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ON METRIC PSEUDO-REGULARITY OF MULTIFUNCTIONS

VAN VU NGUYEN[1,*], HUU TRON NGUYEN[2]

Department of Mathematics and Statistics, Quy Nhon University, Quy Nhon, Binh Dinh, Vietnam

Abstract. In this paper, inspired by the previous work in [Ngai, Tron, Vu, Théra, Set-Val. Var. Anal. 28 (2020), 61-87], we continue to study a general regularity model called *directional metric pseudo-regularity* of set-valued mappings defined on Banach spaces, which is stronger than the one given in the reference above. The characterizations of this model through slopes and coderivative are established.
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1. Introduction

In recent years, directional regularity models of set-valued mappings showed their important role in various mathematical problems, such as in establishing optimality conditions for mathematical programs, in calculating the tangent cones of zero sets, and so on when regularity models do not meet these demands. For instance, when the constraint qualification condition of an optimal problem was violated, a natural question is that "*How do we can still obtain optimality conditions for these problems?*". To this question, Gfrerer [9] used the metric pseudo sub/regularity concepts to obtain optimality conditions for mathematical programs which were degenerated; Ngai, Tron, and Tinh [15] demonstrated that the directional Hölderian metric subregularity can be applied to the tangent cones to the solution set of generalized equations in the degenerate case. As arising from those important roles, in [16], the authors introduced a general concept of metric pseudo-subregularity which unifies some models studied recently by Gfrerer [9] and Ngai, Tron, and Théra [20]. For recent results on directional regular/subregular versions, we refer to [8, 16, 5] and the references therein.

Toward this research line, we here continue to study a generally directional metric pseudo-regular concept, which is stronger than the one mentioned in the work [16]. Moreover, it covers regular models studied recently in [14, 7, 1]. These phenomena have important roles in many areas of mathematics, such as convergence analysis of algorithms, higher/first-order optimality conditions of optimization problems and optimal control problems, calculating of normal/tangent cones, subdifferentials, or coderivative as well as in establishing calculus rules for them, and so on.

*Corresponding author.
E-mail address: nguyenvanvu@qnu.edu.vn (V.V. Nguyen), nguyenhuutron@qnu.edu.vn (H.T. Nguyen).
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The main goal of this paper is of two-folds. First, we introduce our extended model of metric pseudo-regularity which traces back the notion defined and used in [9], and then, we sketch out a simple comparison with the several existing ones in the literature. Second, we study some kinds of sufficient characterizations for the validity of the new notion. Those can be served as a criterion test whether the concrete situation fulfills the model above.

The present paper is organized as follows. In Section 2, we revise some necessary preliminaries of variational analysis and generalized differentiation, which are needed in the following sections. Section 3 is devoted to our model and some characterizations of the directional metric pseudo-regularity in terms of strong slopes of associating functions. Finally, we study some infinitesimal criteria determined through the coderivative concept in Section 4.

2. Preliminaries

Background for the scope of this paper are basic tools from variational analysis and generalized differentiation that can be found in the monograph by Borwein [4], Schirotzek [19], Mordukhovich [12], Ioffe [11], Rockafellar& Wets [18], and Penot [17]. For the convenience of reading, let us recall some essential notations and facts.

We almost work with the objects in a (real) Banach space. If X is such a space, we denote by $\|\cdot\|$ the associated norm and by $d(x,\Omega)$ (or $d_\Omega(x)$ for shortness) the distance from $x \in X$ to the subset Ω of X, that is, $d(x,\Omega) := \inf\{\|x-y\| : y \in \Omega\}$. Given X, we denote by X^* its topological dual (continuous dual), and by $\|\cdot\|^*$ the dual norm of $\|\cdot\|$. As usual, we write $\mathbb{B}_X = \{x \in X : \|x\| \leqslant 1\}$ for the closed unit ball and $\mathbb{B}(x,r) = x + r\mathbb{B}$ the closed ball with center x and radius r, respectively.

Given an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$, $\operatorname{Dom} f$ refers to the domain of f, that is, the set of those points $x \in X$ such that $f(x)$ is finite. The lower semicontinuous envelope of f, written as $\operatorname{cl} f$, is the function defined by $\operatorname{cl} f(x) = \liminf_{u \to x} f(u)$. One next recalls some differential objects which are used in what follows as slopes, subdifferential, and coderivative. The convex subdifferential of f at $x \in \operatorname{Dom} f$ is the set

$$\partial f(x) := \{x^\star \in X^* : \langle x^\star, y - x\rangle \le f(y) - f(x) \text{ for all } y \in X\},$$

with the convention that $\partial f(x) = \emptyset$ when $f(x) = +\infty$.

The slope (or strong slope) $|\nabla f|(x)$ of a lower semicontinuous (lsc) function f at $x \in \operatorname{Dom} f$ is a quantity introduced by De Giovani, Marino & Tosques [6] defined as follows

$$|\nabla f|(x) := \begin{cases} 0 & \text{if } x \text{ is a local minimum of } f, \\ \displaystyle\limsup_{y \to x,\, y \neq x} \frac{[f(x) - f(y)]_+}{\|x - y\|} & \text{otherwise.} \end{cases}$$

where the notation a_+ stands for $\max\{0,a\}$. When $x \notin \operatorname{Dom} f$, we simply set $|\nabla f|(x) = +\infty$.

If f is a convex function defined on a Banach space and x is not a minimum point, then, according to Ioffe [10, Poposition 3.8] (see also Azé & Corvelec [2, Proposition 3.2]),

$$|\nabla f|(x) = \sup_{y \neq x} \frac{f(x) - f(y)}{\|x - y\|} \quad \text{and} \quad |\nabla f|(x) = d(0, \partial f(x)).$$

Note that the quantity $\sup_{y\neq x}\frac{f(x)-f(y)}{\|x-y\|}$ is sometimes known as non-local/global slope (see, e.g., [20] and the references therein) to f at x. In the sequel, we use the notion of abstract subdifferential operator ∂ from X into the subsets of X^* satisfying the following conditions:

(C1) If $f : X \to \mathbb{R}$ is a convex function which is continuous around $\bar{x} \in X$ and $\beta : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function at $t = f(x)$, then

$$\partial(\beta \circ f)(x) \subseteq \{\beta'(f(x))x^\star \in X^* : \langle x^\star, y-x\rangle \leq f(y)-f(x) \quad \forall y \in X\};$$

(C2) $\partial f(x) = \partial g(x)$ if $f(y) = g(y)$ for all y in a neighborhood of x;

(C3) Let $f_1 : X \to \mathbb{R}\cup\{+\infty\}$ be lsc and $f_2,\ldots,f_n : X \to \mathbb{R}$ be Lipschitz functions. If $f_1 + f_2 + \cdots + f_n$ attains a local minimum at x_0, then for any $\varepsilon > 0$, there exist $x_i \in x_0 + \varepsilon\mathbb{B}_X$, $x_i^\star \in \partial f_i(x_i)$ $(i = 1,2,\ldots,n)$ such that

$$|f_i(x_i) - f_i(x_0)| < \varepsilon,\ \|x_1^\star + x_2^\star + \cdots + x_n^\star\| < \varepsilon.$$

We note that the indicator function δ_C of a closed set C in X is the function defined by $\delta_C(x) = 0$ when $x \in C$ and $\delta_C(x) = +\infty$, otherwise.

Given an abstract subdifferential ∂, the set $N(C,x) := \partial\delta_C(x)$ is called the *normal cone* to C at x associated with ∂.

(C4) $N(C,x)$ is assumed to be a cone for any closed subset C of X.

We further assume that ∂ satisfies the separable property in the following sense:

(C5) If f is a separable function defined on $X \times Y$, that is, $f(x,y) := f_1(x) + f_2(y)$, $(x,y) \in X \times Y$, where $f_1 : X \longrightarrow \mathbb{R}\cup\{+\infty\}$, $f_2 : Y \longrightarrow \mathbb{R}\cup\{+\infty\}$, then there holds

$$\partial f(x,y) = \partial f_1(x) \times \partial f_2(y),\ \text{for all } (x,y) \in X \times Y.$$

It is well known that the proximal subdifferential in Hilbert spaces, the Fréchet subdifferential in Asplund spaces, the viscosity subdifferential in smooth spaces as well as the Ioffe and the Clarke-Rockafellar subdifferentials in the setting of general Banach spaces are the ones verifying conditions (C1)-(C5).

3. Directional Metric Pseudo-Regularity and Strong Slope Characterization

Let us start by recalling the notion of directional metric pseudo-regularity introduced in [9].

Definition 3.1 (directional metric pseudo-regularity)**.** Let $T_i : X \longrightarrow Y_i$ $(i = 1,\ldots,m)$ be given mappings and $\gamma_1,\ldots,\gamma_m \geqslant 1$ be given real numbers. Consider the new mapping $T : X \rightrightarrows Y = Y_1 \times \cdots \times Y_m$ defined by $T(x) = (T_1(x),\ldots,T_m(x))$. One says that T is metrically pseudo-regular of order $\gamma = (\gamma_1,\ldots,\gamma_m)$ in the direction $(u,v) \in X \times Y$ near $(\bar{x},\bar{y}) \in \operatorname{gph} T$ if there are positive real numbers α, β, and κ such that

$$d\big(x, T^{-1}(y)\big) \leqslant \kappa \sum_{i=1}^{m} \frac{d\big(y_i, T_i(x)\big)}{\|x-\bar{x}\|^{\gamma_i - 1}}$$

holds for all $(x,y) \in (\bar{x},\bar{y}) + V_{\alpha,\beta}(u,v)$ with $x \neq \bar{x}$ and $d\big(y_i, T_i(x)\big) \leqslant \beta\|x-\bar{x}\|^{\gamma_i}$, $i = 1,\ldots,m$. Here and in what follows, for each Banach space Z and each $z \in Z$, the notation $V_{\varepsilon,\delta}(z)$ stands for the set $V_{\alpha,\beta}(z) := \big\{z' \in \alpha\mathbb{B}_Z \ \big|\ \big\|\|z\|z' - \|z'\|z\big\| \leqslant \beta\|z\|\|z'\|\big\}$.

To unify and extend regular models existing in the literature, we consider the following general regular model.

Definition 3.2 (directional (γ,h,μ)-MPR)**.** Let $T : X \rightrightarrows Y$ and $\gamma_1,\ldots,\gamma_m \geqslant 1$ be as in Definition 3.1. Let $h_1,\ldots,h_m,\mu : X\times Y \longrightarrow \mathbb{R}_+$ be given real-valued functions, where μ is at least continuous while h satisfies $h_i(x,y) = 0 \Longrightarrow (x,y_i) \in \operatorname{gph} T_i$. The mapping T is said to be metrically pseudo-regular of order $\gamma = (\gamma_1,\ldots,\gamma_m)$ w.r.t h and μ (or (γ,h,μ)-metrically pseudo regular, or (γ,h,μ)-MPR in short) in the direction $(u,v) \in X\times Y$ near $(\bar{x},\bar{y}) \in \operatorname{gph} T$ if there are positive real numbers α, β, and κ such that

$$d\big(x,T^{-1}(y)\big) \leqslant \kappa \sum_{i=1}^{m} \frac{d\big(y_i,T_i(x)\big)}{[h_i(x,y)]^{\gamma_i-1}}, \tag{3.1}$$

holds for all $(x,y) \in (\bar{x},\bar{y}) + V_{\alpha,\beta}(u,v)$ with $d\big(y_i,T_i(x)\big) \leqslant \beta[\mu(x,y)]^{\gamma_i}$, $i = 1,\ldots,m$. In (3.1), we made use of the conventions that $1/0 = +\infty$, $r + (+\infty) = +\infty$ and $0\times(+\infty) = 0$.

When $(u,v) = (0,0)$, we briefly say T to be metrically pseudo-regular of order $\gamma = (\gamma_1,\ldots,\gamma_m)$ w.r.t h and μ near $(\bar{x},\bar{y}) \in \operatorname{gph} T$. Definition 3.2 covers almost regular phenomenon existing separately in the literature. Below we demonstrate that it is really a generalization of many regular properties in variational analysis and optimization.

(1) By letting $h_i(x,y) := \|x-\bar{x}\|$ and $\mu(x,y) := \|x-\bar{x}\|$, we recover the notion of directional metric pseudo-regularity in Definition 3.1.
(2) Taking $h_i(x,y) := \|x-\bar{x}\|$ and $\mu(x,y) := \|(x,y)-(\bar{x},\bar{y})\|$, Definition 3.2 subsumes the notion of directional metric Holder regularity mentioned in the work [20]. This property is an extention of the Hölder metric regularity property studied by Frankowska and Quicampoix in [7], which is well-known in the community of optimization and variational analysis.

Remark 3.1. Fixing an element $(\bar{x},\bar{y}) \in X\times Y$ and setting up $\overline{T}(\cdot) := T(\cdot+\bar{x})-\bar{y}$, we find $(\bar{x},\bar{y}) \in \operatorname{gph} T \Longleftrightarrow (0,0) \in \operatorname{gph}\overline{T}$. For simplicity, throughout the paper, we focus on the case $\bar{x} = 0$ and $\bar{y} = 0$ only. (If the present situation does not satisfy, we simply consider $\overline{T}$ instead of T itself.) Further, by an abuse of terminology, let us skip the reference point $(0,0)$ thereafter when dealing with the property of directional (γ,h,μ)-MPR.

To establish some useful characterizations for the property defined in Definition 3.2, we introduce the extended real-valued function $\psi_T : X\times Y \longrightarrow \mathbb{R}\cup\{+\infty\}$ given through the following formula

$$\psi_T(x,y) := \begin{cases} \displaystyle\sum_{h_i(x,y)>0} \frac{d_{\operatorname{gph} T_i}(x,y_i)}{h_i(x,y)^{\gamma_i-1}}, & \text{if } \sum_{i=1}^m h_i(x,y) > 0 \\ 0, & \text{if } \sum_{i=1}^m h_i(x,y) = 0 \text{ and } y \in T(x) \\ +\infty, & \text{otherwise.} \end{cases}$$

We begin with the following proposition.

Proposition 3.1. *Let γ, h and μ be as in Definition 3.2 for which μ is Lipschitz near $(\bar{x},\bar{y}) = (0,0)$ and h is at least continuous such that*

$$(0,0) \in h_1^{-1}(0) = \ldots = h_m^{-1}(0), \tag{3.2a}$$

$$\limsup_{\alpha\downarrow 0,\beta\downarrow 0} \left\{ \sup_{(x,y)\in V_{\alpha,\beta}(u,v)} \frac{\mu(x,y)}{h_i(x,y)} \right\} < +\infty, \forall i = 1,\ldots,m. \tag{3.2b}$$

Suppose that there exist some positive real sequences $\tau_k \downarrow 0$, $\alpha_k \downarrow 0$, $\beta_k \downarrow 0$ together with some sequences $(x_k,y_k) \notin \operatorname{gph} T$ fulfilling the following conditions

$$(x_k,y_k) \in V_{\alpha_k,\beta_k}(u,v), \tag{3.3a}$$

$$d\big(y_{ik},T_i(x_k)\big) \leqslant \beta_k[\mu(x_k,y_k)]^{\gamma_i}, i = 1,\ldots,m, \tag{3.3b}$$

$$\sum_{h_i(x_k,y_k)>0} \frac{d\big(y_{ki},T_i(x_k)\big)}{[h_i(x_k,y_k)]^{\gamma_i-1}} < \tau_k d\big(x_k,T^{-1}(y_k)\big), \tag{3.3c}$$

$$\limsup_{k\to\infty} \frac{\beta_k}{\tau_k} < +\infty. \tag{3.3d}$$

Then, it is possible to find out $(x'_k,y'_k) \in X\times Y\setminus \operatorname{gph} T$, $(\tau'_k,\alpha'_k,\beta'_k) \in (0,+\infty)\times(0,+\infty)\times(0,+\infty)$ such that

(i) $\lim_{k\to\infty} \tau'_k = \lim_{k\to\infty} \alpha'_k = \lim_{k\to\infty} \beta'_k = 0$*;*
(ii) $|\Gamma\psi_T(\cdot,y'_k)|(x'_k) \leqslant \tau'_k$*;*
(iii) $(x'_k,y'_k) \in V_{\alpha'_k,\beta'_k}(u,v)$*;*
(iv) $\limsup_{k\to\infty} \dfrac{\psi_T(x'_k,y'_k)}{\mu(x'_k,y'_k)} = 0.$

Here, In (ii), the notation $\Gamma_T(\cdot,y'_k)(x'_k)$ stands for the non-local/global slope to the function $\psi_T(\cdot,y'_k)$ taken at the reference point x'_k.

Proof. Ignoring a few first indexes k if necessary, we can assume that μ is Lipschitz with modulus $L_\mu > 0$ on the ball $\alpha_k\mathbb{B}_{X\times Y}$ for all k. Observe that $d_{\operatorname{gph} T_i}(x,y_i) \leqslant d\big(y_i,T_i(x)\big)$. Invoking (3.3b) and (3.3c), one has

$$d_{\operatorname{gph} T_i}(x_k,y_{ik}) \leqslant \beta_k[\mu(x_k,y_k)]^{\gamma_i}, \psi_T(x_k,y_k) < \tau_k d\big(x_k,T^{-1}(y_k)\big). \tag{3.4}$$

Since equality $\sum_{i=1}^m h_i(x,y) = 0$ holds true only on the case $(x,y) \in \operatorname{gph} T$, one sees that (3.4) implies $\sum_{i=1}^m h_i(x_k,y_k) > 0$. By setting

$$\varepsilon_k := \min\left\{ \tau_k d\big(x_k,T^{-1}(y_k)\big), (1+1/k^2)\beta_k \sum_{h_i(x_k,y_k)>0} \frac{[\mu(x_k,y_k)]^{\gamma_i}}{[h_i(x_k,y_k)]^{\gamma_i-1}} \right\} > 0, \tag{3.5}$$

we have $\psi_T(x_k,y_k) < \varepsilon_k$. Applying the Ekeland variational principle to the lsc function $\psi_T(\cdot,y_k)$, it is possible to find out some element $\hat{x}_k \in X$ such that

- $\|\hat{x}_k - x_k\| \leqslant \lambda_k = \varepsilon_k/\sqrt{\tau_k}$;
- $\psi_T(x_k,y_k) - (\varepsilon_k/\lambda_k)\|\hat{x}_k - x_k\| \geqslant \psi_T(\hat{x}_k,y_k)$;
- $\hat{x}_k$ minimizes $\psi_T(\cdot,y_k) + (\varepsilon_k/\lambda_k)\|\cdot - \hat{x}_k\|$ over the whole space X.

The extremal property of $\hat{x}_k$ permits us to write

$$\left|\Gamma\psi_T(\cdot,y_k)\right|(\hat{x}_k) \leqslant \frac{\varepsilon_k}{\lambda_k}. \tag{3.6}$$

If $y_k \in T(\hat{x}_k)$, then $\hat{x}_k$ belongs to the preimage $T^{-1}(y_k)$, which yields $d\big(x_k, T^{-1}(y_k)\big) \leqslant \|x_k - \hat{x}_k\| \leqslant \lambda_k$. As a result, we obtain $\varepsilon_k \leqslant \tau_k d\big(x_k, T^{-1}(y_k)\big) \leqslant \tau_k\lambda_k = \sqrt{\tau_k}\varepsilon_k$, which is equivalent to the inequality $\tau_k \geqslant 1$. This implicitly shows that the inclusion $y_k \in T(\hat{x}_k)$ is only valid for finitely many k because of $\tau_k \downarrow 0$. Thus, after relabeling the indexes, we can say $y_k \notin T(\hat{x}_k)$ for all k. Let us choose $x'_k = \hat{x}_k$ and $y'_k = y_k$ and define

$$\tau'_k = \frac{\varepsilon_k}{\lambda_k}, \tag{3.7a}$$

$$\alpha'_k = \|(\hat{x}_k, y_k)\|, \tag{3.7b}$$

$$\beta'_k = \begin{cases} \big\|\|(\hat{x}_k,y_k)\|^{-1}(\hat{x}_k,y_k) - \|(u,v)\|^{-1}(u,v)\big\|, & \text{if } \|(u,v)\| \neq 0, \\ \|(\hat{x}_k,y_k)\|, & \text{otherwise.} \end{cases} \tag{3.7c}$$

Under such selection, we have $(x'_k, y'_k) \notin \operatorname{gph} T$, which yields $h_i(x'_k, y'_k) > 0$ for any $i = 1, \ldots, m$.

We are now going to verify (i), (ii), (iii), and (iv). The fact (ii) is obviously followed from the definition of τ'_k and (3.6), while (3.7) implies (iii) directly. The rest of our proof needs the next useful estimation

$$\limsup_{k\to\infty} \left\{ \frac{\lambda_k\sqrt{\tau_k}}{\beta_k\mu(x_k,y_k)} \right\} < +\infty. \tag{3.8}$$

Indeed, invoking (3.5), we have that

$$\lambda_k = \frac{1}{\sqrt{\tau_k}}\varepsilon_k \leqslant \left(1 + \frac{1}{k^2}\right) \frac{\beta_k\mu(x_k,y_k)}{\sqrt{\tau_k}} \times \sum_{h_i(x_k,y_k)>0} \left(\frac{\mu(x_k,y_k)}{h_i(x_k,y_k)}\right)^{\gamma_i - 1}. \tag{3.9}$$

Hence, (3.8) is derived by (3.2b) and (3.9). Let us proceed the proof of (i). We first immediately infer $\lim_{k\to\infty}\tau'_k = 0$ from the choice of λ_k. On the other hand, assumption (3.2b) permits us to write $\mu(0,0) = 0$. By exploiting the Lipschitz continuity of μ, the sequence of ratios $\mu(x_k,y_k)/\|(x_k,y_k)\|$ is thereby bounded from above. Combining with (3.8), we deduce

$$\limsup_{k\to\infty} \frac{\lambda_k\sqrt{\tau_k}}{\beta_k\|(x_k,y_k)\|} \leq M, \tag{3.10}$$

for some positive real M. From assumption (3.3d), it holds that

$$\begin{aligned} \limsup_{k\to\infty} \frac{\|(x'_k,y'_k) - (x_k,y_k)\|}{\|(x_k,y_k)\|} &\leqslant \limsup_{k\to\infty} \frac{\lambda_k}{\|(x_k,y_k)\|} \leqslant \limsup_{k\to\infty} \frac{M\beta_k}{\sqrt{\tau_k}} \\ &= \limsup_{k\to\infty} \left(M\sqrt{\beta_k}\sqrt{\frac{\beta_k}{\tau_k}} \right) = 0. \end{aligned}$$

Thus

$$\limsup_{k\to\infty} \frac{\|(x'_k,y'_k) - (x_k,y_k)\|}{\|(x_k,y_k)\|} = 0. \tag{3.11}$$

As a consequence, we obtain $\frac{\|(x'_k,y'_k)\|}{\|(x_k,y_k)\|} \to 1$, which implies $\alpha'_k = \|(x'_k,y'_k)\| \to 0$. To reach an upper bound for β'_k, let us introduce some new elements as follows

$$u_k = \|(x_k,y_k)\|^{-1}x_k, \quad v_k = \|(x_k,y_k)\|^{-1}y_k,$$
$$u'_k = \|(x'_k,y'_k)\|^{-1}x'_k, \quad v'_k = \|(x'_k,y'_k)\|^{-1}y'_k.$$

In view of expression (3.7b), a few simple manipulations give us

$$\begin{aligned} \|(u,v)\|\beta'_k &= \big\|\|(u,v)\|(u'_k,v'_k)-(u,v)\big\| \\ &\leqslant \frac{\|(x_k,y_k)\|}{\|(x'_k,y'_k)\|}\big\|\|(u,v)\|(u_k,v_k)-(u,v)\big\| \\ &\quad + \frac{\|(x'_k,y'_k)-(x_k,y_k)\|}{\|(x_k,y_k)\|} + \|(u,v)\|\left|1-\frac{\|(x_k,y_k)\|}{\|(x'_k,y'_k)\|}\right|. \end{aligned} \tag{3.12}$$

Taking into account $(x_k,y_k) \in V_{\alpha_k,\beta_k}(u,v)$, we reach the following

$$\big\|\|(u,v)\|(u_k,v_k)-(u,v)\big\| \leqslant \|(u,v)\|\beta_k. \tag{3.13}$$

Hence, combining the definition of β'_k along with (3.12) and (3.13), we obtain

$$\|(u,v)\|\beta'_k \leqslant \|(u,v)\|\frac{\|(x_k,y_k)\|}{\|(x'_k,y'_k)\|}\beta_k + \frac{\|(x'_k,y'_k)-(x_k,y_k)\|}{\|(x_k,y_k)\|} \qquad + \|(u,v)\|\left|1-\frac{\|(x_k,y_k)\|}{\|(x'_k,y'_k)\|}\right|. \tag{3.14}$$

As a result, the limit $\lim_{k\to\infty}\beta'_k = 0$ is derived by (3.7b), (3.11), and (3.14).

Finally, we prove (iv). But this evidently comes from (3.8), (3.3d), and the following facts

- $\psi_T(x'_k,y'_k) \leqslant \psi_T(x_k,y_k) < \lambda_k\sqrt{\tau_k}$;
- $\mu(x'_k,y'_k) \geqslant \mu(x_k,y_k) - L_\mu\|(x'_k,y'_k)-(x_k,y_k)\|$;
- $\|(x'_k,y'_k)-(x_k,y_k)\| \leqslant \lambda_k$.

Summarily, the proof of Proposition 3.1 is thereby completed. □

Remark 3.2. It is possible to see that assumptions (3.2a)–(3.2b) are the key point for the arguments in the proof of Proposition 3.1. It measures how fast the gauge function μ varies (as (x,y) tends to the reference point $(0,0)$ along the ray determined by direction (u,v)) in comparison with each controlling one h_i. Such a condition does hold trivially if we are in the situation of the metric pseudo-regularity in terms of [9], since all ratios μ/h_i are constant.

Based on Proposition 3.1, we are now going to present the first result related to the directional (γ,h,μ)-MPR concept described in Definition 3.2. For some Banach space Z, we write $z \xrightarrow{\hat{z}} 0$ to emphasize that z converges to the origin along the direction $\hat{z}$. Precisely, it means

$$\|z\| \to 0 \text{ and } \big\|\|\hat{z}\|z - \|z\|\hat{z}\big\| = o(\|z_k\|\|\hat{z}\|).$$

Theorem 3.1 (Strong slope characterization for directional (γ,h,μ)-MPR)**.** *Let γ,h and μ be as in Definition 3.2. Suppose that all the hypotheses (3.2a), (3.2b) in Proposition 3.1 are valid as well. Then the mapping T is (γ,h,μ)-MPR in the direction (u,v) if the following condition*

$$\liminf_{\substack{(x,y)\xrightarrow{(u,v)}(0,0),\\ \psi_T(x,y)/\mu(x,y)\to 0,\\ (x,y)\notin \operatorname{gph} T}} |\Gamma\psi_T(\cdot,y)|(x) > 0$$

does hold.

Proof. Suppose on the contrary that T fails to be (γ,h,μ)-MPR in direction (u,v). Then, for each $k=1,2,\dots$, there exit positive parameters $\alpha_k \leqslant \frac{1}{k^2}$, $\beta_k \leqslant \frac{1}{k^2}$, and a pair of elements $(x_k,y_k)\in V_{\alpha_k,\beta_k}(u,v)$ such that

$$d\big(y_{ik},T_i(x_k)\big)\leqslant \beta_k[\mu(x_k,y_k)]^{\gamma_i},$$
$$\sum_{h_i(x_k,y_k)>0}\frac{d\big(y_{ik},T_i(x_k)\big)}{[h_i(x_k,y_k)]^{\gamma_i-1}}<\frac{1}{k}d\big(x_k,T^{-1}(y_k)\big).$$

Applying now Proposition 3.1, we can find new sequences of positive parameters $\tau'_k\downarrow 0$, $\alpha'_k\downarrow 0$, $\beta'_k\downarrow 0$ together with the sequence of elements $(x'_k,y'_k)\in V_{\alpha'_k,\beta'_k}(u,v)\setminus \operatorname{gph}T$ obeying the conditions below

- $\psi_T(x'_k,y'_k)/\mu(x'_k,y'_k)\to 0$;
- $|\Gamma\psi_T(\cdot,y'_k)|(x'_k)\leqslant \tau'_k$.

Those reach contradiction, since

$$\liminf_{\substack{(x,y)\xrightarrow{(u,v)}(0,0),\\ \psi_T(x,y)/\mu(x,y)\to 0,\\ (x,y)\notin\operatorname{gph}T}}\Big\{|\Gamma\psi_T(\cdot,y)|(x)\Big\}\leqslant\liminf_{k\to\infty}\Big\{|\Gamma\psi_T(\cdot,y'_k)|(x'_k)\Big\}\leqslant\liminf_{k\to\infty}\big\{\tau'_k\big\}.$$

Hence, we reach the desired conclusion. $\square$

Theorem 3.1 can be handled as a technical test for further consideration of the directional metric pseudo-regularity. As commented in Remark 3.2, this test is according to the quantitative condition (3.2b). Yet it is sometimes not easy to verify such a condition in practice. The rest of this section will deal with the particular case $m=1$ where one may avoid the occurrence of (3.2b). To this end, we need a preparatory result.

Proposition 3.2. *Let $\mu,h: X\times Y\longrightarrow\mathbb{R}_+$ be the real-valued functions vanishing at the origin of $X\times Y$ such that*

- *μ is Lipschitz continuous around the origin;*
- *h is lsc and at least continuous w.r.t the first argument x with $h^{-1}(0)\subset\operatorname{gph}T$.*

Further, let $\gamma\geqslant 1$ be a given number, and let (α_k), (β_k), and (τ_k) be the positive real sequences converging to zero with $\limsup_{k\to\infty}\{\beta_k/\tau_k\}<+\infty$. Suppose that there exists some sequence $(x_k,y_k)\in X\times Y$ satisfying all conditions (3.3a)-(3.3c) with $m=1$, $\gamma_1=\gamma$, $h_1=h$, and $(u,v)\in X\times Y$. Then, there exist sequences $(\alpha'_k,\beta'_k,\tau'_k)\in(0,+\infty)\times(0,+\infty)\times(0,+\infty)$ and $(x'_k,y'_k)\in X\times Y$ fulfilling the followings statements:

(i) $(x'_k,y'_k)\in V_{\alpha'_k,\beta'_k}(u,v)$;

(ii) $\lim_{k\to\infty}\tau'_k=\lim_{k\to\infty}\alpha'_k=\lim_{k\to\infty}\beta'_k=0$;

(iii) $\limsup_{k\to\infty}\dfrac{d_{\operatorname{gph}T}(x'_k,y'_k)}{[\mu(x'_k,y'_k)]^{\gamma}}=0$;

(iv) $|\nabla\psi_T(\cdot,y'_k)|(x'_k)\leqslant d_{\operatorname{gph}T}(x'_k,y'_k)|\nabla[h(\cdot,y'_k)]^{1-\gamma}|(x'_k)+\tau'_k[h(x'_k,y'_k)]^{1-\gamma}$.

Proof. By virtue of the assumption of Proposition 3.2, the distance function $d_{\operatorname{gph}T}$ obeys the estimation

$$d_{\operatorname{gph}T}(x_k,y_k)<\min\left\{\tau_k[h(x_k,y_k)]^{\gamma-1}d\big(x_k,T^{-1}(y_k)\big),\left(1+\frac{1}{k^2}\right)\beta_k[\mu(x_k,y_k)]^{\gamma}\right\}. \qquad (3.16)$$

Pick ε_k to be the quantity in the right-hand side of (3.16). Applying the Ekeland variational principle (see, e.g., [3, 4]), we obtain a point $\tilde{x}_k \in X$ which satisfies the following conditions:

- $\|\tilde{x}_k - x_k\| \leqslant \lambda_k = \varepsilon_k/\sqrt{\tau_k}$;
- $d_{\operatorname{gph} T}(\tilde{x}_k, y_k) \leqslant d_{\operatorname{gph} T}(x_k, y_k) - \varepsilon_k/\lambda_k \|\tilde{x}_k - x_k\|$;
- the function $d_{\operatorname{gph} T}(\cdot, y_k) + \varepsilon_k/\lambda_k \|\cdot - \tilde{x}_k\|$ attains its global minimum at $\tilde{x}_k$.

The same arguments as in the preceding proposition give us $(\tilde{x}_k, y_k) \notin \operatorname{gph} T$. In particular, $\|(\tilde{x}_k, y_k)\| > 0$. Let us proceed the proof by setting $x'_k = \tilde{x}_k$, $y'_k = y_k$, and

$$\begin{aligned}
\tau'_k &= \frac{\varepsilon_k}{\lambda_k},\\
\alpha'_k &= \|(x'_k, y'_k)\|,\\
\beta'_k &= \begin{cases} \left\| \|(x'_k, y'_k)\|^{-1}(x'_k, y'_k) - \|(u,v)\|^{-1}(u,v) \right\|, & \text{if } \|(u,v)\| \neq 0,\\ \|(x'_k, y'_k)\|, & \text{otherwise.} \end{cases}
\end{aligned}$$

Next, we check all the conclusions of Proposition 3.2. The first assertion (i) is clear by the definition of x'_k and y'_k. To verify item (ii), it remains to focus on the two limits $\limsup_{k\to\infty} \alpha'_k$ and $\limsup_{k\to\infty} \beta'_k$ only, because of $\tau'_k = \sqrt{\tau_k} \to 0$. To this end, we observe that the following inequality

$$\limsup_{k\to\infty} \left\{ \frac{\lambda_k \sqrt{\tau_k}}{\beta_k [\mu(x_k, y_k)]^\gamma} \right\} < +\infty \tag{3.17}$$

does hold by the definition of λ_k. Thus, by mimicking the strategy as in the proof of Proposition 3.1, we obtain

$$\limsup_{k\to\infty} \left\{ \frac{\lambda_k}{\|(x_k, y_k)\|^\gamma} \right\} = 0. \tag{3.18}$$

Since

$$\left| \|(x'_k, y'_k)\| - \|(x_k, y_k)\| \right| \leqslant \|(x'_k, y'_k) - (x_k, y_k)\| = \|\tilde{x}_k - x_k\| \leqslant \lambda_k,$$

relation (3.18) yields $\lim_{k\to\infty} \left(\|(x'_k, y'_k)\| / \|(x_k, y_k)\| \right) = 1$. Consequently, repeating the arguments as in the proof of Proposition 3.1, the two limits $\limsup_{k\to\infty} \alpha'_k = \limsup_{k\to\infty} \beta'_k = 0$ follow. We are now in a position to check (iii). When k is large both (x'_k, y'_k) and (x_k, y_k) are in a same neighborhood of the origin, the Lipschitz continuity of μ could be applied. Precisely, one has

$$|\mu(x'_k, y'_k) - \mu(x_k, y_k)| \leqslant L_\mu \|(x'_k, y'_k) - (x_k, y_k)\| = L_\mu \|x'_k - x_k\| \leqslant L_\mu \lambda_k, \tag{3.19}$$

where L_μ stands for a Lipschitz modulus to μ. Combining (3.19) with (3.17), we arrive at

$$\lim_{k\to\infty} \frac{\mu(x'_k, y'_k)}{\mu(x_k, y_k)} = 1$$

since $\beta_k/\sqrt{\tau_k} \to 0$. Thus, (iii) is a consequence of the choice of (x'_k, y'_k).

The remainder of our proof is to deal with (iv). We need the following relation

$$|\nabla \psi_T(\cdot, y'_k)|(x'_k) \leqslant [h(x'_k, y'_k)]^{1-\gamma} \frac{\varepsilon_k}{\lambda_k} + d_{\operatorname{gph} T}(x'_k, y'_k) |\nabla [h(\cdot, y'_k)]^{1-\gamma}|(x'_k). \tag{3.20}$$

Indeed, letting $\hat{\gamma} = \gamma - 1 \geqslant 0$ and utilizing the optimality of x'_k, one has

$$\psi_T(x'_k, y'_k) h(x'_k, y'_k)^{\hat{\gamma}} \leqslant \psi_T(x, y'_k) h(x, y'_k)^{\hat{\gamma}} + \frac{\varepsilon_k}{\lambda_k} \|x - x'_k\|.$$

Rearranging this inequality, we obtain

$$h(x'_k,y'_k)^{\hat{\gamma}}[\psi_T(x'_k,y'_k)-\psi_T(x,y'_k)] \leqslant \frac{\varepsilon_k}{\lambda_k}\|x-x'_k\| \\ + \psi_T(x,y'_k)h(x'_k,y'_k)^{\hat{\gamma}}h(x,y'_k)^{\hat{\gamma}}[h(x'_k,y'_k)^{-\hat{\gamma}}-h(x,y'_k)^{-\hat{\gamma}}]. \quad (3.21)$$

Dividing both sides of (3.21) to $\|x-x'_k\| \neq 0$ and passing to the limit as $x \to x'_k$, one has

$$h(x'_k,y'_k)^{\hat{\gamma}}|\nabla\psi_T(\cdot,y'_k)|(x'_k) \leqslant \frac{\varepsilon_k}{\lambda_k} + d_{\operatorname{gph}T}(x'_k,y'_k)|\nabla[h(\cdot,y'_k)]^{-\hat{\gamma}}|(x'_k), \quad (3.22)$$

which is obviously equivalent to (3.20). Returning to the main process, we reach (iv) by invoking (3.20). □

With the help of the previous results, we present a variant of the strong slope characterization given in Theorem 3.1 which might sometimes be more advantageous for applications. The next theorem is in this sense.

Theorem 3.2 (A refinement test when $m=1$)**.** *Let $\gamma, h,$ and μ be as in the hypotheses of Proposition 3.2. Suppose that the quantity $|\nabla[h(\cdot,y)]^{1-\gamma}|(x)$ is finite whenever $(x,y) \notin \operatorname{gph}T$ sufficiently close to $(0,0)$. Then T is (γ,h,μ)-MPR in the direction (u,v) if the following limit*

$$\liminf_{\substack{(x,y)\xrightarrow{(u,v)}(0,0),\\ \frac{d_{\operatorname{gph}T}(x,y)}{[\mu(x,y)]^{\gamma}}\to 0,\\ (x,y)\notin\operatorname{gph}T}} \left\{[h(x,y)]^{\gamma-1}\left[|\nabla\psi_T(\cdot,y)|(x) - d_{\operatorname{gph}T}(x,y)|\nabla[h(\cdot,y)]^{1-\gamma}|(x)\right]\right\} \quad (3.23)$$

is strict positive.

Proof. The proof is in a similar strategy as the one in Theorem 3.1, that can be traced back to Proposition 3.2. Assume that T is not (γ,h,μ)-MPR in the direction (u,v). Due to the definition, we can find out for each k a pair of positive constants $\alpha_k, \beta_k \leqslant 1/k$ and some $(x_k,y_k) \in V_{\alpha_k,\beta_k}(u,v)$ such that

$$d\big(y_k, T(x_k)\big) \leqslant \beta_k[\mu(x_k,y_k)]^{\gamma}$$

and

$$d\big(y_k, T(x_k)\big) < \frac{1}{k} d\big(x_k, T^{-1}(y_k)\big)[h(x_k,y_k)]^{\gamma-1}.$$

Following Proposition 3.2, we see that there exists some sequences $\tau'_k \downarrow 0$, $\alpha'_k \downarrow 0$, $\beta'_k \downarrow 0$, and $(x'_k,y'_k) \in V_{\alpha'_k,\beta'_k}(u,v) \setminus \operatorname{gph}T$ such that

- $\limsup_{k\to\infty} \frac{d_{\operatorname{gph}T}(x'_k,y'_k)}{[\mu(x'_k,y'_k)]^{\gamma}} = 0$;
- $|\nabla\psi_T(\cdot,y'_k)|(x'_k) \leqslant d_{\operatorname{gph}T}(x'_k,y'_k)|\nabla[h(\cdot,y'_k)]^{1-\gamma}|(x'_k) + 2\tau'_k[h(x'_k,y'_k)]^{1-\gamma}$.

Recall from the proof of Proposition 3.2 that $\|(x'_k,y'_k)\| \leqslant \alpha'_k$ and $(x'_k,y'_k) \notin \operatorname{gph}T$. Thus, it follows for large k that the quantity $|\nabla[h(\cdot,y'_k)]^{1-\gamma}|(x'_k)$ is finite. Consequently, we obtain

$$[h(x'_k,y'_k)]^{\gamma-1}\left[|\nabla\psi_T(\cdot,y'_k)|(x'_k) - d_{\operatorname{gph}T}(x'_k,y'_k)|\nabla[h(\cdot,y'_k)]^{1-\gamma}|(x'_k)\right] \leqslant 2\tau'_k. \quad (3.24)$$

Passing to the limit in (3.24), the quantity

$$\liminf_{k\to\infty}\left\{[h(x'_k,y'_k)]^{\gamma-1}\left[|\nabla\psi_T(\cdot,y'_k)|(x'_k) - d_{\operatorname{gph}T}(x'_k,y'_k)|\nabla[h(\cdot,y'_k)]^{1-\gamma}|(x'_k)\right]\right\}$$

is evidently non-positive. This contradicts the positivity of the limit in (3.23). □

Remark 3.3. It is obvious that the finiteness of quantity $|\nabla[h(\cdot,y)]^{1-\gamma}|(x)$ for (x,y) is outside $\operatorname{gph} T$ but nearby $(0,0)$ plays a crucial role for the validity of the test given in Theorem 3.2. Such a condition is of course trivial if $h(x,y)=\|x-\bar{x}\|$ as in Definition 3.1. More general, it should be valid when h is locally Lipschitz around the reference point w.r.t the first variable x uniformly in the others.

4. Infinitesimal Characterization for Directional Metric Pseudo-Regularity

The present section is devoted to some infinitesimal criteria for the directional (γ,h,μ)-MPR in Definition 3.2. We assume hereinafter that, all Banach spaces under consideration, particularly, X and Y_i, are smooth whenever being necessary. This supposition is aimed to use the smooth variational principle and its consequent facts; see [3]. As mentioned in [3], the preceding assumption may be widely available such as in the class of reflexive Banach spaces. Let us recall that [12] the ε-normal cone $\widehat{N}^{\varepsilon}(\Omega;(x,y))$ to a closed subset Ω of $X\times Y$ at some point $(x,y)\in\Omega$ is formed by the collection of all elements $(u^\star,v^\star)\in X^*\times Y^*$ such that

$$\limsup_{\substack{\|(u,v)\|\downarrow 0,\\ (x,y)+(u,v)\in\Omega}} \left\{\frac{\langle (u^\star,v^\star),(u,v)\rangle}{\|(u,v)\|}\right\} \leqslant \varepsilon. \tag{4.1}$$

Replacing ε with 0 in (4.1), one obtains the corresponding Fréchet normal cone $\widehat{N}(\Omega;(x,y))$. The Mordukhovich limiting normal cone $N(\Omega;(x,y))$ to Ω at $(x,y)\in\Omega$ is defined as follows. For a pair $(u^\star,v^\star)\in X^*\times Y^*$, $(u^\star,v^\star)$ belongs to $N(\Omega;(x,y))$ if there are a sequence $(x_k,y_k)\to(x,y)$ with $(x_k,y_k)\in\Omega$ and a scalar one $\varepsilon_k\downarrow 0$ satisfying the statement

$$\text{for each } k \text{ it has } (u_k^\star,v_k^\star)\in\widehat{N}^{\varepsilon_k}(\Omega;(x_k,y_k)) \text{ such that } (u_k^\star,v_k^\star)\xrightarrow{w^*}(u^\star,v^\star).$$

Here and in what follows, the notation $w_k\xrightarrow{w^*}w$ means that the sequence w_k converges to w in the weak-star topology of the corresponding space. Finally, the coderivative to $T: X\longrightarrow Y$ at reference point $(x,y)\in\operatorname{gph} T$ is the mapping $D^*T(x,y): Y^*\rightrightarrows X^*$ given by

$$x^\star\in D^*T(x,y)(y^\star)\Longleftrightarrow (x^\star,-y^\star)\in N(\operatorname{gph} T;(x,y)).$$

Note that the limiting normal cone is corresponding to the (basic/limiting) subdifferential, which satisfies our preceding conditions (C1)–(C5). We refer to the monographs [12, 13] for further discussions about the generalized differentiation w.r.t. set-valued maps and their applications. In this paper, we make use of coderivative to study the infinitesimal characterization of the property stated before in Definition 3.2.

Proposition 4.1. *Let γ, μ, h, and T be as in Theorem 3.1 and suppose further that h is locally Lipschitz w.r.t. x uniformly in y near $(\bar{x},\bar{y})=(0,0)$. If T is failed to satisfy Definition 3.2, then there exist some sequences $t_k\downarrow 0$, $(u_k,v_k)\in X\times Y$, and $(u_k^\star,v_k^\star)\in X^*\times Y^*$ such that*

(1) $\|(u_k,v_k)\|=1$, $\lim\limits_{k\to\infty}\big\|\|(u,v)\|(u_k,v_k)-(u,v)\big\|=0$;
(2) $(u_k^\star,-h_1(t_ku_k,t_kv_k)^{-\hat{\gamma}_1}v_{k1}^\star,\dots,-h_m(t_ku_k,t_kv_k)^{-\hat{\gamma}_m}v_{km}^\star)$ *is an element of* $\widehat{N}(\operatorname{gph} T,(t_ku_k,t_kv_k))$;
(3) $\lim_{k\to\infty}\|u_k^\star\|=0$;
(4) $\lim_{k\to\infty}\max_{i=1,\dots,m}\|v_{ki}^\star\|=1$.

Proof. Suppose that the product spaces $Y = Y_1 \times \cdots \times Y_m$ and $X \times Y$ is endowed with the maximum norm

$$\|(x,y)\| = \max\{\|x\|, \|y\|\} = \max\left\{\|x\|, \max_{i=1,\dots,m} \|y_i\|\right\}$$

for $x \in X$, $y = (y_1, \dots, y_m) \in Y$. By the assumption, we find out some positive real sequences $\alpha_k \downarrow 0$, $\beta_k \downarrow 0$, $\tau_k \downarrow 0$ together with the elements $(x_k, y_k) \in X \times Y$ such that

$$\begin{aligned} d\big(y_{ki}, T(x_k)\big) \leqslant \beta_k[\mu(x_k,y_k)]^{\gamma_i}, \\ \sum_{h_i(x_k,y_k)>0} \frac{d\big(y_{ki}, T(x_k)\big)}{[h_i(x_k,y_k)]^{\hat{\gamma}_i}} < \tau_k d\big(x_k, T^{-1}(y_k)\big), \end{aligned} \tag{4.2}$$

where $\hat{\gamma}_i := \gamma_i - 1 \geqslant 0$ for $i = 1, \dots, m$. Note from (4.2) that none of elements (x_k, y_k) is zero to μ or h_i. Let

$$\eta_{ki} := \Big(1 + \frac{1}{k^2}\Big)\beta_k[\mu(x_k,y_k)]^{\gamma_i}, \quad \eta_k := \min_{i=1,\dots,m} \eta_{ki}$$

we select $\hat{y}_{ki} \in T_i(x_k)$ such that $\|y_{ki} - \hat{y}_{ki}\| < \eta_k$ and

$$\sum_{h_i(x_k,y_k)>0} \frac{\|y_{ki} - \hat{y}_{ki}\|}{[h_i(x_k,y_k)]^{\hat{\gamma}_i}} < \tau_k d\big(x_k, T^{-1}(y_k)\big).$$

We next define a lsc function by the formulae

$$f_k(x,y) := \sum \frac{\|y_i - y_{ki}\|}{[h_i(x,y_k)]^{\hat{\gamma}_i}} + \delta_{\operatorname{gph} T}(x,y)$$

where the convention $0/0 = 0$ is used. It is straightforward to see that, for

$$\varepsilon_k := \min\left\{\tau_k d\big(x_k, T^{-1}(y_k)\big), \Big(1 + \frac{1}{k^2}\Big)\beta_k \sum_{h_i(x_k,y_k)>0} \frac{[\mu(x_k,y_k)]^{\gamma_i}}{[h_i(x_k,y_k)]^{\hat{\gamma}_i}}\right\},$$

there holds $f_k(x_k, \hat{y}_k) < \varepsilon_k$. To proceed, we invoke Ekeland's variational principle (see, e.g., [4]) for the lsc f_k and obtain some $(x'_k, y'_k) \in X \times Y$ such that

- $\|(x'_k, y'_k) - (x_k, \hat{y}_k)\| \leqslant \lambda_k := \varepsilon_k / \sqrt{\tau_k}$;
- $f_k(x'_k, y'_k) \leqslant f_k(x_k, \hat{y}_k) - \varepsilon_k/\lambda_k \|(x'_k, y'_k) - (x_k, \hat{y}_k)\|$;
- the function $(x,y) \longmapsto f_k(x,y) + \varepsilon_k/\lambda_k \|(x,y) - (x'_k, y'_k)\|$ admits (x'_k, y'_k) as a global minimum.

The minimal property to (x'_k, y'_k) yields $(x'_k, y'_k) \in \operatorname{gph} T$. Note from the definition of ε_k that

$$d\big(x_k, T^{-1}(y_k)\big) > \frac{1}{\tau_k}\varepsilon_k = \frac{1}{\sqrt{\tau_k}}\lambda_k \geqslant \frac{1}{\sqrt{\tau_k}}\|(x'_k, y'_k) - (x_k, \hat{y}_k)\| \geqslant \frac{1}{\sqrt{\tau_k}}\|x'_k - x_k\|.$$

Therefore, the Lipschitz continuity of the distance $d\big(\cdot, T^{-1}(y_k)\big)$ yields

$$d\big(x'_k, T^{-1}(y_k)\big) \geqslant d\big(x_k, T^{-1}(y_k)\big) - \|x'_k - x_k\| > \left(\frac{1}{\sqrt{\tau_k}} - 1\right)\|x'_k - x_k\|. \tag{4.3}$$

In particular, (4.3) implies $y_k \notin T(x'_k)$. Thus $y'_k \neq y_k$. We define some positive quantities

$$r_k := \min\left\{\varepsilon_k, \alpha_k \min_{i=1,\dots,m} h_i(x_k,y_k)^{\gamma_i}, \beta_k \min_{i=1,\dots,m} \|\hat{y}_{ki} - y_{ki}\|, \alpha_k \min_{y'_{ki} \neq y_{ki}} \|y'_{ki} - y_{ki}\|\right\},$$

$$\rho_k := \tau_k,$$

and make use of the local fuzzy sum rule [3]. As a consequence, it is possible to find out $(x_k^{1i},y_k^{1i}),(x_k^2,y_k^2),(x_k^3,y_k^3)$ in the ball $\mathbb{B}_{X\times Y}((x_k',y_k'),r_k)$ together with the elements $(x_k^{1i\star},y_k^{1i\star})$, $(x_k^{2\star},y_k^{2\star})$, $(x_k^{3\star},y_k^{3\star})$ to $X^*\times Y^*$ fulfilling the following conditions

$$\begin{gathered}(x_k^{1i\star},y_k^{1i\star})\in\partial_F f_{ki}(x_k^{1i},y_k^{1i})\text{ for } f_{ki}(x,y)=\|y_i-y_{ki}\|[h_i(x,y_k)]^{-\hat{\gamma}_i};\\ (x_k^{2\star},y_k^{2\star})\in\partial_F\delta_{\operatorname{gph}T}(x_k^2,y_k^2);\\ (x_k^{3\star},y_k^{3\star})\in\partial_F\left(\|(\cdot,\cdot)-(x_k',y_k')\|\right)(x_k^3,y_k^3);\\ \operatorname{diam}\left\{(x_k^{11},y_k^{11}),\ldots,(x_k^{1m},y_k^{1m}),(x_k^2,y_k^2),(x_k^3,y_k^3)\right\}\times\\ \times\max\left\{\|(\tilde{x}_k^{11\star},\tilde{y}_k^{11\star})\|,\ldots,\|(x_k^{1m\star},y_k^{1m\star})\|,\|(x_k^{2\star},y_k^{2\star})\|,\tfrac{\varepsilon_k}{\lambda_k}\|(x_k^{3\star},y_k^{3\star})\|\right\}\leqslant\rho_k;\\ \left\|(x_k^{11\star},y_k^{11\star})+\cdots+(x_k^{1m\star},y_k^{1m\star})+(x_k^{2\star},y_k^{2\star})+\frac{\varepsilon_k}{\lambda_k}(x_k^{3\star},y_k^{3\star})\right\|\leqslant\rho_k.\end{gathered}$$

In view of $(x_k^{2\star},y_k^{2\star})\in\partial_F\delta_{\operatorname{gph}T}(x_k^2,y_k^2)$ we can write $(x_k^{2\star},y_k^{2\star})\in\widehat{N}(\operatorname{gph}T,(x_k^2,y_k^2))$. □

Next, we need a few auxiliary facts.

Lemma 4.1. *The following limit*

$$\lim_{k\to\infty}\frac{\|(x_k^{1i},y_k^{1i})\|}{\|(x_k,y_k)\|}=1 \tag{4.5}$$

does hold for every $i=1,\ldots,m$. The similar limits

$$\lim_{k\to\infty}\frac{\|(x_k^2,y_k^2)\|}{\|(x_k,y_k)\|}=\lim_{k\to\infty}\frac{\|(x_k^3,y_k^3)\|}{\|(x_k,y_k)\|}=1 \tag{4.6}$$

are also valid as well. Further,

$$\lim_{k\to\infty}\frac{h_i(x_k^{1i},y_k)}{h_i(x_k,y_k)}=1\quad\textit{for all } i=1,\ldots,m. \tag{4.7}$$

Proof of Lemma 4.1. To the goal of proving (4.5) and (4.6), let us first find an upper bound for $\|(x_k',y_k')-(x_k,y_k)\|$ that

$$\|(x_k',y_k')-(x_k,y_k)\|\leqslant\|(x_k',y_k')-(x_k,\hat{y}_k)\|+\|\hat{y}_k-y_k\|\leqslant\lambda_k+\eta_k. \tag{4.8}$$

(Recall that the norm on Y is supposed to be the maximum of the norms on its components Y_i.) Repeating the analogous arguments as in proof of Proposition 3.1, we have (see (3.10))

$$\limsup_{k\to\infty}\frac{\lambda_k\sqrt{\tau_k}}{\beta_k\|(x_k,y_k)\|}<+\infty.$$

The latter implies $\lambda_k=o(\|(x_k,y_k)\|)$ as $k\to\infty$ by means of $\limsup_{k\to\infty}(\beta_k/\tau_k)<+\infty$. According to the Lipschitz continuity of μ, it holds, for $i=1,\ldots,m$,

$$\limsup_{k\to\infty}\frac{\eta_{ki}}{\beta_k\|(x_k,y_k)\|^{\gamma_i}}<+\infty,$$

which particularly gives us $\eta_k=o(\|(x_k,y_k)\|)$ as $k\to\infty$. The latter implies that

$$\lim_{n\to\infty}\frac{\|(x_k',y_k')-(x_k,y_k)\|}{\|(x_k,y_k)\|}=0.$$

Hence, we deduce

$$\lim_{n\to\infty}\frac{\|(x'_k,y'_k)\|}{\|(x_k,y_k)\|}=1. \tag{4.9}$$

Observe that $r_k=o(\|(x_k,y_k)\|)$ by the definition of r_k. Consequently, (4.5) and (4.6) are derived in the same way as (4.9), since the ball $\mathbb{B}_{X\times Y}((x'_k,y'_k),r_k)$ consists of $(x_k^{1i},y_k^{1i}),(x_k^2,y_k^2),(x_k^3,y_k^3)$.

Finally, it is sufficient to check (4.7) for $i=1$ only. Invoking the supposition concerning Lipschitz continuity to h w.r.t. the first variable x we can prove that

$$\limsup_{k\to\infty}\frac{|h_i(x_k^{1i},y_k)-h_i(x_k,y_k)|}{\|x_k^{1i}-x_k\|}=\limsup_{k\to\infty}\frac{|h_i(x_k^{1i},y_k)-h_i(x_k,y_k)|}{\|(x_k^{1i},y_k)-(x_k,y_k)\|}<+\infty. \tag{4.10}$$

By the choice of (x_k^{1i},y_k^{1i}), we have

$$\|x_k^{1i}-x'_k\|\leqslant\|x_k^{1i}-x'_k\|\leqslant\|(x_k^{1i},y_k^{1i})-(x'_k,y'_k)\|\leqslant r_k\leqslant\alpha_k[h_i(x_k,y_k)]^{\gamma_i}.$$

On the other hand, since $\|(x'_k,y'_k)-(x_k,\hat{y}_k)\|\leqslant\lambda_k$, we obtain

$$\begin{aligned}\|x'_k-x_k\|&\leqslant\|(x'_k,y'_k)-(x_k,\hat{y}_k)\|\\&\leqslant\lambda_k=\frac{\varepsilon_k}{\sqrt{\tau_k}}\\&\leqslant\Big(1+\frac{1}{k^2}\Big)\beta_k\sum_{h_i(x_k,y_k)>0}\frac{\mu(x_k,y_k)^{\gamma_i}}{[h_i(x_k,y_{ki})]^{\hat{\gamma}_i}}\\&\leqslant\Big(1+\frac{1}{k^2}\Big)\frac{\beta_k}{\sqrt{\tau_k}}\mu(x_k,y_k)\sum_{h_i(x_k,y_k)>0}\left(\frac{\mu(x_k,y_k)}{h_i(x_k,y_{ki})}\right)^{\hat{\gamma}_i}.\end{aligned}$$

Taking into account assumption (3.2b) and (3.3d), the limit $\lim_{k\to\infty}\frac{\|x'_k-x_k\|}{h_i(x_k,y_k)}=0$ hold. Thus, the aforementioned estimate $\|x_k^{1i}-x'_k\|\leqslant\alpha_k[h_i(x_k,y_k)]^{\gamma_i}$ shows that

$$\lim_{k\to\infty}\frac{\|x_k^{1i}-x_k\|}{h_i(x_k,y_k)}=0. \tag{4.11}$$

Summarily, the limit in (4.7) is derived from (4.10) and (4.11).

Lemma 4.2. *Let* $t_k=\|(x_k^2,y_k^2)\|>0$ *and define* $u_k=(1/t_k)x_k^2$, $v_{ki}=(1/t_k)y_{ki}^2$ *for* $i=1,\dots,m$. *Then* $\|(u_k,v_k)\|=1$ *and*

$$\lim_{k\to\infty}\|\|(u,v)\|(u_k,v_k)-(u,v)\|=0. \tag{4.12}$$

Proof of Lemma 4.2. The fact that $t_k>0$ is due to (4.6), since $(x_k,y_k)\notin\operatorname{gph}T$. The equality $\|(u_k,v_k)\|=1$ is trivial by the definition of u_k and v_k. To establish (4.12), we shall evaluate the magnitude of the norm $\|(u'_k,v'_k)\|$ for $(u'_k,v'_k)=\|(u,v)\|(u_k,v_k)-(u,v)$. Indeed, letting $(\tilde{u}_k,\tilde{v}_k):=\|(u,v)\|(u_k,v_k)-\|(u_k,v_k)\|(u,v)$, we have

$$\begin{aligned}t_k(u'_k,v'_k)&=\|(u,v)\|(x_k^2,y_k^2)-\|(x_k^2,y_k^2)\|(u,v)\\&=(\tilde{u}_k,\tilde{v}_k)+\|(u,v)\|\big[(x_k^2,y_k^2)-(x_k,y_k)\big]+\big[-\|(x_k^2,y_k^2)\|+\|(x_k,y_k)\|\big](u,v).\end{aligned} \tag{4.13}$$

In view of $(x_k,y_k)\in V_{\alpha_k,\beta_k}(u,v)$, we have $\|(\tilde{u}_k,\tilde{v}_k)\|\leqslant\beta_k\|(x_k,y_k)\|\|(u,v)\|$.

On the other hand, the estimation $\|(x'_k,y'_k)-(x_k,y_k)\|\leqslant\lambda_k+\eta_k$ follows from (4.8). Thus we deduce

$$\|(x_k^2,y_k^2)-(x_k,y_k)\|\leqslant\lambda_k+\eta_k+r_k=o(\|(x_k,y_k)\|). \tag{4.14}$$

Combining (4.13) with (4.14), we reach the estimate

$$t_k\|(u_k',v_k')\| \leqslant \big[\beta_k\|(x_k,y_k)\|+2(\lambda_k+\eta_k+r_k)\big]\|(u,v)\|.$$

This provides the desired conclusion, since $t_k=\|(x_k^2,y_k^2)\|\sim\|(x_k,y_k)\|$.

Lemma 4.3. *Define* $u_k^\star:=x_k^{2\star}$, $v_{ki}^\star:=-h_i(t_ku_k,t_kv_k)^{\hat{\gamma}_i}y_{ki}^{2\star}$, *and* $\tilde{v}_{ki}^\star:=h_i(t_ku_k,t_kv_k)^{-\hat{\gamma}_i}v_{ki}^\star$, $i=1,\dots,m$. *Then*

$$(u_k^\star,-\tilde{v}_{k1}^\star,\dots,-\tilde{v}_{km}^\star)\in\widehat{N}(\operatorname{gph}T,(t_ku_k,t_kv_k)); \tag{4.15a}$$

$$\lim_{k\to\infty}\|u_k^\star\|=0, i=1,\dots,m; \tag{4.15b}$$

$$\lim_{k\to\infty}\max_{i=1,\dots,m}\|v_{ki}^\star\|=1, i=1,\dots,m. \tag{4.15c}$$

Proof of Lemma 4.3. Inclusion (4.15a) is clear. We continue to establish (4.15b). Recall from the choice of $(x_k^{1i\star},y_k^{1i\star})$, $(x_k^{2\star},y_k^{2\star})$ and $(x_k^{3\star},y_k^{3\star})$ that

$$\left\|\sum_{i=1}^m(x_k^{1i\star},y_k^{1i\star})+(x_k^{2\star},y_k^{2\star})+\frac{\varepsilon_k}{\lambda_k}(x_k^{3\star},y_k^{3\star})\right\|\leqslant\rho_k. \tag{4.16}$$

Observe that $(x_k^{3\star},y_k^{3\star})\in\partial_F(\|(\cdot,\cdot)-(x_k',y_k')\|)(x_k^3,y_k^3)$, $(x_k^{3\star},y_k^{3\star})$ satisfies

$$\|(x_k^{3\star},y_k^{3\star})\|\leqslant 1, \langle(x_k^{3\star},y_k^{3\star}),(x_k^3,y_k^3)-(x_k',y_k')\rangle=\|(x_k^3,y_k^3)-(x_k',y_k')\|. \tag{4.17}$$

The remainder is to deal with $(x_k^{1i\star},y_k^{1i\star})\in\partial_F f_{ki}(x_k^{1i},y_k^{1i})$. Applying the product rule to the function $f_{ki}=\|\cdot-y_{ki}\|[h_i(\cdot,y_k)]^{-\hat{\gamma}_i}$, we obtain $(\hat{x}_k^{1i},\hat{y}_k^{1i}),(\tilde{x}_k^{1i},\tilde{y}_k^{1i})\in\mathbb{B}_{X\times Y}((x_k^{1i},y_k^{1i}),r_k)$, which together with $(0,\hat{y}_k^{1i\star})\in\partial_F(\|\cdot-y_{ki}\|)(\hat{x}_k^{1i},\hat{y}_k^{1i})$ and $(\tilde{x}_k^{1i\star},0)\in\partial_F(h_i(\cdot,y_k)^{-\hat{\gamma}_i})(\tilde{x}_k^{1i},\tilde{y}_k^{1i})$ yields

$$\left\|(x_k^{1i\star},y_k^{1i\star})-h_i(x_k^{1i},y_k)^{-\hat{\gamma}_i}(0,\hat{y}_k^{1i\star})-\|y_{ki}^{1i}-y_{ki}\|(\tilde{x}_k^{1i\star},0)\right\|\leqslant\rho_k. \tag{4.18}$$

In particular, we have

$$\|x_k^{1i\star}\|\leqslant\|y_{ki}^{1i}-y_{ki}\|\|\tilde{x}_k^{1i\star}\|+\rho_k. \tag{4.19}$$

Taking the fact $(\tilde{x}_k^{1i\star},0)\in\partial_F(h_i(\cdot,y_k)^{-\hat{\gamma}_i})(\tilde{x}_k^{1i},\tilde{y}_k^{1i})$ into account, the chain rule can be used here to derive

$$\tilde{x}_k^{1i\star}=-\hat{\gamma}_ih_i(\check{x}_k^{1i},y_k)^{-\hat{\gamma}_i-1}\check{x}_k^{1i\star}=-\hat{\gamma}_ih_i(\check{x}_k^{1i},y_k)^{-\gamma_i}\check{x}_k^{1i\star}, \tag{4.20}$$

for some $\check{x}_k^{1i}\in\mathbb{B}_X(x_k^{1i},r_k)$ and $\check{x}_k^{1i\star}\in\partial_F(h_i(\cdot,y_k))(\check{x}_k^{1i})\subset X^*$. Note from the Lipschitz continuity of h_i that $(\check{x}_k^{1i\star})$ is bounded. Observe that $\|\check{x}_k^{1i}-x_k^{1i}\|\leqslant r_k$. Repeating the arguments in Lemma 4.1 and then invoking (4.7), we can write

$$\lim_{k\to\infty}\frac{h_i(\check{x}_k^{1i},y_k)}{h_i(x_k,y_k)}=1. \tag{4.21}$$

We know from the choice of (x_k^{1i},y_k^{1i}) that $(x_k^{1i},y_k^{1i})\in\mathbb{B}_{X\times Y}((x_k',y_k'),r_k)$, and hence

$$\begin{aligned}\|y_{ki}^{1i}-y_{ki}\|&\leqslant\|y_k^{1i}-y_{ki}'\|+\|y_{ki}'-y_{ki}\|\\&\leqslant\|(x_k^{1i},y_k^{1i})-(x_k',y_k')\|+\|y_{ki}'-y_{ki}\|\\&\leqslant r_k+\|y_{ki}'-y_{ki}\|\end{aligned}$$

for every i. On account of $f_k(x'_k, y'_k) \leqslant f_k(x_k, \hat{y}_k)$, we deduce

$$\begin{aligned}\|y'_{ki} - y_{ki}\| &= h_i(x'_k, y'_k)^{\hat{\gamma}_i} \frac{\|y'_{ki} - y_{ki}\|}{h_i(x'_k, y'_k)^{\hat{\gamma}_i}} \\ &\leqslant h_i(x'_k, y'_k)^{\hat{\gamma}_i} f_k(x'_k, y'_k) \leqslant h_i(x'_k, y'_k)^{\hat{\gamma}_i} \varepsilon_k \\ &\leqslant \Big(1 + \frac{1}{k^2}\Big) \beta_k h_i(x'_k, y'_k)^{\hat{\gamma}_i} \mu(x_k, y_k) \sum \left(\frac{\mu(x_k, y_k)}{h_i(x'_k, y'_k)^{\hat{\gamma}_i}} \right)^{\hat{\gamma}_i}.\end{aligned} \tag{4.22}$$

Involving the inclusion $(x_k, y_k) \in V_{\alpha_k, \beta_k}(u, v)$, assumption

$$\limsup_{\alpha \downarrow 0, \beta \downarrow 0} \left\{ \sup_{(x,y) \in V_{\alpha,\beta}(u,v)} \frac{\mu(x,y)}{h_i(x,y)} \right\} < +\infty$$

gives us

$$\limsup_{k \to \infty} \frac{\|y'_{ki} - y_{ki}\|}{h_i(x_k, y_k)^{\gamma_i}} = 0, \quad \text{for } i = 1, \ldots, m. \tag{4.23}$$

Combining (4.22), (4.23) and the fact that $r_k = o\big(h_i(x_k, y_k)^{\gamma_i}\big)$, we conclude that

$$\limsup_{k \to \infty} \frac{\|y^{1i}_{ki} - y_{ki}\|}{h_i(x_k, y_k)^{\gamma_i}} = 0, \quad \text{for } i = 1, \ldots, m.$$

In view of Lemma 4.1 and (4.21), we find $\limsup_{k \to \infty} \frac{\|y^{1i}_{ki} - y_{ki}\|}{h_i(x_k, y_k)^{\gamma_i}} = 0$, which follows from (4.19) and (4.20) that $\|x^{1i\star}_k\| \to 0$ as $k \to \infty$. Finally, the limit (4.15b) is inferred from (4.16) and (4.17), since $\rho_k \to 0$ and $\varepsilon_k/\lambda_k = \sqrt{\tau_k} \to 0$. The proof of (4.15c) is almost similar. Recall first the relation below (see (4.18))

$$\big\|(x^{1i\star}_k, y^{1i\star}_k) - h_i(x^{1i}_k, y_k)^{-\hat{\gamma}_i}(0, \hat{y}^{1i\star}_k) - \|y^{1i}_{ki} - y_{ki}\|(\tilde{x}^{1i\star}_k, 0)\big\| \leqslant \rho_k,$$

that

$$\big\|h_i(x^{1i}_k, y_k)^{\hat{\gamma}_i} y^{1i\star}_k - \hat{y}^{1i\star}_k\big\| \leqslant h_i(x^{1i}_k, y_k)^{\hat{\gamma}_i} \rho_k \xrightarrow{k \to \infty} 0. \tag{4.24}$$

By the selection of $\hat{y}^{1i\star}_{ki}$, it follows that $\hat{y}^{1i\star}_{ki} \in \partial_F(\|\cdot - y_{ki}\|)(\hat{y}^{1i}_{ki})$. As a result, we deduce

$$\|\hat{y}^{1i\star}_{ki}\| \leqslant 1, \langle \hat{y}^{1i\star}_{ki}, \hat{y}^{1i}_{ki} - y_{ki} \rangle = \|\hat{y}^{1i}_{ki} - y_{ki}\|; i = 1, \ldots, m. \tag{4.25}$$

We shall justify that there is at least one i for which $\hat{y}^{1i}_{ki} - y_{ki} \neq 0$. Indeed, taking an index i that $y'_{ki} \neq y_{ki}$. (Such an index exists because of $y'_k \neq y_k$.) By the choice of y^{1i}_k we have

$$\begin{aligned}\|\hat{y}^{1i}_k - y'_{ki}\| &\leqslant \|(\hat{x}^{1i}_k, \hat{y}^{1i}_k) - (x'_k, y'_k)\| \\ &\leqslant \|(\hat{x}^{1i}_k, \hat{y}^{1i}_k) - (x^{1i}_k, y^{1i}_k)\| + \|(x^{1i}_k, y^{1i}_k) - (x'_k, y'_k)\| \\ &\leqslant 2r_k,\end{aligned}$$

and hence

$$\|\hat{y}^{1i}_k - y_{ki}\| \geqslant \big|\|\hat{y}^{1i}_k - y'_{ki}\| - \|y'_{ki} - y_{ki}\|\big| \geqslant \Big(1 - \frac{2r_k}{\|y'_{ki} - y_{ki}\|}\Big) \|y'_{ki} - y_{ki}\|.$$

Taking into account $r_k \leqslant \alpha_k \|y'_{ki} - y_{ki}\|$, the latter immediately implies $\|\hat{y}^{1i}_k - y_{ki}\| > 0$. This is our desired conclusion. According to $\hat{y}^{1i}_k \neq y_{ki}$, we deduce from (4.25) that $\hat{y}^{1i\star}_{ki} \in Y^*_i$ is actually a unit vector. Combining (4.17) with (4.24) and passing to the limit in (4.16), we reach (4.15c). The proof is done.

Back to our main process, we obtain the full demonstration for Proposition 4.1 after involving Lemma 4.1, Lemma 4.2, and Lemma 4.3 above. Proposition 4.1 could be viewed as a based-point necessary condition for the failure of (γ,h,μ)-MPR. Inspiring from such condition, let us now introduce a useful notion, which is crucial for the criterion later.

Definition 4.1 (Based point (γ,h,μ)-critical set)**.** For a triplet (γ,h,μ), and the mapping T, we define the (γ,h,μ)-critical set $SCr_{\gamma,h,\mu}T(\bar{x},\bar{y})(u,v) \subset X^* \times Y^*$ w.r.t T and a pair $(\bar{x},\bar{y}) \in \operatorname{gph} T$ relative to the direction $(u,v) \in X \times Y$ as follows. We have $(x^\star,y^\star)$ belonging to $SCr_{\gamma,h,\mu}T(\bar{x},\bar{y})$ if it is possible to find out some sequences $t_k \downarrow 0$, $(x_k,y_k) \in \operatorname{gph} T$ and $(x_k^\star,y_k^*) \in X^* \times Y^*$ fulfilling the following conditions:

(1) for $u_k := \frac{1}{t_k}(x_k-\bar{x})$ and $v_k := \frac{1}{t_k}(y_k-\bar{y})$, one simultaneously has $\|(u_k,v_k)\| = 1$ as well as $\lim_{k\to\infty} \|\|(u,v)\|(u_k,v_k)-(u,v)\| = 0$;

(2) $(x_k^\star, -h_1(x_k)^{1-\gamma_1}y_{k1}^\star, \ldots, -h_m(x_k)^{1-\gamma_m}y_{km}^\star) \in \widehat{N}(\operatorname{gph} T,(x_k,y_k))$;

(3) $y_k^\star \xrightarrow{w^*} y^\star$ and $x_k^\star \xrightarrow{X^*} x^\star$.

Making use of the preceding notion, we reach the first result characterizing the MPR property by involving Proposition 4.1.

Theorem 4.1 ((γ,h,μ)-MPR via (γ,h,μ)-critical set)**.** *Let (γ,h,μ) be in the statement of Proposition 4.1. Then T is (γ,h,μ)-MPR at $(\bar{x},\bar{y}) = (0,0) \in \operatorname{gph} T$ in the direction (u,v) if*

$$SCr_{\gamma,h,\mu}T(\bar{x},\bar{y})(u,v) \cap \left(\{0\} \times Y^*\right) = \{(0,0)\}. \tag{4.26}$$

Proof. We mimic the strategy of Proposition 4.1. Assume on the contrary that T fails to be (γ,h,μ)-MPR at $(0,0) \in \operatorname{gph} T$ in the direction (u,v). Following Proposition 4.1, we obtain some sequences $t_k \downarrow 0$ and $(u_k,v_k) \in X \times Y$ as well as $(u_k^\star,v_k^\star) \in X^* \times Y^*$ such that

(1) $\|(u_k,v_k)\| = 1$, $\lim_{k\to\infty} \|\|(u,v)\|(u_k,v_k)-(u,v)\| = 0$;

(2) the element $(u_k^\star, -h_1(t_ku_k,t_kv_k)^{-\hat{\gamma}_1}v_{k1}^\star, \ldots, -h_m(t_ku_k,t_kv_k)^{-\hat{\gamma}_m}v_{km}^\star)$ belongs to the normal cone $\widehat{N}(\operatorname{gph} T,(t_ku_k,t_kv_k))$;

(3) $\lim_{k\to\infty} \|u_k^\star\| = 0$;

(4) $\lim_{k\to\infty} \max_{i=1,\ldots,m} \|v_{ki}^\star\| = 1$.

Denote $\sigma_k := \|v_k^\star\|$. In view of (4), we have $\sigma_k > 0$ unless several indexes. Let us now define $x_k := t_ku_k$ $y_{ki} := t_kv_{ki}$, $x_k^\star := \frac{1}{\sigma_k}u_k^\star$, and $y_{ki}^\star := \frac{1}{\sigma_k}v_{ki}^\star$, $i = 1,\ldots,m$. Since $y_k^\star$ has unit norm, then sequence $(y_k^\star)$ converges in the weak-star topology of Y^* to some unit vector $y^\star$. Then it is evident to check according to Definition 4.1 that

$$(0,y^\star) \in SCr_{\gamma,h,\mu}T(0,0)(u,v), \tag{4.27}$$

which is a contradiction. This completes our proof. □

Remark 4.1. As a particular case, let us suppose now $h_i(x,y) = \|x-\bar{x}\|$ and $\mu(x,y) = \|x-\bar{x}\|$. Then it is straightforward to see that for $\gamma_i \geqslant 1$ the triplet (γ,h,μ) satisfies the conditions of Theorem 4.1. Thus, the infinitesimal test (4.26) subsumes the one studied in the aforementioned work [9]. Especially, when X is finitely dimensional, if we take $\gamma_i = 1$ and $(u,v) = (0,0)$, (4.26) recovers the well-known Mordukhovich criterion for metric regularity in [12].

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