \to \mathbb{R}$ be Fréchet differentiable $\bar{x} \in X$. If $l$ is either subdifferentiable at $\bar{x}$, or convex, then $l$ is lower semicontinuous at $\bar{x}$.*

*Proof.* Since $l$ is convex and Fréchet differentiable $\bar{x} \in X$, then $l$ is subdifferentiable at $\bar{x}$. Thus there exists a real number $r \geq 0$ such that $(\nabla l(\bar{x}), r) \in \partial^w l(\bar{x})$, where $\nabla l(\bar{x})$ denotes the Fréchet derivative of $l$ at $\bar{x}$. It can be easily seen that $l(x) - l(\bar{x}) \geq \langle \nabla l(\bar{x}), x-\bar{x}\rangle - r\|x-\bar{x}\|$ for all $x \in X$. By passing the limit inferior of the both sides of the last inequality when $x \to \bar{x}$, it holds that $\liminf_{x\to\bar{x}} l(x) \geq l(\bar{x})$, which ensures the lower semicontinuity of $l$ at $\bar{x}$. □

**Remark 3.3.** Observe that the converse of Corollary 3.1 is not true. In fact, let $l$ be given as in Remark 3.2, which guarantees the lower semicontinuity of $l$ at $\bar{x} = 0$, while $l$ does not subdifferentiable at $\bar{x} = 0$, where $\partial l(0) = \emptyset$.

**Theorem 3.3.** *Let $l : X \to \mathbb{R}$, $C$ be a subset of $X$, and $\bar{x} \in C$. If $l$ attains a global maximum on a subset $C$ at $\bar{x}$, then*

$$N^a_C(\bar{x}) \supset \partial^w_C l(\bar{x}). \tag{3.2}$$

*Especially, for the case $C = X = \mathbb{R}^n$, we have*

$$N^a_X(\bar{x}) \supset N^{triv}_X(\bar{x}) \supset \partial^w l(\bar{x}). \tag{3.3}$$

*Proof.* Without loss of generality, we may assume that

$$\sup_{x\in C\setminus\{\bar{x}\}} \frac{\left|l(\bar{x})-l(x)\right|}{\|x-\bar{x}\|} < +\infty.$$

Taking into account Theorem 3.1 (ii) above, one sees that $l$ is weakly subdifferentiable on $C$ at $\bar{x}$. By taking $(\xi,r)\in\partial_C^w l(\bar{x})$, one can reach, for every $x\in C$, that $l(x)-l(\bar{x})\geq\langle\xi,x-\bar{x}\rangle-r\|x-\bar{x}\|$, which combined with $l$ attains a global maximum on a subset $C$ at $\bar{x}$. This guarantees

$$0\geq\langle\xi,x-\bar{x}\rangle-r\|x-\bar{x}\| \text{ for all } x\in C. \tag{3.4}$$

Thus $(\xi,r)\in N_C^a(\bar{x})$, and inclusion (3.2) is fulfilled. For the case $C=X=\mathbb{R}^n$, it follows from inequality (3.4) that $\langle\xi,h\rangle\leq r\|h\|$ for all $h\in X$, which means that $\|\xi\|\leq r$ (by taking $h=\xi\in X$). Thus $(\xi,r)\in N_X^{triv}(\bar{x})\subset N_X^a(\bar{x})$, which proves inclusion (3.3). This completes the proof. □

**Remark 3.4.** Inclusion (3.3) may be strict and the converse of Theorem 3.3 may fail. For example, let us can take $C=X=\mathbb{R}$, $\bar{x}=0\in\mathbb{R}$ and let $l:X\to\mathbb{R}$ be given by $l(x)=-2023|x|$ for every $x\in\mathbb{R}$. Then $l$ attains a global maximum at $\bar{x}$. An easy computation gives that $N_X^a(0)=\{(\xi,r)\in\mathbb{R}\times\mathbb{R}_+|\|\xi\|\leq r\}$ and $\partial^w l(0)=\{(\xi,r)\in\mathbb{R}\times\mathbb{R}_+|\|\xi\|\leq r-2023\}$. Therefore, $\partial^w l(0)\neq N_X^a(0)$. We observe that if $l$ is defined by

$$l(x)=\begin{cases}0 & \text{if } x\in\mathbb{Q},\\ 1 & \text{otherwise,}\end{cases}$$

then $\partial^w l(0)=N_X^a(0)=\{(\xi,r)\in\mathbb{R}\times\mathbb{R}_+|\|\xi\|\leq r\}$, while $l$ attains only a global minimum at $\bar{x}=0$.

**Theorem 3.4.** *Let $l:X\to\mathbb{R}$, $C$ be a subset of $X$, and $\bar{x}\in C$. Assume that $l$ is a constant function on a subset $C$. Then $l$ is weakly subdifferentiable on $C$ at $\bar{x}$ and furthermore,*

$$N_C^a(\bar{x})=\partial_C^w l(\bar{x})\supset\partial^w l(\bar{x}). \tag{3.5}$$

*Especially, for the case $C=X=\mathbb{R}^n$, one has*

$$N_X^a(\bar{x})=N_X^{triv}(\bar{x})=\partial^w l(\bar{x}). \tag{3.6}$$

*Proof.* The last inclusion in (3.5) is trivial because $C\subset X$. Since $l$ is a constant function on a subset $C$, which can be verified that $(0,r)\in\partial_C^w l(\bar{x})$ for every $r\geq 0$. Hence, $l$ is weakly subdifferentiable on $C$ at $\bar{x}$. For every $(\xi,r)\in\partial_C^w l(\bar{x})$ and $x\in C$, one can obtain that

$$f(x)-f(\bar{x})\geq\langle\xi,x-\bar{x}\rangle-r\|x-\bar{x}\|\Longleftrightarrow 0\geq\langle\xi,x-\bar{x}\rangle-r\|x-\bar{x}\|,$$

that is, $(\xi,r)\in N_C^a(\bar{x})$. Thus (3.5) is valid. In addition, if $C=X=\mathbb{R}^n$, then it follows from (3.3) that

$$N_X^a(\bar{x})\supset N_X^{triv}(\bar{x})\supset\partial^w l(\bar{x}). \tag{3.7}$$

Because $l$ is a constant function, it attains a global minimum at $\bar{x}$. Applying Theorem 3.1 (i) for the case that $C=X$, one sees that $l$ is weakly subdifferentiable at $\bar{x}$. Using inclusion (3.1) yields that $N_X^a(\bar{x})\subset\partial^w l(\bar{x})$, which together with (3.7) guarantees that (3.6) is satisfied. □

**Proposition 3.1.** *Let $C$ be a subset of $X$, $\bar{x}\in C$, and a distance function $d_C:X\to\mathbb{R}_+$ to the subset $C$ be defined by $d_C(x)=\inf\{\|x-c\|\,\big|\,c\in C\}$ for all $x\in X$. Then*

(i) $\partial_C^w d_C(\bar{x})=N_C^a(\bar{x})=N_{clC}^a(\bar{x})$.

(ii) *If, in addition, $X$ is a finite dimensional space, then* $\partial^w d_C(\bar{x})\subset\Big\{(\xi,r)\in N_C^q(\bar{x})\,\big|\,\|\xi\|\leq r+1\Big\}$.

*Proof.* In fact, it is known that $d_C(x) = 0$ if and only if $x \in clC$. Making use of the result of Theorem 3.4 deduces that conclusion (i) is valid. For the case that $X$ is a finite dimensional space, for a pair $(\xi, r) \in \partial^w d_C(\bar{x})$, which is arbitrarily taken, we have $d_C(x) - d_C(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\|$ for every $x \in X = cl\,C \cup (X \setminus cl\,C)$, which implies that

$$\begin{cases} 0 \geq \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \text{ for every } x \in cl\,C, \\ \|x - \bar{x}\| \geq \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \text{ for every } x \in X \setminus cl\,C. \end{cases}$$

As a direct consequence, it follows that $(\xi, r) \in N^a_{clC}(\bar{x}) = N^a_C(\bar{x})$ satisfies $0 \geq \langle \xi, h \rangle - (r+1)\|h\|$ for every $h \in X$. By picking $h = \xi$, it yields that $h \in X$ due tp $X^* = X$. We further observe that $\|\xi\| = \sqrt{\langle \xi, \xi \rangle} \leq \sqrt{(r+1)\|\xi\|}$, which leads to $\|\xi\| \leq r+1$. This completes the proof. □

**Theorem 3.5.** *(***Necessary and sufficient optimality conditions***) Let $l : X \to \mathbb{R}$ be a scalar function, $C$ be a nonempty subset of $X$, and $\bar{x} \in C$. Then the following statements hold*

(i) *If $l$ attains a global minimum on $C$ at $\bar{x}$, then there exists sequence $((\xi_n, r_n))_{n \geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n \to +\infty} \xi_n = 0$ such that $(\xi_n, r_n) \in \partial^w_C l(\bar{x})$ for every $n \geq 1$. Additionally, if $\lim\limits_{n \to +\infty} r_n = 0$, then the necessary optimality condition becomes the sufficient optimality condition.*

(ii) *If $l$ attains a global minimum on $C$ at $\bar{x}$, then there exists sequence $((\xi_n, r_n))_{n \geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n \to +\infty} \xi_n = 0$ such that $(\xi_n, r_n) \in \partial^w_C l(\bar{x}) + N^a_C(\bar{x})$ for every $n \geq 1$. Additionally, if $\lim\limits_{n \to +\infty} r_n = 0$, then the necessary optimality condition becomes the sufficient optimality condition.*

*Proof.* (i) Since $l$ attains a global minimum on $C$ at $\bar{x}$, in view of Theorem 3.1 (i), we see that $l$ is weakly subdifferentiable on $C$ at $\bar{x}$ and moreover, $l(x) - l(\bar{x}) \geq \langle \xi_n, x - \bar{x} \rangle - r_n\|x - \bar{x}\|$ for all $x \in C$, $\xi_n = 0$, $r_n \geq 0$, and $n \geq 1$. Thus there exists $((\xi_n, r_n))_{n \geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n \to +\infty} \xi_n = 0$ such that $(\xi_n, r_n) \in \partial^w_C l(\bar{x})$ for any $n \geq 1$. Conversely, we suppose that there exists $((\xi_n, r_n))_{n \geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n \to +\infty} \xi_n = 0$ and $\lim\limits_{n \to +\infty} r_n = 0$, satisfying $(\xi_n, r_n) \in \partial^w_C l(\bar{x})$ for all $n \geq 1$. By virtue of the weak subdifferential notion, one obtains $l(x) - l(\bar{x}) \geq \langle \xi_n, x - \bar{x} \rangle - r_n\|x - \bar{x}\|$ for all $x \in C$. By passing the limit above as $n \to +\infty$, it holds that $l(x) - l(\bar{x}) \geq 0 = \langle 0, x - \bar{x} \rangle - 0\|x - \bar{x}\|$ for all $x \in C$. Therefore, $\bar{x} \in C$ is a global minimum of $l$ on $C$, as required.

(ii) Because $(0,0) \in N^a_C(\bar{x})$, it holds that $\partial^w_C l(\bar{x}) \subset \partial^w_C l(\bar{x}) + N^a_C(\bar{x})$, which together with result of (i), we arrive at the desired conclusion. Conversely, we assume that there exists $((\xi_n, r_n))_{n \geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n \to +\infty} \xi_n = 0$ and $\lim\limits_{n \to +\infty} r_n = 0$ satisfying $(\xi_n, r_n) \in \partial^w_C l(\bar{x}) + N^a_C(\bar{x})$ for all $n \geq 1$. We set $(\eta_n, s_n) \in \partial^w_C l(\bar{x})$ such that $(\xi_n - \eta_n, r_n - s_n) \in N^a_C(\bar{x})$ for all $n \geq 1$. Taking arbitrary $x \in C$, we can reach the result $\langle \xi_n, x - \bar{x} \rangle - r_n\|x - \bar{x}\| \leq \langle \eta_n, x - \bar{x} \rangle - s_n\|x - \bar{x}\|$ for all $n \geq 1$, which together with $l(x) - l(\bar{x}) \geq \langle \eta_n, x - \bar{x} \rangle - s_n\|x - \bar{x}\|$ for all $n \geq 1$ guarantees $l(x) - l(\bar{x}) \geq \langle \xi_n, x - \bar{x} \rangle - r_n\|x - \bar{x}\|$ for all $n \geq 1$. Since $\lim\limits_{n \to +\infty} \xi_n = 0$ and $\lim\limits_{n \to +\infty} r_n = 0$, one can achieve the inequality, $l(x) \geq l(\bar{x})$ for every $x \in C$, and the claim follows. □

**Corollary 3.2.** *Let $l : X \to \mathbb{R}$ be a scalar function, $C$ be a nonempty subset of $X$, and $\bar{x} \in C$. Then, the following statements holds*

(i) *The function $l$ attains a global minimum on $C$ at $\bar{x}$ if and only if there exists a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty}(\xi_n, r_n) = (0,0)$ such that $(\xi_n, r_n) \in \partial_C^w l(\bar{x})$ for every $n \geq 1$.*

(ii) *The function $l$ attains a global minimum on $C$ at $\bar{x}$ if and only if there exists a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty}(\xi_n, r_n) = (0,0)$ such that $(\xi_n, r_n) \in \partial_C^w l(\bar{x}) + N_C^a(\bar{x}))$ for every $n \geq 1$.*

*Proof.* From the proof of Theorem 3.5, we obtain the desired conclusion immediately. □

**Remark 3.5.** We mention that if $\lim\limits_{n\to+\infty} r_n = r_0 > 0$, then the converse of Theorem 3.5 may fail. Indeed, we can take $C = [-1,1] \subset X = \mathbb{R}$, $\bar{x} = 0 \in C$, and consider the function $l : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -|x|$ for all $x \in \mathbb{R}$. An easy computation shows that

$$\partial_{[-1,1]}^w l(0) = \{(\xi, r) \in \mathbb{R} \times \mathbb{R}_+ || \xi| \leq r - 1\}$$

and

$$N_{[-1,1]}^a(0) = \{(\xi, r) \in \mathbb{R} \times \mathbb{R}_+ || \xi| \leq r\}.$$

Thus, either for any sequence $(\xi_n, r_n) \in \partial_{[-1,1]}^w l(0)$ $(\forall n \geq 1)$ with $\lim\limits_{n\to+\infty} \xi_n = 0$, it holds that $|\xi_n| \leq r_n - 1$ $(\forall n \geq 1)$, which means that $\lim\limits_{n\to+\infty} r_n := r_0 \geq 1$, or, for any sequence

$$(\xi_n, r_n) \in \partial_{[-1,1]}^w l(0) + N_{[-1,1]}^a(0) \ \ (\forall n \geq 1)$$

with $\lim\limits_{n\to+\infty} \xi_n = 0$, it results in $|\xi_n| \leq r_n - 1$ for all $n \geq 1$, that is, $\lim\limits_{n\to+\infty} r_n := r_0 \geq 1$, while $\bar{x} = 0$ is not a global minimum of $l$ on $C$.

**Theorem 3.6.** *(**Necessary optimality conditions**) Let $l : X \to \mathbb{R}$ be a scalar function, $C$ be a subset of $X$, and $\bar{x} \in C$. Suppose that $\sup\limits_{x\in C\setminus\{\bar{x}\}} \frac{\left|l(\bar{x})-l(x)\right|}{\|x-\bar{x}\|} < +\infty$. Then the following assertions hold:*

(i) *If the function $l$ attains a global maximum on $C$ at $\bar{x}$, then there exists a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} \xi_n = 0$ satisfying $(\xi_n, r_n) \in \partial_C^w l(\bar{x})$ for every $n \geq 1$.*

(ii) *If the function $l$ attains a global maximum on $C$ at $\bar{x}$, then there exists a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty}(\xi_n, r_n) = (0,0)$ satisfying $(\xi_n, r_n) \in \partial_C^w l(\bar{x}) + N_C^a(\bar{x})$ for every $n \geq 1$.*

*Proof.* (i): Observe that $\sup_{x\in C\setminus\{\bar{x}\}} \frac{\left|l(\bar{x})-l(x)\right|}{\|x-\bar{x}\|} < +\infty$. Since $l$ attains a global maximum on a subset $C$ at $\bar{x}$, taking into account Theorem 3.1 (ii), we obtain that $l$ is weakly subdifferentiable on $C$ at $\bar{x}$, that is, there exists a subgradient pair $(\xi_0, r_0) \in X^* \times \mathbb{R}_+$ satisfying $(\xi_0, r_0) \in \partial_C^w l(\bar{x})$. By virtue of the proof of Theorem 3.1 (ii) once again, it holds that $\xi_0 = 0$ and $r_0 < +\infty$. We set, for all $n \geq 1$, $\xi_n := 0$ and $r_n := r_0 + \frac{1}{2^n}$. Then, it is easy to verify that $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} \xi_n = 0$ satisfying $(\xi_n, r_n) \in \partial_C^w l(\bar{x})$.

(ii): Since $\partial_C^w l(\bar{x})$ and $N_C^a(\bar{x})$ are nonempty convex sets, it follows from $(0,0) \in N_C^a(\bar{x})$ and case (i) that $\frac{1}{2^n}(\xi_0, r_0) + \frac{2^n-1}{2^n}(0,0) \in \partial_C^w l(\bar{x}) + N_C^a(\bar{x})$. In what follows, we define the sequence of subgradient pairs $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$, where $\xi_n = \frac{\xi_0}{2^n}$, $r_n = \frac{r_0}{2^n}$, $n = 1, 2, \ldots$. It is not difficult to verify that $\lim_{n\to+\infty}(\xi_n, r_n) = (0,0)$, where $(\xi_n, r_n) \in \partial_C^w l(\bar{x}) + N_C^a(\bar{x})$. This completes the proof. $\square$

The following example demonstrates that the converse of the Theorem 3.6 may fail.

**Example 3.1.** Let $C = \{x \in X \,|\, \|x\| \leq 1\}$, $\bar{x} = 0 \in C$, and the real-valued function $l : X \to \mathbb{R}$ be defined by

$$l(x) = \begin{cases} 0 & \text{if } \|x\| < 1, \\ \|x\| & \text{otherwise.} \end{cases}$$

Then, we compute that

$$\sup_{x\in C\setminus\{\bar{x}\}} \frac{\left|l(\bar{x}) - l(x)\right|}{\|x - \bar{x}\|} = \sup_{x\in C\setminus\{\bar{x}\}} \frac{\|x\|}{\|x\|} = 1 < +\infty.$$

Additionally, $\partial_C^w l(\bar{x}) = N_C^a(\bar{x}) = \left\{(\xi, r) \in X^* \times \mathbb{R}_+ \,|\, \|\xi\| \leq r\right\}$. For every $x \in X$, $\|x\| = 1$, one can achieve that $l(x) = \|x\| = 1 > 0$. Hence $\bar{x} = 0$ is not a global maximum of $l$ on $C$ because $l(\bar{x}) = 0 < 1$. For $\underline{x} \in C$ fixed, consider a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$, which is given by

$$\begin{aligned} &\xi_n : X \longrightarrow \mathbb{R} \\ &\quad x \mapsto \xi_n(x) = \frac{\|\underline{x}\| - 1}{n},\ n = 1, 2, \ldots, \end{aligned}$$

and $r_n = \frac{1}{n}$ for all $n = 1, 2, \ldots$. It is obvious that $\|\xi_n\| = \frac{1-\|\underline{x}\|}{n} \longrightarrow 0$ and $r_n = \frac{1}{n} \longrightarrow 0$ as $n \to +\infty$, which prove $\lim_{n\to+\infty}(\xi_n, r_n) = (0,0)$. For every $n \geq 1$, one has $\langle \xi_n, x - \bar{x}\rangle - r_n\|x - \bar{x}\| \leq 0 \leq l(x) = l(x) - l(\bar{x})$ for all $x \in X$. Therefore, $(\xi_n, r_n) \in \partial^w l(\bar{x}) \subset \partial_C^w l(\bar{x}) \subset \partial_C^w l(\bar{x}) + N_C^a(\bar{x})$ for all $n = 1, 2, \ldots$.

In the sequel, the problem of minimizing the vector function $l : X \longrightarrow Y$ over the nonempty subset $C \subset X$ is considered.

**Definition 3.2.** ([1, 18]) Let $C$ be nonempty subset of $X$ and $\bar{x} \in X$.

(i) A vector function $l : X \to Y$ has a weakly efficient solution at $\bar{x} \in C$ on $C$ if $l(x) - l(\bar{x}) \notin -\mathrm{int}Q$ for all $x \in C$.

(ii) A vector function $l : X \to Y$ has an efficient solution at $\bar{x} \in C$ on $C$ if $l(x) - l(\bar{x}) \notin -(Q \setminus \{0\})$ for all $x \in C$.

The following theorems characterize the class of augmented weakly subdifferentiable functions.

**Theorem 3.7.** *Let $l : X \to Y$, where $\mathrm{int}Q \neq \emptyset$, $C$ be a subset of $X$, and $\bar{x} \in C$. If $l$ has a weakly efficient solution on $C$ at $\bar{x}$, then $l$ is augmented weakly subdifferentiable on $C$ at $\bar{x}$. Furthermore, there exists $P \in \mathscr{P}^w$ satisfying $N_C^a(\bar{x}) \subset \partial_C^w(P_0 l)(\bar{x})$.*

*Proof.* Suppose that $l$ has a weakly efficient solution at $\bar{x}$ on a subset $C$. Since $\mathrm{int}Q \neq \emptyset$, there exists $e \in \mathrm{int}Q$. Consider the Gerstewitz mapping $P_1 : Y \to \mathbb{R}$ being given as in the proof of Proposition 2.1 (i). Then, $P_1 \in \mathscr{P}^w$ satisfies the variational inequality $\langle P_1, l(x) - l(\bar{x})\rangle \geq 0$ for every $x \in C$. Thus, the scalar function $P_{10} l : X \to \mathbb{R}$ attains a global minimum at $\bar{x}$ on a subset

$C$. Based on the result of Theorem 3.1 (i), we assert that $P_{1}0l$ is weakly subdifferentiable at $\bar{x}$ on a subset $C$, that is, there exists a weak subgradient $(\xi, r) \in \partial_C^w(P_{1}0l)(\bar{x})$. It follows that

$$(\xi, r) \in \partial_C^w(P_{1}0l)(\bar{x}) \Longleftrightarrow \langle P_1, l(x) - l(\bar{x})\rangle \geq \langle \xi, x - \bar{x}\rangle - r\|x - \bar{x}\| \ \forall x \in C.$$

Consequently, $(\xi, P_1, r) \in \partial_{a,C}^w l(\bar{x})$, which means that $l$ is augmented weakly subdifferentiable at $\bar{x}$ on a subset $C$. According to Theorem 3.1 (i) once again, we obtain the desired inclusion for $P := P_1$, which terminates the proof. □

**Theorem 3.8.** *Let $l : X \to Y$, where $\emptyset \neq Q \setminus \{0\} \subset \mathrm{int}H$ with $H$ be a pointed convex cone in $Y$, $C$ be a subset of $X$, and $\bar{x} \in C$. If $l$ has an efficient solution on $C$ at $\bar{x}$, then $l$ is augmented weakly subdifferentiable on $C$ at $\bar{x}$. Furthermore, there exists $P \in \mathscr{P}_w$ satisfying $N_C^a(\bar{x}) \subset \partial_C^w(P_0l)(\bar{x})$.*

*Proof.* Notice that $l$ has an efficient solution at $\bar{x}$ on $C$. Since $H$ is a pointed convex cone in $Y$ such that $\emptyset \neq Q \setminus \{0\} \subset \mathrm{int}H$, we see that there exists $\hat{e} \in \mathrm{int}H$. Consider the Gerstewitz mapping $P_2 : Y \to \mathbb{R}$ being given as in the proof of Proposition 2.1 (ii). Then, $P_2 \in \mathscr{P}_w$ satisfies the variational inequality $\langle P_2, l(x) - l(\bar{x})\rangle \geq 0$ for every $x \in C$. Therefore, $P_{2}0l : X \to \mathbb{R}$ attains a global minimum at $\bar{x}$ on $C$. Arguing similarly as in Theorem 3.7, we obtain the desired conclusion. □

**Theorem 3.9.** *(**Necessary and sufficient optimality conditions**) Let $l : X \to Y$ be a vector function, $C$ be a nonempty subset of $X$, and $\bar{x} \in C$. Then the following statements hold:*

(i) *If $\mathrm{int}Q \neq \emptyset$ and the vector function $l$ has a weakly efficient solution on $C$ at $\bar{x}$, then there exist $P \in \mathscr{P}^w$ and a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} \xi_n = 0$ such that $(\xi_n, r_n) \in \partial_C^w(P_0l)(\bar{x})$ for any $n \geq 1$. If, in addition, $\lim\limits_{n\to+\infty} r_n = 0$, then the necessary optimality condition becomes the sufficient optimality condition.*

(ii) *If $\mathrm{int}Q \neq \emptyset$ and the vector function $l$ has a weakly efficient solution on $C$ at $\bar{x}$, then there exist $P \in \mathscr{P}^w$ and a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} \xi_n = 0$ such that $(\xi_n, r_n) \in \partial_C^w(P_0l)(\bar{x}) + N_C^a(\bar{x})$ for any $n \geq 1$. If, in addition, $\lim\limits_{n\to+\infty} r_n = 0$, then the necessary optimality condition becomes the sufficient optimality condition.*

(iii) *If there exists a pointed convex cone $H$ with $\emptyset \neq Q \setminus \{0\} \subset \mathrm{int}H$ and the vector function $l$ has an efficient solution on $C$ at $\bar{x}$, then there exist $P \in \mathscr{P}_w$ and a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} \xi_n = 0$ such that $(\xi_n, r_n) \in \partial_C^w(P_0l)(\bar{x})$ for any $n \geq 1$. If, in addition, $\lim\limits_{n\to+\infty} r_n = 0$, then the necessary optimality condition becomes the sufficient optimality condition.*

(iv) *If there exists a pointed convex cone $H$ with $\emptyset \neq Q \setminus \{0\} \subset \mathrm{int}H$ and the vector function $l$ has an efficient solution on $C$ at $\bar{x}$, then there exist $P \in \mathscr{P}_w$ and a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} \xi_n = 0$ such that $(\xi_n, r_n) \in \partial_C^w(P_0l)(\bar{x}) + N_C^a(\bar{x})$ for any $n \geq 1$. If, in addition, $\lim\limits_{n\to+\infty} r_n = 0$, then the necessary optimality condition becomes the sufficient optimality condition.*

*Proof.* (i) Thanks to the proof of Theorem 3.7, one can find $P_1 \in \mathscr{P}^w$ such that $P_{1}0l : X \to \mathbb{R}$ attains a global minimum at $\bar{x}$ on a subset $C$. Following the result of Theorem 3.5 (i), one sees that there exists a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} \xi_n = 0$ such that $(\xi_n, r_n) \in \partial_C^w(P_10l)(\bar{x})$ for every $n \geq 1$. Also, if $\lim\limits_{n\to+\infty} r_n = 0$, then the necessary optimality condition becomes the sufficient optimality condition.

(ii) Its proof is similar with the proof of (i).

(iii) Based on the proof of Theorem 3.8, there exists $P_2 \in \mathscr{P}_w$ such that $P_{20}l : X \to \mathbb{R}$ attains a global minimum at $\bar{x}$ on set $C$. According to the proof of (i), there exists a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} \xi_n = 0$ such that, for every $n \geq 1$, $(\xi_n, r_n) \in \partial_C^w(P_20l)(\bar{x})$. Additionally, if $\lim\limits_{n\to+\infty} r_n = 0$, then the necessary optimality condition becomes the sufficient optimality condition.

(iv) Since it is similar with the proof of Case (ii), we here omit the proof. □

**Corollary 3.3.** *Let $l : X \to Y$ be a vector function, $C$ be a subset of $X$, and $\bar{x} \in C$.*

*If* $\operatorname{int}Q \neq \emptyset$*, then the following statements hold:*

(i) *The vector function $l$ has a weakly efficient solution on $C$ at $\bar{x}$ if and only if there exist $P \in \mathscr{P}^w$ and a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} (\xi_n, r_n) = (0,0)$ such that $(\xi_n, r_n) \in \partial_C^w(P_0l)(\bar{x})$ for every $n \geq 1$.*

(ii) *The vector function $l$ has a weakly efficient solution on $C$ at $\bar{x}$ if and only if there exist $P \in \mathscr{P}^w$ and a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} (\xi_n, r_n) = (0,0)$ such that $(\xi_n, r_n) \in \partial_C^w(P_0l)(\bar{x}) + N_C^a(\bar{x}))$ for every $n \geq 1$.*

*If there exists a pointed convex cone $H$ with $\emptyset \neq Q \setminus \{0\} \subset \operatorname{int}H$, then the following statements hold:*

(iii) *The vector function $l$ has an efficient solution on $C$ at $\bar{x}$ if and only if there exist $P \in \mathscr{P}_w$ and a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} (\xi_n, r_n) = (0,0)$ such that $(\xi_n, r_n) \in \partial_C^w(P_0l)(\bar{x})$ for every $n \geq 1$.*

(iv) *The vector function $l$ has an efficient solution on $C$ at $\bar{x}$ if and only if there exist $P \in \mathscr{P}_w$ and a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} (\xi_n, r_n) = (0,0)$ such that $(\xi_n, r_n) \in \partial_C^w(P_0l)(\bar{x}) + N_C^a(\bar{x}))$ for every $n \geq 1$.*

*Proof.* From Theorem 3.8, the desired conclusions can be obtain immediately. □

## 4. Optimality Conditions for Nonsmooth Nonconvex Problems

Our aim in this section is to establish necessary optimality conditions in terms of the weak subdifferentials and augmented normal cones for the efficiency of nonsmooth nonconvex mathematical programming problems with set, inequality, and equality constraints in real normed spaces. It is important to mention that the achieved results in this section are still true for the case $X$ (or $Y$) is a $n$-dimensional Euclidean space.

We focus on the following nonsmooth nonconvex mathematical programming problem with set, inequality, and equality constraints:

$$\begin{aligned} &\min f(x) \\ &\text{subject to } g_i(x) \geq 0,\ i = 1,2,\dots,m, \\ &\qquad\qquad h_j(x) = 0,\ j = 1,2,\dots,p, \\ &\qquad\qquad x \in C, \end{aligned} \tag{NMPP}$$

where $f : X \to Y$, $g_i, h_j : X \to \mathbb{R}$, $i = 1,2,\dots,m$; $j = 1,2,\dots,p$ are given real-valued functions, and $C$ is a nonempty subset of $X$.

**Definition 4.1.** The feasible set of problem (NMPP) is denoted by $K$ and is defined by

$$K := \{x \in C \,|\, g_i(x) \geq 0,\ i = 1,2,\dots,m;\ h_j(x) = 0,\ j = 1,2,\dots,p\}.$$

$\bar{x}$ is said to be a feasible solution to problem (NMPP) if $\bar{x} \in K$.

Let a vector-valued mapping $l = (l_1, l_2, \dots, l_{m+2p}) : X \to \mathbb{R}^{m+2p}$ be given by

$$\begin{aligned} l(x) &:= \Big(l_1(x), l_2(x), \dots, l_{m+2p}(x)\Big) \\ &:= \Big(g_1(x), \dots, g_m(x), h_1(x), \dots, h_p(x), -h_1(x), \dots, -h_p(x)\Big) \text{ for all } x \in X. \end{aligned}$$

For this case, the feasible set to problem (NMPP) can be rewritten as

$$K := \{x \in C \,|\, l_i(x) \geq 0,\ i = 1,2,\dots,m+2p\}.$$

**Definition 4.2.** A vector $\bar{x} \in K$ is said to be a (resp., weakly) efficient solution to problem (NMPP) if $f$ has a (resp., weakly) efficient solution on $K$ at $\bar{x}$.

**Remark 4.1.** It is not difficult to verify that $N^a_C(\bar{x}) \subset N^a_K(\bar{x})$ for every feasible solution $\bar{x}$. Additionally, for the case $Y = \mathbb{R}$, problem (NMPP) is called the problem of minimizing scalar function $f : X \to \mathbb{R}$ over the feasible set $K$ and is denoted by (P).

We start by recalling that

$$sign(y) = \begin{cases} 1, & \text{if } y > 0, \\ -1 & \text{if } y < 0, \\ 0 & \text{if } y = 0, \end{cases}$$

$$\mathscr{P}^w := \{P \in \mathscr{P} | y_2 - y_1 \in \operatorname{int} Q \Longrightarrow \langle P, y_1\rangle < \langle P, y_2\rangle\ \ (\forall y_1, y_2 \in Y)\},$$

$$\mathscr{P}_w := \{P \in \mathscr{P} | y_2 - y_1 \in Q \setminus \{0\} \Longrightarrow \langle P, y_1\rangle < \langle P, y_2\rangle\ \ (\forall y_1, y_2 \in Y)\}.$$

**Theorem 4.1.** *(***Necessary optimality condition for problem (P)***) Let $\bar{x} \in K$. If $f$ attains a global minimum on $K$ at $\bar{x}$, then there exist $\eta := (\eta_1, \eta_2, \dots, \eta_m) \in \mathbb{R}^m_+$, $\gamma := (\gamma_1, \gamma_2, \dots, \gamma_p) \in \mathbb{R}^p$, and a sequence $((\xi_n, r_n))_{n \geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} (\xi_n, r_n) = (0,0)$ such that*

$$(\xi_n, r_n) \in \partial^w_K f(\bar{x}) + \sum_{i=1}^m \eta_i \partial^w_K g_i(\bar{x}) + \sum_{j=1}^p |\gamma_j| \partial^w_K (sign(\gamma_j) h_j)(\bar{x}) + N^a_C(\bar{x})\ (\forall n \geq 1); \tag{4.1}$$

$$\eta_i g_i(\bar{x}) = 0, \qquad i = 1,2,\dots,m; \tag{4.2}$$

$$\gamma_j h_j(\bar{x}) = 0, \qquad j = 1,2,\dots,p. \tag{4.3}$$

*Proof.* Suppose that $f$ attains a global minimum on $K$ at $\bar{x}$. In view of Theorem 3.1 (i), one sees that $f$ is weakly subdifferentiable at $\bar{x}$ on $K$. Since $(0,0) \in N^a_C(\bar{x})$, taking into account Corollary 3.2, one can find a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim\limits_{n\to+\infty} (\xi_n, r_n) = (0,0)$ such that $(\xi_n, r_n) \in \partial^w_K f(\bar{x}) + N^a_C(\bar{x})$. In other words, taking $\eta := (\eta_1, \eta_2, \dots, \eta_m) \in \mathbb{R}^m_+$ satisfies (4.2), which can be verified that $\eta_i g_i$ $(i = 1,2,\dots,m)$ have a global minimum on $K$ at $\bar{x}$. Thus $(0,0) \in \partial^w_K(\eta_i g_i)(\bar{x})$ for $i = 1,2,\dots,m$. We observe that if $g_i(\bar{x}) = 0$ $(i = 1,2,\dots,m)$, then $\eta_i > 0$ $(i = 1,2,\dots,m)$. It further follows that $\partial^w_K(\eta_i g_i)(\bar{x}) = \eta_i \partial^w_K g_i(\bar{x})$ $i = 1,2,\dots,m$; if $\eta_i = 0$ for $i = 1,2,\dots,m$, then $(0,0) \in \partial^w_K(0 g_i)(\bar{x})$ for $i = 1,2,\dots,m$. Thus $(0,0) \in \eta_i \partial^w_K g_i(\bar{x})$ for $i = 1,2,\dots,m$. Similarly to the argument above, one sees that there exists $\gamma^{(1)} := (\gamma^{(1)}_1, \gamma^{(1)}_2, \dots, \gamma^{(1)}_p)$

$\in \mathbb{R}^p_+$ such that $(0,0) \in \partial^w_K(\gamma^{(1)}_j h_j)(\bar{x})$ for $j = 1,2,\dots,p$ and there exists $\gamma^{(2)} := (\gamma^{(2)}_1, \gamma^{(2)}_2, \dots, \gamma^{(2)}_p)$ $\in \mathbb{R}^p_+$ such that $(0,0) \in \partial^w_K(-\gamma^{(2)}_j h_j)(\bar{x})$ for $j = 1,2,\dots,p$. For any $x \in K$, it results $(\gamma^{(1)}_j h_j)(x) - (\gamma^{(1)}_j h_j)(\bar{x}) \geq 0$ for $i = 1,2,\dots,p$, and $(-\gamma^{(2)}_j h_j)(x) - (-\gamma^{(2)}_j h_j)(\bar{x}) \geq 0$ for $i = 1,2,\dots,p$. Therefore, $(\gamma^{(1)}_j h_j - \gamma^{(2)}_j h_j)(x) - (\gamma^{(1)}_j h_j - \gamma^{(2)}_j h_j)(\bar{x}) \geq 0$ for $i = 1,2,\dots,p$, that is, $(0,0) \in \partial^w_K((\gamma^{(1)}_j - \gamma^{(2)}_j)h_j)(\bar{x})$ for $j = 1,2,\dots,p$. By setting $\gamma = \gamma^{(1)} - \gamma^{(2)} \in \mathbb{R}^p$, we see that

$$(0,0) \in \sum_{i=1}^{m} \eta_i \partial^w_K g_i(\bar{x}) + \sum_{j=1}^{p} |\gamma_j| \partial^w_K(sign(\gamma_j)h_j)(\bar{x}),$$

which is due to $(0,0) \in \partial^w_K(\gamma_j h_j)(\bar{x}) = |\gamma_j|\partial^w_K(sign(\gamma_j)h_j)(\bar{x})$ for $j = 1,2,\dots,p$. This proves the conclusions (4.1) and (4.3). □

**Theorem 4.2.** *(***Necessary optimality condition for problem** (NMPP)*) Let $\bar{x} \in K$ and* int$Q \neq \emptyset$. *If $f$ has a weakly efficient solution on $K$ at $\bar{x}$, then there exist $P \in \mathscr{P}^w$, $\eta := (\eta_1, \eta_2, \dots, \eta_m) \in \mathbb{R}^m_+$, $\gamma := (\gamma_1, \gamma_2, \dots, \gamma_p) \in \mathbb{R}^p$, and a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim_{n\to+\infty}(\xi_n, r_n) = (0,0)$ satisfying* (4.2)*,* (4.3)*, and*

$$(\xi_n, r_n) \in \partial^w_K(P_0 f)(\bar{x}) + \sum_{i=1}^{m} \eta_i \partial^w_K g_i(\bar{x}) + \sum_{j=1}^{p} |\gamma_j| \partial^w_K(sign(\gamma_j)h_j)(\bar{x}) + N^a_C(\bar{x}) \ (\forall n \geq 1).$$

*Proof.* By invoking the result of Corollary 3.3, we see that there exists $P \in \mathscr{P}^w$ such that $P_0 f$ attains a global minimum on a feasible subset $K$ at $\bar{x}$. Applying the result of Theorem 4.1, we obtain the desired conclusion immediately. □

**Theorem 4.3.** *(***Necessary optimality condition for problem** (NMPP)*) Let $\bar{x} \in K$ and suppose that there exists a pointed convex cone $H$ with $\emptyset \neq Q \setminus \{0\} \subset$* int$H$. *If $f$ has an efficient solution on $K$ at $\bar{x}$, then there exist $P \in \mathscr{P}_w$, $\eta := (\eta_1, \eta_2, \dots, \eta_m) \in \mathbb{R}^m_+$, $\gamma := (\gamma_1, \gamma_2, \dots, \gamma_p) \in \mathbb{R}^p$, and a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim_{n\to+\infty}(\xi_n, r_n) = (0,0)$ such that $(\xi_n, r_n) \in \partial^w_K f(\bar{x}) + \sum_{i=1}^m \eta_i \partial^w_K g_i(\bar{x}) + \sum_{j=1}^p |\gamma_j| \partial^w_K(sign(\gamma_j)h_j)(\bar{x}) + N^a_C(\bar{x})$ for all $n \geq 1$; $\eta_i g_i(\bar{x}) = 0$ for $i = 1,2,\dots,m$, and $\gamma_j h_j(\bar{x}) = 0$ for $j = 1,2,\dots,p$.*

*Proof.* Arguing similarly as for proving Theorem 4.2, we see that there exists $P \in \mathscr{P}_w$ such that $P_0 f$ attains a global minimum on a feasible subset $K$ at $\bar{x}$. From Theorem 4.1, we obtain the desired conclusion immediately. □

**Theorem 4.4.** *(***Necessary optimality condition for problem (P)***) Let $\bar{x} \in K$ and assume that $\sup_{x\in C\setminus\{\bar{x}\}} \frac{|f(x)-f(\bar{x})|}{\|x-\bar{x}\|} < +\infty$. If $f$ attains a global maximum on $K$ at $\bar{x}$, then there exist $\eta := (\eta_1, \eta_2, \dots, \eta_m) \in \mathbb{R}^m_+$, $\gamma := (\gamma_1, \gamma_2, \dots, \gamma_p) \in \mathbb{R}^p$ and a sequence $((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim_{n\to+\infty}(\xi_n, r_n) = (0,0)$ such that $(\xi_n, r_n) \in \partial^w_K f(\bar{x}) + \sum_{i=1}^m \eta_i \partial^w_K g_i(\bar{x}) + \sum_{j=1}^p |\gamma_j| \partial^w_K(sign(\gamma_j)h_j)(\bar{x}) + N^a_C(\bar{x})$ for all $n \geq 1$, $\eta_i g_i(\bar{x}) = 0$ for $i = 1,2,\dots,m$, and $\gamma_j h_j(\bar{x}) = 0$ for $j = 1,2,\dots,p$.*

*Proof.* Since $K \subset C$, it yields from the initial assumption that $\sup_{x\in K\setminus\{\bar{x}\}} \frac{|f(x)-f(\bar{x})|}{\|x-\bar{x}\|} < +\infty$. Applying Theorem 3.1 (ii), we see that $f$ is weakly subdifferentiable at $\bar{x}$ on $K$. Because $f$ attains a global maximum on $K$ at $\bar{x}$, we find from Theorem 3.6 that there exists a sequence

$((\xi_n, r_n))_{n\geq 1} \subset X^* \times \mathbb{R}_+$ with $\lim_{n\to+\infty}(\xi_n, r_n) = (0,0)$ such that, for all $n \geq 1$, $(\xi_n, r_n) \in \partial_K^w f(\bar{x}) + N_C^a(\bar{x})$. Then, in a similar way to prove Theorem 4.1, one can find Lagrangian multipliers $\eta := (\eta_1, \eta_2, \ldots, \eta_m) \in \mathbb{R}_+^m$ and $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_p) \in \mathbb{R}^p$ satisfying $\eta_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m$; $\gamma_j h_j(\bar{x}) = 0,\ j = 1, 2, \ldots, p$; and the following relation

$$(\xi_n, r_n) \in \partial_K^w f(\bar{x}) + \sum_{i=1}^m \eta_i \partial_K^w g_i(\bar{x}) + \sum_{j=1}^p |\gamma_j| \partial_K^w (sign(\gamma_j) h_j)(\bar{x}) + N_C^a(\bar{x}),$$

which completes the proof. $\square$

**Remark 4.2.** We remark that the results obtained above are still true if the augmented normal cone $N_C^a(\bar{x})$ is removed and replaced by an other augmented normal cone $N_K^a(\bar{x})$. In addition, based on the results achieved in Section 4, if we replace the limit $\lim_{n\to+\infty}(\xi_n, r_n) = (0,0)$ by the limit $\lim_{n\to+\infty}\xi_n = 0$, then the result obtained in all the preceding statements is still true, where the relation

$$(\xi_n, r_n) \in \partial_K^w f(\bar{x}) + \sum_{i=1}^m \eta_i \partial_K^w g_i(\bar{x}) + \sum_{j=1}^p |\gamma_j| \partial_K^w (sign(\gamma_j) h_j)(\bar{x}) + N_C^a(\bar{x}),\ \forall n \geq 1$$

is removed and replaced by the relation

$$(\xi_n, r_n) \in \partial_K^w f(\bar{x}) + \sum_{i=1}^m \eta_i \partial_K^w g_i(\bar{x}) + \sum_{j=1}^p |\gamma_j| \partial_K^w (sign(\gamma_j) h_j)(\bar{x}),\ \forall n \geq 1.$$

## 5. Conclusion

In this paper, we obtained necessary and sufficient optimality conditions for the problem of minimizing/maximizing nonsmooth nonconvex functions $l : X \to Y$ and $l : X \to \mathbb{R}$ over a feasible set $C$ at the vector under consideration by means of the weak subdifferentials, the augmented weak subdifferentials, and the weak subdifferentials. As an application, the necessary optimality conditions for nonsmooth nonconvex mathematical programming problems (P) and (NMPP) via the weak subdifferentials, the augmented weak subdifferentials, and the augmented normal cones in normed spaces were presented. In addition, the characterization of the class of weakly subdifferentiable and augmented weakly subdifferentiable functions are provided accordingly. It is of interest to use these optimality conditions to construct algorithms for finding the (weakly) efficient solutions of the class of nonsmooth nonconvex mathematical programming problems in terms of the weak subdifferentials notion and the augmented normal cones notion in the future.

**Acknowledgments**

The authors are indebted to the editor and the anonymous referee for very useful comments and suggestions on the preceding versions of this paper. This research was supported by The University of Danang - University of Science and Education under Grant No. T2023-KN-24.

## References

[1] S. Boyd, S.P. Boyd, I. Vandenberghe, Convex Optimization, Cambridge University Press, Cambridge, 2004.

[2] A.D. Ioffe, Calculus of dini subdifferentials of functions and contingent conderivatives of set-valued maps, Nonlinear Anal. 8 (1984), 517-539.

[3] M.I. Henig, A cone separation theorem, J. Optim. Theory Appl. 36 (1982), 451-455.
[4] M.I. Henig, Proper efficiency with respect to cones, J. Optim. Theory Appl. 36 (1982), 387-407.
[5] G. Kumar, J.C. Yao, Fréchet subdifferential calculus for interval-valued functions and its applications in nonsmooth interval optimization, J. Nonlinear Var. Anal. 7 (2023), 811-837.
[6] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.
[7] R.T. Rockafellar, Proximal subgradients, marginal values, and augmented Lagrangians in nonconvex optimization, Math. Oper. Res. 6 (1981), 424-436.
[8] R.T. Rockafellar, Lagrange multipliers and optimality, SIAM Rev. 35 (1993), 183-238.
[9] R. Kasimbeyli, M. Mammadov, Optimality conditions in nonconvex optimization via weak subdierentials, Nonlinear Anal. 74 (2011), 2534-2547.
[10] A.Y. Azimov, R.N. Gasimov, On weak conjugacy, weak subdifferentials and duality with zero gap in nonconvex optimization, Int. J. App. Math. 1 (1999), 171-192.
[11] R. Kasimbeyli, M. Mammadov, On weak subdifferentials, directional derivatives, and radial epiderivatives for nonconvex functions, SIAM J. Optim. 20 (2009), 841-855.
[12] R. Kasimbeyli, G. Inceoglu, The properties of the weak subdifferentials, G.U.J. Science. 23 (2010), 49-52.
[13] A.Y. Azimov, R.N. Gasimov, Stability and duality of nonconvex problems via augmented lagrangian, Cybenet. Sys. Anal. 38 (2022), 412-421.
[14] A.Y. Azimov, R.N. Gasimov, Stability and duality of nonconvex problems via augmented Lagrangian, Cybenet. Sys. Anal. 3 (2002), 120-130.
[15] M. Laghdir, M. Echchaabaoui, Pareto subdifferential calculus for convex set-valued mappings and applications to set optimization, J. Appl. Numer. Optim. 4 (2022), 315-339.
[16] G. Inceoglu, Some properties of second-order weak subdifferentials, Turk. J. Math. 45 (2021), 955-960.
[17] R. Kasimbeyli, A nonlinear cone separation theorem and scalarization in nonconvex vector optimization, SIAM J. Optim. 20 (2010), 1591-1619.
[18] E. Constantin, Necessary conditions for weak minima and for strict minima of order two in nonsmooth constrained multiobjective optimization, J. Global. Optim. 80 (2021), 177–193.
[19] X.H. Gong, Scalarization and optimality conditions for vector equilibrium problems, Nonlinear Anal. 73 (2010), 3598-3612.
[20] F. Clarke, A new approach to lagrange multipliers, Math. Oper. Re. 1 (1976), 165-174.
[21] F. Clarke, Optimization and Nonsmooth Analysis, Wiley-Interscience, New York, USA, 1983.
[22] H.P. Benson, An improved definition of proper efficiency for vector maximization with respect to cones, J. Math. Anal. Appl. 71 (1979), 232-241.
[23] Y. Zhai, Q. Wang, T. Tang, Robust duality for robust efficient solutions in uncertain vector optimization problems, Japan J. Ind. Appl. Math. 40 (2023), 907–928.
[24] B. Jiménez, V. Novo, First order optimality conditions in vector optimization involving stable functions, Optimization. 57 (2008), 449-471.
[25] T.V. Su, D. V. Luu, Second-Order Optimality Conditions for Strict Pareto Minima and Weak Efficiency for Nonsmooth Constrained Vector Equilibrium Problems, Numer. Funct. Anal. Optim. 43 (2022), 1732-1759.
[26] D.T. Luc, Theory of Vector Optimization, Lect. Notes in Eco. and Math. Systems, Vol. 319, Springer Verlag, Berlin, 1989.
[27] T.V. Su, New second-order optimality conditions for vector equilibrium problems with constraints in terms of contingent derivatives, Bull. Braz. Math. Soc, New Series. 51 (2020), 371-395.
[28] T.V. Su, D.V. Luu, Higher-order efficiency conditions for constrained vector equilibrium problems, Optimization. 71 (2022), 2613-2642.