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AN INTERIOR-POINT ALGORITHM FOR LINEAR OPTIMIZATION BASED ON A NEW ALGEBRAICALLY EQUIVALENT TRANSFORMATION

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Abstract. In this paper, we presented a full-Newton short-step interior-point algorithm, which is based on a new algebraically equivalent transformation technique, for a linear optimization problem. This technique offers a new type of Newton search direction and a proximity measure for tracing approximately the central-path of the linear optimization problem. We prove, under new defaults, that our algorithm is well-defined and converges to an optimal solution of the linear optimization problem. Moreover, we obtained its currently best known iteration bound, and we presented some numerical results for its evaluation.
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1. INTRODUCTION

Consider the primal linear optimization (LO) problem in its standard format:

$$\min_{x} \left\{ c^T x : Ax = b, x \geq 0 \right\} \tag{$\mathscr{P}$}$$

and its dual

$$\max_{(y,z)} \left\{ b^T y : A^T y + z = c, z \geq 0 \right\}, \tag{$\mathscr{D}$}$$

where $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = m \leq n$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

In the last decades, primal-dual path-following algorithms have been proven to be one of the most effective classes of interior-point methods (IPMs) for LO problems. These algorithms deserve the best known polynomial complexity and work well when the problem size of LO is very large. We refer to [18, 21] for the methodology of these methods.

Recently, researchers developed methods that are based on self-regular barrier and kernel functions. Their associated algorithms lead to the polynomial complexity of such large-update primal-dual IPMs; see, e.g., [3, 4, 6, 15]. Other methods are based on the applications of AET technique of the centering equation of the system which characterizes the central-path. This idea was first introduced by Darvay [8] where a new type of search directions was presented.

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Darvay analyzed a primal-dual full-Newton step algorithm for self-dual LO based on AET induced by the univariate function $\psi(t)=\sqrt{t}$, $t>0$ and obtained the currently best known iteration bound, namely, $\mathscr{O}(\sqrt{n}\log\frac{n}{\varepsilon})$. Later, Achache [1, 3], Wang and Bai [19, 20] successfully extended Darvay's algorithm to convex quadratic optimization (CQO), monotone linear complementarity problems (LCP), semidefinite optimization (SDO), and second-order cone optimization (SCOP). Recently, Haddou et al. [12] analyzed other AETs based on the class of concave smooth functions and derived the best known iteration bound for monotone LCP. The AET based algorithms have also been generalized to weighted primal-dual path-following methods by Darvay [7], Achache [2] for LO and monotone LCP, respectively. Meanwhile, Pan et al. [14] developed an infeasible large-step primal-dual IPM for LO based on the logarithmic function $\psi(t)=\log t$, $t>0$. Besides, Ahmadi et al. [5] presented an infeasible full-Newton step interior-point algorithm for LO based on Darvay's directions. Peng et al. [16] presented a large-update algorithm for LO based on kernel functions where a new search direction was determined. Very recently, in [11], Darvay and Takács proposed an IPM based on the following new univariate function $\psi(t)=t-\sqrt{t}, t>0$ for solving LO where the best polynomial complexity was achieved. By the same function, Darvay et al. [9] proposed a predictor-corrector interior-point algorithm for sufficient LCP. The polynomial complexity of this algorithm was obtained. Furthermore, Kheiferman and Haghighi [13] presented a feasible short-step algorithm for solving the $P_*(\kappa)$-LCP based on a new AET induced by the univariate function $\psi(t)=\frac{\sqrt{t}}{2(1+\sqrt{t})}, t>0$. The complexity of their algorithm deserves the best known complexity as for sufficient LCPs. To better understand the concept of AET based methods, we refer to the thesis [17] and the references therein.

In this paper, we use the AET technique for the centering equation by considering a new univariate function $\psi(t)=t^{\frac{3}{2}}, t>0$. Based on this function, a new type of Newton search direction is offered, which is totally different from those mentioned above. Under new defaults of τ, which defines the size of the neighborhood of the central-path and of θ which determines the rate of decrease of the barrier parameter, we show that the proposed algorithm is well-defined and converges to an optimal solution of LO. Moreover, we obtain the currently best known iteration bound for this algorithm with a short-update method, namely, $\mathscr{O}(\sqrt{n}\log\frac{n}{\varepsilon})$. Finally, some numerical tests on some different sizes of LO problems are provided to evaluate its efficiency.

The paper is organized as follows. In Section 2, we recall in brief the notion of central-path and classical search directions for LO. We present the new search directions for LO related to an algebraically equivalent transformation for the centering equation and the generic primal-dual algorithm. In Section 3, we give the detailed proof of the complexity of the algorithm. In Section 4, some numerical results are reported. In Section 5, the last section, a conclusion and future work are given.

Throughout the paper, the following notations are used. Let $u,v\in\mathbb{R}^n$, $u^Tv=\sum_{i=1}^n u_iv_i$ be their usual inner product. Besides, $\|u\|=\sqrt{u^Tu}$ and $\|u\|_\infty=\max_i|u_i|$ denote the 2-norm and the maximum norm of a vector u, respectively. Given two vectors x and z in $\mathbb{R}^n, xz=(x_iz_i)_{1\le i\le n}$ denotes their coordinate-wise product. Moreover, $\sqrt{x}=(\sqrt{x}_i)_{1\le i\le n}$ and $(\frac{x}{z})=(\frac{x_i}{z_i})_{1\le i\le n}, z_i\neq 0, \forall i=1,\ldots,n$. For two real valued functions $f(t)\ge 0$ and $g(t)\ge 0$, $f(t)=\mathscr{O}(g(t))$ if $f(t)\le kg(t)$ for some positive constant k. Let $x\in\mathbb{R}^n$. $X=diag(x)$ denotes the $n\times n$ diagonal matrix with $X_{ii} = x_i$ for all i. Finally, for a vector $v \in \mathbb{R}^n$, we denote by $\min v = \min_{1\leq i\leq n} v_i$ and the vector of all ones is denoted by e.

2. A Full-Newton Step Feasible Interior-Point (IP) Algorithm for LO

In this section, we review the notion of the central-path and standard search directions for LO. Then we present our new search directions based on a new AET for the centering equations induced by a new univariate function. Finally, the generic full-Newton step algorithm is stated. Throughout the paper, the following assumption holds for both problems $\mathscr{P}$ and $\mathscr{D}$

- Interior-Point-Condition (IPC). There exists (x^0, y^0, z^0) such that:

$$Ax^0 = b, A^T y^0 + z^0 = c, x^0 > 0, z^0 > 0.$$

If IPC holds, then finding an optimal solution of $\mathscr{P}$ and $\mathscr{D}$ is equivalent to solving the following system

$$\begin{cases} Ax & = & b, x \geq 0, \\ A^T y + z & = & c, z \geq 0, \\ xz & = & 0. \end{cases} \tag{2.1}$$

2.1. **Central-path for LO.** The main idea of primal-dual path-following IPMs for LO is to replace the equation $xz = 0$ in (2.1), and the so-called complementarity conditions of $\mathscr{P}$ and $\mathscr{D}$ by the parameterized equation $xz = \mu e$, where μ is a real positive parameter. Then, we have the following system of equations:

$$\begin{cases} Ax & = & b, \ x > 0, \\ A^T y + z & = & c, \ z > 0, \\ xz & = & \mu e. \end{cases} \tag{2.2}$$

Under our assumption, system (2.2) has a unique solution $(x(\mu), y(\mu), z(\mu))$ for any $\mu > 0$, called the μ-center of both problems $\mathscr{P}$ and $\mathscr{D}$. The set of the μ-centers defines a homotopy path which is called the central-path of both problems. If $\mu \mapsto 0$, then the limit of the central-path exists. Since the limit points satisfy the condition $xz = 0$, the limit yields an optimal solution to LO (see, e.g. [18, 21]). A direct application of Newton's method to system (2.2) provides the following linear system for $(\Delta x, \Delta y, \Delta z)$:

$$\begin{cases} A\Delta x & = & 0, \\ A^T \Delta y + \Delta z & = & 0, \\ z\Delta x + x\Delta z & = & \mu e - xz. \end{cases} \tag{2.3}$$

For simplicity, we denote by

$$d_x = \frac{v\Delta x}{x}, d_z = \frac{v\Delta z}{z}, \tag{2.4}$$

where v is the vector defined by: $v := \sqrt{\frac{xz}{\mu}}$. Considering (2.4), we then have

$$z\Delta x + x\Delta z = \mu v(d_x + d_z), \frac{\Delta x \Delta z}{\mu} = d_x d_z. \tag{2.5}$$

Thus system (2.3) is converted to the following scaled linear system:

$$\begin{cases} \bar{A}d_x &= 0, \\ \bar{A}^T\Delta y + d_z &= 0, \\ d_x + d_z &= v^{-1} - v, \end{cases} \tag{2.6}$$

where $\bar{A} = \frac{1}{\mu}AV^{-1}X$ with $X := \mathrm{diag}(x)$ and $V := \mathrm{diag}(v)$. By considering system (2.6), $d_x^T d_z = 0$, which implies the orthogonality of d_x and d_z. This gives the classical search directions for LO.

2.2. **New search directions for LO.** Following [1, 10], we turn to determine the new Newton search directions. The AET method based directions for LO was introduced in [8, 9] and is simply based in replacing the centrality equation $xz = \mu e$ in (2.2) by the new equation $\psi(\frac{xz}{\mu}) = \psi(e)$, where $\psi : (0, +\infty) \to \mathbb{R}$ is a continuously differentiable and invertible function. Then system (2.2) is converted to the following system:

$$\begin{cases} Ax &= b,\ x > 0, \\ A^T y + z &= c,\ z > 0, \\ \psi(\frac{xz}{\mu}) &= \psi(e), \end{cases} \tag{2.7}$$

where ψ is applied coordinatewisely. Applying Newton's method to system (2.7) for a given strictly feasible point (x, y, z) gives us the following linear system:

$$\begin{cases} A\Delta x &= 0, \\ A^T\Delta y + \Delta z &= 0, \\ z\Delta x + x\Delta z &= \dfrac{\mu(\psi(e) - \psi(\frac{xz}{\mu}))}{\psi'\left(\frac{xz}{\mu}\right)}, \end{cases} \tag{2.8}$$

where ψ' denotes the derivative of the function ψ. From (2.4) and (2.5), system (2.8) becomes

$$\begin{cases} \bar{A}d_x &= 0, \\ \bar{A}^T\Delta y + d_z &= 0, \\ d_x + d_z &= p_v \end{cases} \tag{2.9}$$

where $p_v = \frac{\psi(e) - \psi(v^2)}{v\psi'(v^2)}$. System (2.9) determines a family of new Newton search directions given by ψ. Note that, for $\psi(t) = t$, we recuperate the classical Newton search directions given by (2.6). In this paper, as mentioned in the introduction, we consider the AET induced by $\psi(t) = t^{\frac{3}{2}}, t > 0$. This yields

$$p_v = \frac{2}{3}(v^{-2} - v). \tag{2.10}$$

By substituting the value of ψ, system (2.8) becomes

$$\begin{cases} A\Delta x &= 0, \\ A^T\Delta y + \Delta z &= 0, \\ z\Delta x + x\Delta z &= \frac{2\mu}{3}\left((\frac{xz}{\mu})^{-\frac{1}{2}} - \left(\frac{xz}{\mu}\right)\right). \end{cases} \tag{2.11}$$

For the analysis of the algorithm, according to (2.10) we define a norm-based proximity measure $\delta(xz;\mu)$ to the central-path by:

$$\delta(xz;\mu) := \delta(v) = \frac{3}{2}\|p_v\| = \left\|v^{-2} - v\right\|. \tag{2.12}$$

Note that $\delta(v) = 0 \Leftrightarrow v = e \Leftrightarrow xz = \mu e$. Therefore, the value of $\delta(v)$ can be considered as a measure for the distance between any point (x,y,z) and the central-path. The following table summarizes the values of the vector p_v related to some different ψ functions used in the literature.

Functions $\psi(t)$	The vector p_v
$\psi(t) = t$	$(v^{-1} - v)$ (Roos et al. [18]. Standard),
$\psi(t) = \sqrt{t}$	$2(e - v)$ (Darvay [10]),
$\psi(t) = t - \sqrt{t}$	$\frac{2(v-v^2)}{2v-e}, v > \frac{e}{2}$ (Darvay and Takács [11]),
$\psi(t) = \log t$	$(-2v \log v)$ (Pan [14]),
$\psi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$	$(e - v^2)$ (Kheirfam and Haghani [13]),
$\psi(t) = t^{\frac{3}{2}}$	$\frac{2}{3}(v^{-2} - v)$ (Ours).

The value of p_v for different functions ψ.

2.3. **Generic full-Newton step IP algorithm for LO.** The IP full-Newton step algorithm for LO works as follows. First, we use a suitable threshold (default) value $\tau > 0$ with $0 < \tau < 3$, and we suppose that a strictly feasible initial point (x_0, y_0, z_0) exists such that $\delta(x_0 z_0; \mu_0) \leq \tau$ for certain μ_0, which is known. Using the search directions $(\Delta x, \Delta y, \Delta z)$ obtained from (2.11) and taking a full-Newton step, the algorithm produces a new iterate $(x + \Delta x, y + \Delta y, z + \Delta z)$.

Then it updates the barrier parameter μ to $(1-\theta)\mu$ with $0 < \theta < 1$ and solves the Newton system (2.11), and targets a new μ-center and so on. This procedure is repeated until the stopping criterion $x^T z \leq \varepsilon$ is satisfied for $\varepsilon > 0$, a given accuracy parameter. The generic feasible short-step full-Newton step interior-point algorithm for LO is presented below.

Input
An accuracy parameter $\varepsilon > 0$;
An update parameter $\theta, 0 < \theta < 1$ (default value $\theta = \frac{1}{10\sqrt{2n}}$);
A proximity parameter τ, $(0 < \tau < 3)$ (default value $\tau = 1$);
A strictly feasible point (x^0, y^0, z^0), $\mu_0 > 0$, s.t. $\delta(x^0 z^0; \mu^0) \leq \tau$;
Output
$x^T z \leq \varepsilon$;
begin
  Set $x := x^0$; $y := y^0$; $z := z^0$; $\mu := \mu^0$;
  While $n\mu \geq \varepsilon$ **do**
    begin
      $\mu := (1-\theta)\mu$;
  Compute $(\Delta x, \Delta y, \Delta z)$ from (2.11);
  Set $x := x + \Delta x$; $y := y + \Delta y; z := z + \Delta z$;
    endWhile
end.

Algorithm 2.3.

3. Complexity Analysis of Algorithm 2.3

In this section, our interest is to prove the new conditions of defaults that Algorithm 2.3 is well defined and converges locally quadratically to an optimal solution and solves the LO problem in polynomial complexity. We first present some results that are needed later in the analysis of Algorithm 2.3.

Lemma 3.1. *Let $(d_x, \Delta y, d_z)$ be the unique solution of (2.9), and let $\mu > 0$ with $\delta := \delta(xz; \mu)$. Then*

$$\|d_x d_z\|_\infty \leq \frac{\delta^2}{9}, \ \ \|d_x d_z\| \leq \frac{\sqrt{2}\delta^2}{9}. \tag{3.1}$$

Proof. For the first claim, since $d_x d_z = \frac{1}{4}\left((d_x + d_z)^2 - (d_x - d_z)^2\right)$, then we have

$$\|d_x d_z\|_\infty \leq \frac{1}{4}\max\left(\|d_x + d_z\|_\infty^2, \|d_x - d_z\|_\infty^2\right) \leq \frac{1}{4}\max\left(\|d_x + d_z\|^2, \|d_x - d_z\|^2\right).$$

Since $d_x^T d_z = 0$, then $\|d_x + d_z\|^2 = \|d_x - d_z\|^2$. This implies from (2.12) that

$$\|d_x d_z\|_\infty \leq \frac{1}{4}\|d_x + d_z\|^2 = \frac{1}{4}\|p_v\|^2 = \frac{1}{9}\delta^2.$$

For the last claim, we have

$$\begin{aligned}
\|d_x d_z\|^2 &= \frac{1}{16}\left(\left\|(d_x + d_z)^2 - (d_x - d_z)^2\right\|^2\right) \\
&\leq \frac{1}{16}\left(\left\|(d_x + d_z)^2\right\|^2 + \left\|(d_x - d_z)^2\right\|^2\right) \\
&\leq \frac{1}{16}(\|d_x + d_z\|^4 + \|d_x - d_z\|^4) = \frac{1}{8}\|d_x + d_z\|^4 = \frac{1}{8}\|p_v\|^4 = \frac{2}{81}\delta^4.
\end{aligned}$$

Hence $\|d_x d_z\| \leq \frac{\sqrt{2}}{9}\delta^2$. This completes the proof. $\square$

Lemma 3.2. *It holds that $\left|(1-t) - \frac{1}{\sqrt{1-t}}\right| \leq \frac{3t}{\sqrt{1-t}}$ for all $t \in (0, 1)$.*

Proof. For $t \in (0, 1)$, we have

$$\left|(1-t) - \frac{1}{\sqrt{1-t}}\right| = \frac{1}{|1-t|}\left|(1-t)^2 - \sqrt{1-t}\right| = \frac{\left|(1-t)^3 - 1\right|}{(1-t)^2 + \sqrt{1-t}} = \frac{\left|-t(t^2 - 3t + 3)\right|}{(1-t)^2 + \sqrt{1-t}}.$$

Since $\dfrac{1}{(1-t)^2 + \sqrt{1-t}} \leq \dfrac{1}{\sqrt{1-t}}$ and $t^2 - 3t + 3 \leq 3$, $\forall t \in (0, 1)$, it follows that

$$\left|(1-t) - \frac{1}{\sqrt{1-t}}\right| \leq \frac{3t}{\sqrt{1-t}}, \forall t \in (0, 1).$$

This gives the result. $\square$

The following lemma investigates the feasibility of a full-Newton step.

Lemma 3.3. *Let (x, y, z) be a strictly feasible point for both problems $\mathscr{P}$ and $\mathscr{D}$ and $\delta < 3$. Then $x_+ = x + \Delta x > 0$ and $z_+ = z + \Delta z > 0$, that is, x_+ and z_+ are strictly feasible.*

Proof. Observe that, for any $\alpha \in [0, 1]$,

$$x(\alpha)z(\alpha) = (x + \alpha\Delta x)(z + \alpha\Delta z) = xz + \alpha(x\Delta z + z\Delta x) + \alpha^2\Delta x\Delta z.$$

From (2.4) and (2.5), we have

$$x(\alpha)z(\alpha) = \mu((1-\alpha)v^2 + \alpha(v^2 + vp_v + \alpha d_x d_z)). \tag{3.2}$$

Hence $x(\alpha)z(\alpha) > 0$ if $v^2 + vp_v + \alpha d_x d_z > 0$. It follows from $\delta < 3$, (3.1) and (2.10) that

$$v^2 + vp_v + \alpha d_x d_z \geq v^2 + vp_v - \alpha \|d_x d_z\|_\infty e \geq v^2 + vp_v - \alpha \frac{\delta^2}{9} e > \frac{1}{3}v^2 + \frac{2}{3}v^{-1} - e.$$

Hence, $x(\alpha)z(\alpha) > 0$ if $\frac{1}{3}v^2 + \frac{2}{3}v^{-1} - e \geq 0$. Consider the function $g(t) = \frac{1}{3}t^2 + \frac{2}{3}t^{-1} - 1$, $t > 0$. Since g is strictly convex and has a minimum at $t = 1$, then $g(t) \geq g(1) = 0$. Hence, $\frac{1}{3}v^2 + \frac{2}{3}v^{-1} - e \geq 0$. Therefore, $x(\alpha)z(\alpha) > 0$ for $\alpha \in [0,1]$. Since x and z are positive, one finds that $x(\alpha) > 0$ and $z(\alpha) > 0$ for all $\alpha \in [0,1]$. Then $x(1) = x_+$ and $z(1) = z_+$ are positive, i.e., x_+ and z_+ are strictly feasible. This completes the proof. □

For convenience, we may write $v_+ = \sqrt{\frac{x_+ z_+}{\mu}}$.

Lemma 3.4. *If* $\delta < 3$, *then* $\min v_+ \geq \frac{1}{3}\sqrt{9 - \delta^2}$.

Proof. Letting $v_+ = \sqrt{\frac{x_+ z_+}{\mu}}$ and substituting $\alpha = 1$ in (3.2), we have

$$v_+^2 = v^2 + vp_v + d_x d_z = \frac{1}{3}v^2 + \frac{2}{3}v^{-1} + d_x d_z.$$

By Lemma 3.3, we have $\frac{1}{3}v^2 + \frac{2}{3}v^{-1} - e \geq 0$ if $\delta < 3$. This implies that $\frac{1}{3}v^2 + \frac{2}{3}v^{-1} \geq e$. Consequently, $v_+^2 \geq e + d_x d_z$. Next, using (3.1), we deduce that

$$v_+^2 \geq e + d_x d_z \geq (e - \|d_x d_z\|_\infty e) \geq \left(1 - \frac{1}{9}\delta^2\right) e \geq \frac{1}{9}(9e - \delta^2 e).$$

Hence, $\min v_+ \geq \frac{1}{3}\sqrt{9 - \delta^2}$. This completes the proof. □

Lemma 3.5. *Let* (x,y,z) *be a strictly feasible point for both problems* $\mathscr{P}$ *and* $\mathscr{D}$ *and assume* $\delta < 3$. *Then*

$$\delta^+ := \delta(v_+) = \delta(x_+ z_+, \mu) \leq \left(\frac{9}{9 - \delta^2} + \frac{3}{3 + \sqrt{9 - \delta^2}}\right)\left(\frac{1}{3} + \frac{\sqrt{2}}{9}\right)\delta^2.$$

In addition, if $\delta \leq 1$, *then* $\delta^+ \leq \left(\frac{49}{24} - \frac{7}{8}\sqrt{2}\right)\delta^2$, *which means the locally quadratically convergent of the proximity measure.*

Proof. Observe that

$$\delta^+ = \delta(v_+) = \left\|v_+^{-2} - v_+\right\| = \left\|\frac{e - v_+^3}{v_+^2}\right\| = \left\|(e - v_+^2).\left(\frac{e + v_+ + v_+^2}{(e + v_+)v_+^2}\right)\right\|.$$

Consider the function $f(t) = \frac{1+t+t^2}{(1+t)t^2} = \frac{1}{t^2} + \frac{1}{1+t}$. We write

$$\delta^+ = \left\|f(v_+)(e - v_+^2)\right\| \leq \|f(v_+)\|_\infty \left\|e - v_+^2\right\|.$$

Since f is continuous and monotonically decreasing and positive on $(0, +\infty)$, then

$$0 < |f((v_+)_i)| = f((v_+)_i) \leq f(\min v_+) \leq f\left(\frac{1}{3}\sqrt{9 - \delta^2}\right).$$

Consequently, $\|f(v_+)\|_\infty \leq \frac{9}{9-\delta^2}+\frac{3}{3+\sqrt{9-\delta^2}}$. Hence,

$$\delta^+ \leq \left(\frac{9}{9-\delta^2}+\frac{3}{3+\sqrt{9-\delta^2}}\right)\left\|e-v_+^2\right\|.$$

Next, setting $\alpha=1$ in (3.2), we obtain from (2.10) that

$$\left\|e-v_+^2\right\| \leq \left\|e-\frac{1}{3}v^2-\frac{2}{3}v^{-1}\right\|+\|d_xd_z\|.$$

Observe that $\left\|e-\frac{1}{3}v^2-\frac{2}{3}v^{-1}\right\|=\|\varphi(v).\frac{9p_v^2}{4}\|$, where $\varphi(v)=-\frac{v^3(v+2e)}{3(v^2+v+e)^2}$. Consider the function

$$\varphi(t)=-\frac{t^3(t+2)}{3(t^2+t+1)^2}.$$

Since it is continuous and monotonically decreasing and negative on $(0,+\infty)$, then

$$0\leq|\varphi(v_i)|=-\varphi(v_i)\leq\frac{1}{3},\forall i=1,\dots,n.$$

As $\|p_v\|^2=\frac{4}{9}\delta^2, \|\varphi(v)\|_\infty=\max_i|\varphi(v_i)|\leq\frac{1}{3}$, and $\left\|p_v^2\right\|\leq\|p_v\|^2$, it yields

$$\left\|e-\frac{1}{3}v^2-\frac{2}{3}v^{-1}\right\|\leq\|\varphi(v)\|_\infty\frac{9}{4}\|p_v\|^2=\frac{\delta^2}{3}.$$

Due to (3.1), $\|d_xd_z\|\leq\frac{\sqrt{2}}{9}\delta^2$, we have $\left\|e-v_+^2\right\|\leq\left(\frac{1}{3}+\frac{\sqrt{2}}{9}\right)\delta^2$, which implies that

$$\delta^+\leq\left(\frac{9}{9-\delta^2}+\frac{3}{3+\sqrt{9-\delta^2}}\right)\left(\frac{1}{3}+\frac{\sqrt{2}}{9}\right)\delta^2.$$

If $\delta\leq1$, then $\frac{9}{9-\delta^2}\leq\frac{9}{8}$ and $\frac{3}{3+\sqrt{9-\delta^2}}\leq\frac{3}{3+2\sqrt{2}}$. It follows that $\delta^+\leq\left(\frac{49}{24}-\frac{7}{8}\sqrt{2}\right)\delta^2$. This completes the proof. □

The next lemma gives an upper bound for the duality gap after a full-Newton step.

Lemma 3.6. *After a full-Newton step, it holds* $(x_+)^Tz_+\leq 2\mu n$.

Proof. As $v_+^2=\frac{x_+z_+}{\mu}$ and $d_x^Td_z=0$, we then have

$$\begin{aligned}(x_+)^Tz_+ &= \mu e^T(v^2+vp_v+d_xd_z)=\mu\left(\frac{1}{3}e^Tv^2+\frac{2}{3}e^Tv^{-1}+d_x^Td_z\right)\\ &= \mu e^Te+\mu e^T\left(\frac{1}{3}v^2+\frac{2}{3}v^{-1}-e\right)=\mu e^Te+\mu e^T\left(\frac{\frac{1}{3}v^2+\frac{2}{3}v^{-1}-e}{(v^{-2}-v)^2}.\frac{4}{9}p_v^2\right).\end{aligned}$$

Thus $(x_+)^Tz_+=\mu(n+e^T(-\varphi(v).\frac{4}{9}p_v^2))$. Observe that $-\varphi(v)$ is continuous and increasing and positive on $(0,\infty)$. Consequently, $-\varphi(v_i)\leq\frac{1}{3}<1, \forall i=1,\dots,n$, which implies that

$$(x_+)^Tz_+\leq\mu n+\max_i(-\varphi(v_i))\,e^T\frac{4}{9}p_v^2\leq\mu n+e^T\frac{4}{9}p_v^2.$$

As $\frac{4}{9}e^Tp_v^2=\frac{4}{9}\|p_v\|^2=\delta^2$ and $\frac{(x_+)^Tz_+}{\mu}=\|v_+\|^2$, we deduce $\|v_+\|^2=e^Tv_+^2\leq(n+\delta^2)$. Let $\delta\leq1$. Then $e^Tv_+^2\leq(n+1)$ and $(x_+)^Tz_+\leq2\mu n$, since $(n+1)\leq2n$ for all $n\geq1$. This completes the proof. $\square$

The following theorem demonstrates under the defaults $\tau \leq 1$ and $\theta = \frac{1}{10\sqrt{2n}}$ that Algorithm 2.3 is well-defined.

Theorem 3.1. *Let (x,y,z) be a strictly feasible point for both problems $\mathscr{P}$ and $\mathscr{D}$ such that $\delta < 3$ and $\mu_+ = (1-\theta)\mu$, where $\theta \in (0,1)$. Then $\delta(x_+z_+;\mu_+) \leq \frac{3\theta}{\sqrt{1-\theta}}\sqrt{2n} + \delta^+$. In addition, if $\delta \leq 1, \theta = \frac{1}{10\sqrt{2n}}, n \geq 2$, then $\delta(x_+z_+;\mu_+) \leq 1$.*

Proof. As $\sqrt{\frac{x_+z_+}{\mu_+}} = \frac{1}{\sqrt{1-\theta}}v_+$ and $\delta^+ = \left\|v_+^{-2} - v_+\right\|$, we then have,

$$\begin{aligned}
\delta(x_+z_+;\mu_+) &= \left\|\left(\frac{x_+z_+}{\sqrt{1-\theta}\mu}\right)^{-2} - \sqrt{\frac{x_+z_+}{(1-\theta)\mu}}\right\| = \left\|(1-\theta)v_+^{-2} - \frac{1}{\sqrt{1-\theta}}v_+\right\| \\
&\leq \left\|(1-\theta)v_+^{-2} - (1-\theta)v_+\right\| + \left\|(1-\theta)v_+ - \frac{1}{\sqrt{1-\theta}}v_+\right\| \\
&= (1-\theta)\delta^+ + \left|(1-\theta) - \frac{1}{\sqrt{1-\theta}}\right| \|v_+\| \\
&\leq \delta^+ + \left|(1-\theta) - \frac{1}{\sqrt{1-\theta}}\right| \|v_+\|, \text{ for } \theta \in (0,1).
\end{aligned}$$

It follows from Lemma 3.6 that $\|v_+\| \leq \sqrt{2n}$ and hence,

$$\delta(x_+z_+;\mu_+) \leq \delta^+ + \left|(1-\theta) - \frac{1}{\sqrt{1-\theta}}\right| \sqrt{2n}.$$

In view of Lemma 3.2, one has $\delta(x_+z_+;\mu_+) \leq \delta^+ + \frac{3\theta\sqrt{2n}}{\sqrt{1-\theta}}$. As $\delta^+ \leq \left(\frac{49}{24} - \frac{7}{8}\sqrt{2}\right)\delta^2$, one sees that $\delta(x_+z_+;\mu_+) \leq \frac{3\theta\sqrt{2n}}{\sqrt{1-\theta}} + \left(\frac{49}{24} - \frac{7}{8}\sqrt{2}\right)\delta^2$. Since $\delta \leq 1$, then $\delta(x_+z_+;\mu_+) \leq \frac{3\theta\sqrt{2n}}{\sqrt{1-\theta}} + \left(\frac{49}{24} - \frac{7}{8}\sqrt{2}\right)$. Letting $\theta = \frac{1}{10\sqrt{2n}}$, $n \geq 2$, one has $\theta \in \left[0, \frac{1}{9}\right]$. Finally, we conclude that

$$\delta(x_+z_+;\mu_+) \leq \frac{2}{3\sqrt{1-\theta}} + \left(\frac{49}{24} - \frac{7}{8}\sqrt{2}\right).$$

Consider the function $g(\theta) = \frac{2}{3\sqrt{1-\theta}} + \left(\frac{49}{24} - \frac{7}{8}\sqrt{2}\right)$. Since function g is continuous and monotonically increasing on the interval $\left[0, \frac{1}{9}\right]$, then $g(\theta) \in \left[g(0), g(\frac{1}{9})\right]$. This means that the value of $g(\theta)$ does not exceed $g(\frac{1}{9}) = 0.56868$. Hence, $\delta(x_+z_+;\mu_+) \leq 1$. This completes the proof. $\square$

Theorem 3.1 demonstrates that Algorithm 2.3 is well-defined since the strict feasibility of full-Newton iterations and $\delta(xz;\mu) \leq 1$ are maintained during the algorithm process.

3.1. **Iteration bound.** We end this section with a theorem that estimates the number of iterations produced by Algorithm 2.3. By regrouping the results obtained by the previous subsections, we state the following lemma.

Lemma 3.7. *Let x^k and z^k denote the k-ith iteration generated by Algorithm 2.3 with $\mu := \mu^k$. Then $(x^k)^T z^k \leq \varepsilon$ if $k \geq \left[\frac{1}{\theta}\log\frac{2\mu^0 n}{\varepsilon}\right]$.*

Proof. From Lemma 3.6, we have $(x^k)^T z^k \leq 2n\mu^k$. Since $\mu^k = (1-\theta)\mu^{k-1} = (1-\theta)^k \mu^0$, then $(x^k)^T z^k \leq (1-\theta)^k 2\mu^0 n$. Thus inequality $(x^k)^T z^k \leq \varepsilon$ holds if $(1-\theta)^k 2\mu^0 n \leq \varepsilon$. By taking the logarithm, we deduce that $k\log(1-\theta) \leq \log\varepsilon - \log 2n\mu^0 \leq \log\varepsilon - \log 2\mu^0 n$. Due to the inequality $-\log(1-\theta) \geq \theta$ for all $0 < \theta < 1$, we obtain $k\theta \geq \log 2n\mu^0 - \log\varepsilon = \log\frac{2n\mu^0}{\varepsilon}$. This completes the proof. □

Theorem 3.2. *Let $\theta = \frac{1}{10\sqrt{2n}}$ and $\mu^0 = \frac{1}{2}$. Then Algorithm 2.3 requires at most $\mathscr{O}\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$ iterations to obtain an ε-approximated solution of LO.*

Proof. Set $\theta = \frac{1}{10\sqrt{2n}}$ and $\mu^0 = \frac{1}{2}$. Using Lemma 3.7, we obtain the desired conclusion directly. □

4. Numerical Results

In this section, we present numerical experiments on some examples of LO problems with different sizes to evaluate the efficiency of Algorithm 2.3. The experiments are performed with MATLAB 7.9 and carried out on a PC where we set $\varepsilon = 10^{-6}$. The initial primal-dual starting point and the optimal solution LO are denoted by (x_0, y_0, z_0) and (x^*, y^*, z^*), respectively. Moreover, the number of iterations and the elapsed time are denoted by *Iter* and *CPU*, respectively. In our implementation, we take μ_0 to be in the relaxed set $\{0.5, 0.05, 0.005\}$ in order to enforce the barrier parameter μ to tend to zero and $\theta = \frac{1}{10\sqrt{2n}}, n \geq 2$.

<u>Problem 1.</u> Consider the following primal-dual pair $(\mathscr{P},\mathscr{D})$, where

$$A = \begin{bmatrix} 3 & -5 & 2 & -8 & -5 & 2 & 6 & 10 & 5 & 2 \\ 2 & -8 & -6 & -7 & 6 & 5 & 5 & 6 & -9 & -8 \\ 6 & 1 & 9 & 5 & -2 & 3 & 10 & -5 & 6 & -7 \\ 6 & 3 & -2 & 1 & -4 & 7 & -8 & 10 & 8 & 4 \\ 1 & -3 & -1 & 5 & -8 & -4 & 4 & 1 & 4 & -2 \\ 4 & -1 & 3 & -2 & 5 & -8 & 4 & 1 & -6 & 0 \\ 4 & -2 & 6 & -8 & 8 & 8 & -2 & -1 & 6 & 7 \\ 8 & 0 & -9 & -5 & 5 & 4 & -8 & -6 & 5 & 5 \end{bmatrix}, \; c = \begin{bmatrix} 103 \\ -39 \\ -131 \\ -140 \\ 236 \\ 134 \\ -164 \\ 7 \\ -49 \\ 70 \end{bmatrix}, \; b = \begin{bmatrix} 49 \\ 0 \\ 22 \\ 84 \\ -42 \\ -14 \\ 108 \\ 15 \end{bmatrix}.$$

The initial point for this example is taken as:

$$\begin{aligned} x_0 &= [1,2,2,1,4,5,2,4,2,4]^T, \\ z_0 &= [8,10,5,4,9,1,4,6,5,1]^T, \\ y_0 &= [-9,10,-6,5,0,3,10,7]^T. \end{aligned}$$

An optimal solution of this problem is:

$$\begin{aligned} x^* &= [2.2808, 0.7890, 3.7253, 0.0618, 1.5198, 5.4796, 0, 2.8514, 0, 3.6362]^T, \\ z^* &= [0,0,0,0,0,0,12.3145,0,9.7292,0]^T, \\ y^* &= [-9.7262, 9.9978, -5.6856, 6.2811, -1.0117, 4.1407, 9.8503, 6.7071]^T. \end{aligned}$$

The numerical results are summarized in Table 1.

$\mu \rightarrow$	0.5		0.05		0.005	
$\theta \downarrow$	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*
$\frac{1}{10\sqrt{2n}}$	581	0.115945	479	0.099233	377	0.084222

Table 1. Numerical results of Example 1 with size $(8,10)$.

Problem 2. Consider the primal-dual pair $(\mathscr{P},\mathscr{D})$, where

$$
\begin{aligned}
A &= \begin{bmatrix}
4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},\\
c &= [-1,1,1,1,1,-1,3,1,1,1,1,1]^T, b = [2,0,0,0,0,0]^T.
\end{aligned}
$$

The initial point is taken as:

$$
\begin{aligned}
x_0 &= [1,1,1,1,1,1,1,1,1,1,1,1]^T,\\
z_0 &= [1,1,1,1,1,1,1,1,1,1,1,1]^T,\\
y_0 &= [-1,-1,-1,-1,-1,-1]^T.
\end{aligned}
$$

The optimal solution of this problem is:

$$
\begin{aligned}
x^* &= [0.8571,0.7143,0.5714,0.4286,0.2857,0.1429,0,0,0,0,0,0]^T,\\
z^* &= [0,0,0,0,0,0,4,1,1,1,1,1]^T,\\
y^* &= [0.5,1.5,2,2,1.5,0.5]^T.
\end{aligned}
$$

The number of iterations and the elapsed time with our direction and with update parameter θ and with the relaxed barrier parameter μ are summarized in Table 2.

$\mu \rightarrow$	0.5		0.05		0.005	
$\theta \downarrow$	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*
$\frac{1}{10\sqrt{2n}}$	757	0.121179	646	0.147071	534	0.129456

Table 2. Numerical results of Example 2 with size $(6,12)$.

Problem 3. Consider the primal-dual pair $\mathscr{P}$-$\mathscr{D}$, where

$$
\begin{aligned}
A = (a_{ij}) &= \begin{cases} 0 \text{ if } i \neq j \text{ and } j \neq i+m\\ 1 \text{ if } i = j \text{ and } j = i+m, \end{cases}\\
c &= [-1,-1,\ldots,-1]^T \in \mathbb{R}^n,\\
b &= [2,2,\ldots 2]^T \in \mathbb{R}^m.
\end{aligned}
$$

For this example, the initial point is taken as:

$$
\begin{aligned}
x_0 &= [1,1,\ldots,1]^T \in \mathbb{R}^n,\\
z_0 &= [1,1,\ldots,1]^T \in \mathbb{R}^n\\
y_0 &= [-2,-2,\ldots -2]^T \in \mathbb{R}^m.
\end{aligned}
$$

A primal-dual optimal solution for this problem is:

$$
\begin{aligned}
x^* &= [1,1,\ldots,1]^T \in \mathbb{R}^n\\
z^* &= [0,0,\ldots,0]^T \in \mathbb{R}^n\\
y^* &= [-1,-1,\ldots,-1]^T \in \mathbb{R}^m.
\end{aligned}
$$

The numerical results obtained are stated in Table 3.

$\mu \rightarrow$	0.5		0.05		0.005	
Size (m,n)	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*
$(10,20)$	1012	0.268046	867	0.204202	723	0.175834
$(50,25)$	1695	2.043330	1466	1.784198	1237	1.519320
$(100,50)$	2499	13.863291	2174	12.415493	1850	12.870055

Table 3. Numerical results of Example 3.

From the numerical results, we see that Algorithm 2.3 with full-Newton steps obtains an optimal solution to these testing problems. Unfortunately, its convergence is very slow. The reason is due to the fact that the theoretical choice of θ is dependent to the problem size n, so θ approaches zero when the size becomes very large. This means that the rate of decrease $(1-\theta)$ in the sequence of barrier parameter $\{\mu_k\}$ becomes close to one.

4.1. **A numerical amelioration of Algorithm 2.3.** In this subsection, in order to improve the performances of Algorithm 2.3, we choose θ as a constant which belongs to the set $\{0.1,...,0.9\}$. Further, to keep iterates positive through the algorithm, we use a damped Newton step $x+\alpha\Delta x, y+\beta\Delta y$ and $z+\beta\Delta z$, where (α,β) is a step-size along the Newton direction such that $x+\alpha\Delta x>0$ and $z+\beta\Delta z>0$.

Our next task is to determine the pair (α,β) by suggesting a novel strategy. Let us define the quantities α_+,α_- and β_+,β_- as follows:

$$\alpha_+ = \begin{cases} +\infty & \text{if } \Delta x_i \leq 0, \forall i \\ \min_i\left(\frac{x_i}{\Delta x_i}\right) & \text{otherwise,} \end{cases} \quad \alpha_- = \begin{cases} -\infty & \text{if } \Delta x_i \geq 0, \forall i \\ \max_i\left(\frac{x_i}{\Delta x_i}\right) & \text{otherwise} \end{cases}$$

$$\beta_+ = \begin{cases} +\infty & \text{if } \Delta z_i \leq 0, \forall i \\ \min_i\left(\frac{z_i}{\Delta z_i}\right) & \text{otherwise,} \end{cases} \quad \beta_- = \begin{cases} -\infty & \text{if } \Delta z_i \geq 0, \forall i \\ \max_i\left(\frac{z_i}{\Delta z_i}\right) & \text{otherwise.} \end{cases}$$

Next, we choose our step-size (α,β) as the optimal solution of the following linear optimization:

$$\max_{(\alpha,\beta)} \left\{\alpha z^T\Delta x + \beta x^T\Delta z \backslash\ \rho\alpha_- < \alpha < \rho\alpha_+,\ \rho\beta_- < \alpha < \rho\beta_+ \right\}$$

where $\rho \in (0,1)$. Therefore, it is easily seen that α and β are as follows:

$$\alpha = \begin{cases} \rho\alpha_+ & \text{if } z^T\Delta x \geq 0, \\ \rho\alpha_- & \text{if } z^T\Delta x \leq 0, \end{cases} \quad \beta = \begin{cases} \rho\beta_+ & \text{if } x^T\Delta z \geq 0, \\ \rho\alpha_- & \text{if } x^T\Delta z \leq 0. \end{cases}$$

Now, the numerical experiments of Algorithm 2.3 with our modifications are summarized in Tables 4,5 and 6. Here, in our implementation, we take the same initial points

$\mu \rightarrow$	0.5		0.05		0.005	
$\theta \downarrow$	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*
0.1	125	0.208605	103	0.166964	81	0.153294
0.2	59	0.126565	49	0.113812	39	0.115195
0.3	37	0.102837	31	0.098312	24	0.043022
0.5	19	0.041055	16	0.035695	13	0.035055
0.7	11	0.034788	9	0.034158	8	0.031726
0.8	9	0.033709	7	0.033861	6	0.033468
0.9	6	0.029882	5	0.029857	4	0.029949

Table 4. Numerical results of Example 1.

$\mu \rightarrow$	0.5		0.05		0.005	
$\theta \downarrow$	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*
0.1	125	0.170406	103	0.150345	59	0.117155
0.3	37	0.100296	31	0.091114	18	0.080982
0.4	26	0.089121	22	0.083294	13	0.075129
0.5	19	0.081956	16	0.078503	9	0.072496
0.6	15	0.084861	12	0.074116	7	0.070835
0.8	9	0.072436	7	0.073210	4	0.069784
0.9	6	0.074519	5	0.073519	3	0.269412

Table 5. Numerical results of Example 2.

$\theta \rightarrow$	0.1		0.5		0.9	
Size (m,n)	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*
$(50,100)$	206	1.501503	32	0.422011	10	0.240236
$(100,200)$	220	5.931682	34	1.065663	10	0.450525
$(500,1000)$	250	399.084762	38	62.542417	12	22.490726
$(750,1500)$	258	1317.049438	40	241.300418	12	77.149317
$(1000,2000)$	264	3403.263618	40	550.292878	12	182.611720

Table 6. Numerical results of Example 3.

5. Conclusion and future work

In this paper, we presented a full-Newton short-step interior-point algorithm, which is based on a new algebraically equivalent transformation technique, for linear optimization (LO). This technique offers a new type of Newton search direction and a proximity measure for tracing approximately the central-path of LO. We proved, under the new default τ, which defines the size of the neighborhood of the central-path and the new default θ, which determines the rate of decrease on the barrier parameter that our algorithm is well-defined and converges to an optimal solution of LO. Moreover, we obtained its currently best known iteration bound, and we presented some numerical results for its evaluation. It is of interest to continuous to extend our proposed AET algorithm to other problems, such as convex quadratic optimization, monotone LCP, and symmetric cone problems.

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