Appl. Set-Valued Anal. Optim. 7 (2025), No. 2, pp. 165-177
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.7.2025.2.02

# FINITE CONVERGENCE CRITERIA FOR NORMALIZED NASH EQUILIBRIUM THROUGH WEAK SHARPNESS AND LINEAR CONDITIONING

ASRIFA SULTANA*, SHIVANI VALECHA

*Department of Mathematics, Indian Institute of Technology Bhilai, Durg - 491002, Chhattisgarh, India*

**Abstract.** The generalized Nash equilibrium problem play a significant role in modeling and analyzing several complex economics problems. In this paper, we consider jointly convex generalized Nash games which were introduced by Rosen. We study two important aspects related to these games, which include the weak sharpness property for the set of normalized Nash equilibria and the linear conditioning technique for regularized Nikaido-Isoda function. First, we define the weak sharpness property for the set of normalized Nash equilibria, and then we provide its characterization in terms of the regularized gap function. Furthermore, we provide the sufficient conditions under which the linear conditioning criteria for regularized Nikaido-Isoda function becomes equivalent to the weak sharpness property for the set of normalized Nash equilibria. We demonstrate that an iterative algorithm used for determining a normalized Nash equilibrium for jointly convex generalized Nash games terminates finitely under the weak sharpness and linear conditioning assumptions. Finally, we estimate the number of steps needed to determine a solution for the specific type of jointly convex generalized Nash game as an application.
**Keywords.** Jointly convex generalized Nash equilibrium problem; Non-cooperative games; Normalized Nash equilibria; Nikaido-Isoda function; Variational inequality.
**2020 Mathematics Subject Classification.** 65K10, 90C30, 91A10.

## 1. Introduction

The concept of Nash equilibrium initiated by John F. Nash, was defined for non-cooperative games consisting of finite number of players [1]. In Nash equilibrium problems (NEP), the feasible strategy set for each player is assumed to be fixed. Later on, Arrow-Debreu [2] extended NEP and developed the notion of generalized Nash equilibrium problem (GNEP) in which the strategy sets of players are allowed to rely on the strategies selected by rivals. Recently, the concept of GNEP has attracted considerable attention due to its applicability in electricity markets [3], microeconomics [4], environmental sustainability issues [5] and many other fields. For historical background on GNEP and a brief description of recent developments in this field, one can refer to an article by Facchinei-Kanzow [6].

Let us recall the formal definition of GNEP from [2]. Consider the set $\{1,2,\cdots N\}$ containing $N$-players participating in a non-cooperative game. Suppose that player $i$ controls a strategy vector $x_i \in \mathbb{R}^{n_i}$, where $\sum_{i=1}^{N} n_i = n$. Assume that the vector $x = (x_i)_{i=1}^{N} = (x_{-i}, x_i) \in \mathbb{R}^n$ indicates the

*Corresponding author.
E-mail address: asrifa@iitbhilai.ac.in (A. Sultana), shivaniv@iitbhilai.ac.in (S. Valecha).
Received 5 March 2023; Accepted 7 September 2024; Published online 25 March 2025.

strategies of all $N$-players, where $x_{-i} \in \mathbb{R}^{n-n_i}$ is strategy vector for all players except $i$. For any strategy vector $x_{-i}$ of rivals, the feasible strategy set of the player $i$ is restricted to $K_i(x_{-i}) \subseteq \mathbb{R}^{n_i}$ and one intends to choose the strategy $x_i \in K_i(x_{-i})$ which is solution of the following problem $P_i(x_{-i})$, given as

$$P_i(x_{-i}): \quad \min_{z_i} \; \theta_i(x_{-i}, z_i) \;\text{ subject to } z_i \in K_i(x_{-i}),$$

considering that $\theta_i : \mathbb{R}^n \to \mathbb{R}$ indicates the loss function of player $i$. The symbol $Sol_i(x_{-i})$ indicates the solution set for $P_i(x_{-i})$, and the vector $\bar{x} = (\bar{x})_{i=1}^N$ is known as *generalized Nash equilibrium* if $\bar{x}_i \in Sol_i(\bar{x}_{-i})$ for each $i = 1, 2, \cdots N$. In one of the preliminary works by Rosen [7], an important class of GNEP was discussed, which has been recently termed as "jointly convex generalized Nash equilibrium problem" [8]. For a given strategy vector $x_{-i}$ of the rivals, the feasible strategy set $K_i(x_{-i})$ of player $i$ in a jointly convex GNEP is given as

$$K_i(x_{-i}) = \{x_i \text{ in } \mathbb{R}^{n_i} \mid (x_{-i}, x_i) \text{ lies in } X\}, \tag{1.1}$$

where $X \subseteq \mathbb{R}^n$ is closed and convex.

Before the formal introduction of the jointly convex GNEP in [8], the early literature provided the characterization of classical GNEP with respect to a quasi-variational inequality (QVI) problem [6, 9], but this characterization is not useful enough because when compared to variational inequality (VI) problems, and the literature of QVI is not equipped with well developed solving techniques (see [9] for more details). We recollect that, for a given map $F : X \to \mathbb{R}^n$ (where $X \subseteq \mathbb{R}^n$), the variational inequality problem $VI(F,X)$ corresponds to determine $x^* \in X$ satisfying, $\langle F(x^*), y - x^* \rangle \geq 0$ for each $y \in X$. In the following result, Facchinei et al. [8] stated that, under suitable conditions, some solutions of jointly convex GNEP can be obtained by solving an associated VI problem.

**Theorem 1.1.** [8] *Let $X \subset \mathbb{R}^n$ be non-empty, convex, and closed. For any $i$, let the function $\theta_i : \mathbb{R}^{n-n_i} \times \mathbb{R}^{n_i} \to \mathbb{R}$ be continuously differentiable and $\theta_i(x_{-i}, .)$ be a convex function for each $x_{-i} \in \mathbb{R}^{n-n_i}$. Then any $\bar{x} \in X$ solving $VI(F,X)$ is an equilibrium for jointly convex GNEP, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is considered as $F(x) = \prod_{i=1}^N (\nabla_{x_i} \theta_i(x))$.*

Here, $(\nabla_{x_i} \theta_i(x))$ denotes the partial derivative of the function $\theta_i$ w.r.t. variable $x_i$. According to Theorem 1.1, it is not necessary that any equilibrium of jointly convex GNEP necessarily solves the associated variational inequality $VI(F,X)$. In fact, the equilibrium points for jointly convex GNEP which are obtained by solving $VI(F,X)$ are specifically termed as *normalized Nash equilibrium* (NNE) [8, 10]. Several iterative methods have been developed to find normalized Nash equilibrium for jointly convex GNEP over the past decade (see, e.g., [6, 10, 11, 12]). Therefore, investigating the conditions under which these iterative methods converge after finite number of steps is an important aspect. In the available literature, one can find various approaches related to finite termination of iterative methods used for solving optimization problems [13, 14, 15], variational inequality problems [16, 17], equilibrium problems [18, 19] and multi-criteria optimization [20]. In this direction, Burke-Ferris [14] studied the notion of the weak sharp minima to investigate the finite termination of algorithms for solving optimization problems with non-unique solution set. It extends the notion of the sharp minimum which was studied in [13, 15] for the optimization problems with a unique solution. Later, Patricksson in [16] and Marcotte-Zhu in [17] studied the weak sharpness property for the solution set of variational inequalities in order to derive finite convergence results for iterative methods used in solving variational inequalities. On the other hand, Moudafi [18] developed $\theta$-conditioning criterion for equilibrium problems, which extends and unifies several existing concepts including the weak sharp minima for optimization problems. Recently, Nguyen et al. [19] introduced the weak sharpness property for the solution set of equilibrium problems that is equivalent to the 1-conditioning (or linear conditioning) criterion. Moreover, the authors in [18, 19] demonstrated the finite termination of an iterative method for solving equilibrium problems. As far as we know, there is no detailed study on the finite termination of iterative methods for solving GNEP by using the weak sharpness property or linear conditioning criterion.

In this work, we focuses on two important aspects of the finite convergence analysis of jointly convex GNEP, including the weak sharpness property for the set of normalized Nash equilibria and the linear conditioning technique for regularized Nikaido-Isoda functions. In particular, we first give formal definitions of the weak sharpness property for normalized Nash equilibria, and then we provide its characterization with respect to regularized gap functions (used for determining NNE). Furthermore, we provide the suitable conditions under which linear conditioning of the regularized Nikaido-Isoda function is equivalent to the weak sharpness property for the set of normalized Nash equilibria. As an application, we obtain sufficient conditions under which any sequence calculated by employing a proximal point algorithm [20, 21] converges after finitely many iterations to an equilibrium solving the jointly convex GNEP. Finally, we apply our results to estimate the number of steps needed to determine NNE for a specific type of games consisting of $N$-players where the cost function for every player is assumed to be quadratic.

This article is arranged in following sequence: Section 2 recalls basic definitions and useful notations which we need later. We introduce the weak sharpness property for the set of normalized Nash equilibria and give its characterization with respect to regularized gap functions in Section 3. In Section 4, we first provide the sufficient conditions under which linear conditioning of the regularized Nikaido-Isoda function is equivalent to the weak sharpness property of the set of NNE. Further, in Section 5 we use this equivalence to show the finite termination of an iterative algorithm for determining NNE. Finally, we employ our results to estimate the number of steps needed to determine the normalized Nash equilibrium for some specific type of games.

## 2. Preliminaries and Notations

For any non-empty set $X \subseteq \mathbb{R}^n$, the set $X^\circ$ indicates polar set (see [22])

$$X^\circ = \{x^* \in \mathbb{R}^n |\, \langle x^*, z\rangle \leq 0 \text{ for any } z \in X\}. \tag{2.1}$$

If $X$ is an empty set, then $X^\circ = \mathbb{R}^n$. The set $N_X(x)$ indicates normal cone [22] corresponding to $X \subseteq \mathbb{R}^n$ at any $x \in \mathbb{R}^n$,

$$N_X(x) = (X - x)^\circ = \begin{cases} \{x^* \in \mathbb{R}^n |\, \langle x^*, z - x\rangle \leq 0,\ \forall\, z \in X\}, & \text{if } x \in \mathbb{R}^n \\ \emptyset, & \text{otherwise,} \end{cases}$$

and the tangent cone [22] corresponding to $X$ at any $x$ is $T_X(x) = [N_X(x)]^\circ$. Alternatively,

$$T_X(x) = \{\, h \in \mathbb{R}^n |\text{ for any } t_k \downarrow 0, \exists\, h_k \to h \text{ s.t. } x + t_k h_k \in X \text{ for any } k\}.$$

Let $X \subseteq \mathbb{R}^n$ be non-empty, closed, and convex. Then the projection of any $z \in \mathbb{R}^n$ onto $X$ is

$$P_X(z) = \{x \in X \mid \|z-x\| = \inf_{w \in X} \|z-w\|\}.$$

A bifunction $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is pseudomonotone [19, 23] on $X \subseteq \mathbb{R}^n$ if $\psi(x,z) \leq 0$ whenever $\psi(z,x) \geq 0$ for each couple $x,z \in X$. Let $X^* \subset X$. We call $\psi$ pseudomonotone on $X^*$ w.r.t. $X$ (renaming the monotonicity condition assumed in [21, Definition 1.2]) if, for any $z \in X^*$ and $x \in X$, $\psi(x,z) \leq 0$ whenever $\psi(z,x) \geq 0$. We call $\psi$ negatively psuedomonotone if $-\psi$ is psuedomontone.

We make the following assumptions about the feasible strategy map and loss function of each player $i$ throughout this article.

(A1) the strategy map $K_i$ is considered as per (1.1), where $X \neq \emptyset$ is closed and convex set in $\mathbb{R}^n$;

(A2) the loss function $\theta_i : \mathbb{R}^{n-n_i} \times \mathbb{R}^{n_i} \to \mathbb{R}$ is continuously differentiable and $\theta_i$ is convex in $x_i$ for each $x_{-i} \in \mathbb{R}^{n-n_i}$.

Further, $X^* \subset X$ denotes the set of NNE for the given GNEP.

Let us recall the concept of the*Nikaido-Isoda* (NI) function $\psi : X \times X \to \mathbb{R}$ (see [24])

$$\psi(x,y) = \sum_{i=1}^{N} \Big[\theta_i(x_{-i},x_i) - \theta_i(x_{-i},y_i)\Big].$$

A gap function $V : X \to \mathbb{R}$ for determining NNE (see [6]) can be constructed as follows by using NI function

$$V(x) = \sup_{y \in X} \psi(x,y). \tag{2.2}$$

The authors in [10] noticed the fact that it is not necessary that the supremum in (2.2) exists at unique $y \in X$ for any given $x \in X$, which results into non-differentiabilty of the gap function $V$. Consider that $a$ is any positive real number, a regularized gap function $V_a$ is introduced in [10] for determining NNE, which becomes differentiable at each point in $X$ under certain conditions. The function $V_a$ is formed as $V_a(x) = \max_{y \in X} \psi_a(x,y)$, where $\psi_a : X \times X \to \mathbb{R}$ is the regularized NI function [10]

$$\psi_a(x,y) = \sum_{i=1}^{N} \left[\theta_i(x_{-i},x_i) - \theta_i(x_{-i},y_i) - \frac{a}{2}\|x_i - y_i\|^2\right].$$

Now, we provide the sufficient conditions under which regularized NI function $\psi_a$ becomes negatively pseudomonotone over the solution set $X^*$. In this regard, we have the below-mentioned result as a modification of [21, Theorem 2.2].

**Lemma 2.1.** [21] *Let $X^* \subseteq X$ be a set of normalized Nash equilibria. Let $\psi_a : X \times X \to \mathbb{R}$ be a regularized NI function and $\psi_a(.,y)$ be a convex function for any $y \in X$. Then $\psi_a(x,x^*) \geq 0$ for each $x^*$ in $X^*$ and $x$ in $X$.*

**Remark 2.1.** One can deduce that $\psi_a$ is negatively pseudomonotone on $X^*$ w.r.t. $X$ if $\psi_a(.,y)$ is convex for any $y \in X$. In fact, for each pair $x^* \in X^*, x \in X$, we know that $\psi_a(x^*,x) \leq 0$ due to [10, Definition 3.1]. Further, $\psi_a(x,x^*) \geq 0$ follows if we assume that $\psi_a(.,y)$ is convex for any $y \in X$.

Suppose that $V_a : X \to \mathbb{R}$ is considered as $V_a(x) = \max_{y\in X} \psi_a(x,y)$ and $y^a(x)$ is the unique maximizer of $\psi_a(x,y)$ for given $x$ in $X$. Then the following result by Heusinger-Kanzow [10] states that $V_a$ is a differentiable regularized gap function for determining NNE if certain conditions hold.

**Lemma 2.2.** [10] *Suppose assumptions (A1) and (A2) holds. Then*

a) $V_a(x) \geq 0$ *for each* $x \in X$*;*
b) $x^*$ *is a NNE iff* $x^* \in X$ *satisfies* $V_a(x^*) = 0$*;*
c) $V_a$ *being continuously differentiable yields*

$$\begin{aligned}\nabla V_a(x) = &\sum_{i=1}^{N} \Big[\nabla\theta_i(x_{-i},x_i) - \nabla\theta_i(x_{-i},y_i^a(x))\Big] - a(x-y^a(x)) \\ &+ \Big[\nabla_{x_1}\theta_1(x_{-1},y_1^a(x)),\cdots,\nabla_{x_N}\theta_N(x_{-N},y_N^a(x))\Big]^T.\end{aligned}$$

## 3. Weakly Sharp Set of Normalized Nash Equilibria

In [14], Burke-Ferris initiated the notion of the weak sharp minima to study the finite convergence of several iterative algorithms for convex optimization problems with non-unique solutions. It extends the notion of the sharp minima, which was studied in [13, 15] for the optimization problems admitting a unique solution. Suppose that $f : \mathbb{R}^n \to \mathbb{R}\cup\{-\infty,\infty\}$ is an extended real-valued function. A set $Y^* \subset \mathbb{R}^n$ is known as a set of weak sharp minimas for the minimization problem $\min_{x\in Y} f(x)$ if there exists $\beta > 0$ satisfying $f(x) \geq f(x^*) + \beta\, d(x,Y^*)$ for all $x \in Y$ and all $x^* \in Y^*$. Later, Pactricksson [16] and Marcotte-Zhu [17] studied the weak sharpness property for the solution set of VI problems. Consider $X \subseteq \mathbb{R}^n$ and a map $F : X \to \mathbb{R}^n$. Then variational inequality problem $VI(F,X)$ is to determine $x^* \in X$ with $\langle F(x^*), y-x^*\rangle \geq 0$ for each $y \in X$. According to [16], the solution set $X^*$ of $VI(F,X)$ fulfills the weak sharpness property if there exists some positive real $\beta$ such that

$$-F(x^*) + \beta B \subset \bigcap_{x\in X^*} [T_X(x)\cap N_{X^*}(x)]^\circ,\ \forall x^* \in X^*, \tag{3.1}$$

where $B$ is the unit ball in $\mathbb{R}^n$.

According to Theorem 1.1, the set of normalized Nash equilibria for convex GNEP is nothing but the set of solutions for variational inequality problem $VI(F,X)$, where $F$ is defined as $F(x) = \prod_{i=1}^N (\nabla_{x_i}\theta_i(x))$. Hence, we define the weak sharpness property for the set of normalized Nash equilibria by following Patricksson [16] and Marcotte-Zhu [17].

**Definition 3.1.** Let $X^* \subseteq X$ denote the set of normalized Nash equilibria for a given convex $GNEP$. Then the set $X^*$ fulfills weakly sharpness property if there exists some $\gamma > 0$ such that

$$-\prod_{i=1}^{N}(\nabla_{x_i}\theta_i(x^*)) + \gamma B \subseteq \bigcap_{x\in X^*} [T_X(x)\cap N_{X^*}(x)]^\circ,\ \forall x^* \in X^* \tag{3.2}$$

where $B$ is the unit ball in $\mathbb{R}^n$.

In [17], Marcotte-Zhu provided the suitable conditions under which the solution set of $VI(F,X)$ fulfills the weak sharpness property iff there exists some $\beta > 0$ satisfying

$$G(y) \geq \beta d(y,X^*), \quad \forall y \in X, \tag{3.3}$$

where $G$ denotes dual gap function, $G(y) = \sup_{w\in X}\langle F(w), y-w\rangle$.

Inequality (3.3) is due to Marcotte-Zhu [17] motivating us to investigate the necessary and sufficient conditions under which the weak sharpness property for the set of NNE can be characterized in terms of a regularized gap function $V_a$ (see Lemma 2.2 for the definition of $V_a$).

**Theorem 3.1.** *Let $\psi_a : X\times X \to \mathbb{R}$ be a regularized NI function and $\psi_a(.,y)$ be a convex function for any $y\in X$. Then, the set of normalized Nash equilibria $X^*\subseteq X$ fulfills the weak sharpness property iff there exists $\gamma>0$ satisfying*

$$V_a(x) \geq \gamma d(x, X^*),\ \forall x\in X. \tag{A}$$

*Proof.* Consider a function $F:\mathbb{R}^n\to\mathbb{R}^n$ formed as $F(x) = \prod_{i=1}^{N}(\nabla_{x_i}\theta_i(x))$. We aim to prove that (A) holds if the set $X^*$ consisting of NNE fulfills the weak sharpness property. Suppose that $x^*$ in $X^*$ is chosen arbitrarily. According to (3.2), we obtain some $\gamma>0$ satisfying

$$-F(x^*)+\gamma b \in [T_X(x^*)\cap N_{X^*}(x^*)]^\circ \text{ for each } b \text{ in } B. \tag{3.4}$$

We show that inclusion (3.4) holds for $x^*$ iff

$$\langle F(x^*), z\rangle \geq \gamma\|z\| \text{ for each } z\in T_X(x^*)\cap N_{X^*}(x^*). \tag{3.5}$$

Suppose inclusion (3.4) holds. Then, the following inequality holds for each $b\in B$ according to (2.1) $\langle -F(x^*)+\gamma b, z\rangle \leq 0$ for each $z\in T_X(x^*)\cap N_{X^*}(x^*)$. We observe that inequality (3.5) is fulfilled for any $z$ in $T_X(x^*)\cap N_{X^*}(x^*)$ by substituting $b=\frac{z}{\|z\|}$ in the above inequality.

Conversely, let inequality (3.5) hold. Then, for any $z\in T_X(x^*)\cap N_{X^*}(x^*)$ and $b\in B=\{r\in\mathbb{R}^n |\, \|r\|\leq 1\}$, it follows from the Cauchy-Schwarz inequality and (3.5) that

$$\langle -F(x^*)+\gamma b, z\rangle \leq -\langle F(x^*), z\rangle + \gamma\|b\|\|z\| \leq -\langle F(x^*), z\rangle + \gamma\|z\| \leq 0.$$

Since $z\in T_X(x^*)\cap N_{X^*}(x^*)$ is arbitrary, we have $-F(x^*)+\gamma b\in [T_X(x^*)\cap N_{X^*}(x^*)]^\circ$ for any $b\in B$ according to (2.1).

Now, we verify that $V_a$ is a convex function by following [27, Proposition 3.1.1 (a)]. For each $x$ in $X$, we always have a unique maximizer of $\psi_a(x,.)$ since it is strictly concave (see [10, Theorem 3.3 (c)]). Assume that $y^a(x)$ denotes the unique maximizer of $\psi_a(x,.)$ for $x$ in $X$, that is, $\psi_a(x, y^a(x)) = \max_{y\in X}\psi_a(x,y)$. If $x,w\in X$ and $t\in[0,1]$ are arbitrary, then we have by assuming $tx+(1-t)w = w_t$ that

$$\begin{aligned}
V_a(w_t) &= \max_{y\in X}\psi_a(w_t,y) = \psi_a(w_t, y^a(w_t)) \\
&\leq t\psi_a(x, y^a(w_t)) + (1-t)\psi_a(w, y^a(w_t)) \quad [\because \psi_a(\cdot,y) \text{ is convex for any } y\in X] \\
&\leq t\psi_a(x, y^a(x)) + (1-t)\psi_a(w, y^a(w)) \quad [\text{by definition of } y^a(x) \text{ and } y^a(w)] \\
&= tV_a(x) + (1-t)V_a(w).
\end{aligned}$$

From Lemma 2.2, $V_a$ is a continuously differentiable function. We show that $\nabla V_a(x^*) = F(x^*)$ for any $x^*\in X^*$. In fact, due to Lemma 2.2, we have

$$\nabla V_a(x) = \sum_{i=1}^{N}[\nabla\theta_i(x_{-i},x_i) - \nabla\theta_i(x_{-i}, y_i^a(x))] + \prod_{i=1}^{N}[\nabla_{x_i}\theta_i(x_{-i}, y_i^a(x))] - a(x-y^a(x)) \tag{3.6}$$

for any $x\in X$. According to [10, Proposition 3.4], a point $x^*$ lies in a set of NNE $X^*$ iff $x^* = y^a(x^*)$. Thus, by equation (3.6), we obtain

$$F(x^*) = \nabla V_a(x^*) \text{ for each } x^* \text{ in } X^*. \tag{3.7}$$

Now, let $x \in X$ be arbitrary. Since $X^*$ is closed and convex, a unique point $x^* \in X^*$ exists which satisfies $x^* = P_X(x)$ and $(x - x^*) \in T_X(x^*) \cap N_{X^*}(x^*)$. Finally, by using to convexity and differentiability of the function $V_a$, we have

$$V_a(x) \geq V_a(x^*) + \langle \nabla V_a(x^*), x - x^* \rangle = \langle F(x^*), x - x^* \rangle \geq \gamma \|x - x^*\| = \gamma d(x, X^*).$$

Since $V_a(x^*) = 0$ (see Lemma 2.2), the equality in second step holds due to (3.7) and last inequality holds due to (3.5).

Conversely, suppose that there is some $\gamma > 0$ such that (A) holds for the set of NNE $X^*$. Then, we aim to show that $X^*$ satisfies (3.1). Taking $x^* \in X^*$ arbitrarily, we observe that $T_X(x^*) \cap N_{X^*}(x^*) = \{0\}$ implies $-F(x^*) + \gamma B \subset [T_X(x^*) \cap N_{X^*}(x^*)]^\circ$. If some non-zero vector $s$ is in $T_X(x^*) \cap N_{X^*}(x^*)$, then $\langle s, s \rangle > 0$ and hence, $x^* + s \notin X^*$. This implies that there exists a hyperplane $\mathscr{H}_s$ containing the point $x^*$ and orthogonal to $s$ which separates $x^* + s$ from $X^*$. Since $s \in T_X(x^*)$, we obtain a sequence $s_k$ converging to $s$ for any sequence $t_k \downarrow 0$ such that $x^* + t_k s_k \in X$. Furthermore, one can obtain a $\tilde{k} \in \mathbb{N}$ so that $\mathscr{H}_s$ separates $x^* + t_k s_k$ from $X^*$ for each $k > \tilde{k}$. Hence,

$$d(x^* + t_k s_k, X^*) \geq d(x^* + t_k s_k, \mathscr{H}_s) = \frac{t_k \langle s, s_k \rangle}{\|s\|}, \ \forall\, k > \tilde{k}. \tag{3.8}$$

Since $V_a(x) \geq \gamma d(x, X^*)$ for each $x$ in $X$ and $V_a(x^*) = 0$ for each $x^*$ in $X^*$, we observe that following condition holds due to inequality (3.8)

$$\frac{V_a(x^* + t_k s_k) - V(x^*)}{t_k} \geq \gamma \frac{\langle s, s_k \rangle}{\|s\|}, \ \forall\, k > \tilde{k}.$$

Now, we obtain $\langle \nabla V_a(x^*), s \rangle \geq \gamma \|s\|$ for any $0 \neq s \in T_X(x^*) \cap N_{X^*}(x^*)$ by using differentiability of the function $V_a$. Eventually, $\langle \gamma b - \nabla V_a(x^*), s \rangle \leq 0$ is fulfilled for any $b \in B, s \in T_X(x^*) \cap N_{X^*}(x^*)$. This fact, together with equation (3.7) implies

$$-F(x^*) + \gamma B \subseteq [T_X(x^*) \cap N_{X^*}(x^*)]^\circ \text{ for each } x^* \text{ in } X^*. \tag{3.9}$$

Since $V_a$ is convex and differentiable, we see that $\nabla V_a$ is constant over $X^*$ from [25, Lemma 1]. Hence, $\nabla V_a(x^*) = C = F(x^*)$ for each $x^*$ in $X^*$ for some $C \in \mathbb{R}$. Finally, one can observe that the fact $F$ is constant over $X^*$ along with inclusion (3.9) yields (3.2). $\square$

**Remark 3.1.** Observe that Marcotte-Zhu [17] characterized the weakly sharp solution set of VI problems by using a dual gap function. Furthermore, Liu-Wu [26] studied the similar type of characterization in terms of primal gap functions by assuming that the primal gap function is Gateaux differentiable. We provide the error bound in terms of regularized gap functions for determining NNE, which is differentiable according to [10]. In a similar way, one can obtain such type of error bound without any differentiability assumption by considering the regularized primal gap function of VI problems.

## 4. Linearly Conditioned Regularized Nikaido - Isoda Function

Moudafi initiated the notion of $\theta$-conditioned bifunction in [18]. Using this concept, the author demonstrated that, under some suitable conditions, the proximal point method converges for equilibrium problem after finite number of steps. Let $T : X \times X \to \mathbb{R}$ be a bifunction defined on a set $X \subseteq \mathbb{R}^n$. Recall that an equilibrium problem is to find some $x^* \in X$ such that $T(x^*, z) \geq 0$ for every $z \in X$. Let $X^*$ denote the set of solutions for the equilibrium problem above. Then, $T$ is said to be $\theta$-conditioned [18] for some $\theta > 0$ if there exists $\gamma > 0$ such that $-T(x, P_{X^*}(x)) \geq \gamma[d(x, X^*)]^\theta$ for every $x \in X$. Furthermore, $T$ is said to be linearly conditioned if $\theta = 1$. According to [10, Definition 3.1], we know that any point $x^* \in X$ is NNE if

$$-\psi_a(x^*, y) \geq 0, \text{ for every } y \in X. \tag{4.1}$$

Therefore, regularized NI function becomes linearly conditioned if there exists some $\gamma > 0$ such that

$$\psi_a(x, P_{X^*}(x)) \geq \gamma d(x, X^*), \text{ for every } x \in X. \tag{4.2}$$

Next, we consider the relation between linear conditioning of regaularized NI functions and weak sharpness property for the set of NNE. In following result, we state the suitable conditions that ensure the linear conditioning of regaularized Nikaido-Isoda function [18] is equivalent to the weak sharpness property for the set of NNE.

**Theorem 4.1.** *Let $\psi_a : X \times X \to \mathbb{R}$ be a regularized NI function and $\psi_a(., y)$ be a convex function for any $y \in X$. Then $X^* \subset X$ consisting of NNE fulfills weakly sharpness property iff $\psi_a$ is linearly conditioned.*

*Proof.* Assume that $X^*$ fulfills the weak sharpness property. It follows by (3.2) that

$$\gamma B - F(x^*) \subseteq [T_X(x^*) \cap N_{X^*}(x^*)]^\circ, \text{ for each } x^* \text{ in } X^*.$$

From the proof of Theorem 3.1, one sees that $F(x^*) = \nabla V_a(x^*)$ for each $x^* \in X^*$. Hence,

$$\gamma B - \nabla V_a(x^*) \subseteq [T_X(x^*) \cap N_{X^*}(x^*)]^\circ, \text{ for each } x^* \text{ in } X^*. \tag{4.3}$$

Let $x$ be arbitrary point in $X$. Since $X^*$ is a closed and convex set, a unique point $x^* \in X^*$ with $P_{X^*}(x) = x^*$ can be obtained. We now claim that $\psi_a$ is linearly conditioned. In fact, on considering $P_{X^*}(x) = x$, this claim follows trivially. If $P_{X^*}(x) \neq x$, then inclusion (4.3) along with the fact $(x - x^*) \in T_X(x^*) \cap N_{X^*}(x^*)$ implies

$$\left\langle \frac{\gamma(x - x^*)}{\|x - x^*\|} - \nabla V_a(x^*), x - x^* \right\rangle \leq 0,$$

which yields

$$\gamma \|x - x^*\| - \langle \nabla V_a(x^*), x - x^* \rangle \leq 0. \tag{4.4}$$

From [10, Prop. 3.4], any point $x^*$ is in $X^*$ iff $x^* = y_a(x^*)$, where $y_a(x^*)$ solves the following maximization problem $V_a(x^*) = \max_{y \in X} \psi_a(x^*, y)$. Thus, by applying Danskin's theorem [9] for any $x^*$ in $X^*$, one has $\nabla V_a(x^*) = \nabla_x[\psi_a(x^*, y)]_{(y = y_a(x^*))}$, where $\nabla_x \psi_a(x^*, y)$ denotes the partial derivative of the function $\psi_a$ w.r.t. $x$ at the point $x = x^*$. Now, we have following due to inequality (4.4)

$$\gamma d(x, X^*) \leq \langle \nabla_x \psi_a(x^*, x^*), x - x^* \rangle. \tag{4.5}$$

Since $\psi_a(., y)$ is convex for every $y \in X$ and $\psi_a(x^*, x^*) = 0$, it yields

$$\langle \nabla_x \psi_a(x^*, x^*), x - x^* \rangle \leq \psi_a(x, x^*). \tag{4.6}$$

Finally, inequality (4.5) along with (4.6) yields $\psi_a(x, P_{X^*}(x)) \geq \gamma d(x, X^*)$.

Conversely, we assume that $\psi_a$ is linearly conditioned. We prove that $X^*$ fulfills the weak sharpness criterion. Suppose that $x \in X$ is arbitrary. For each $x$ in $X$, a vector $y_a(x)$ exists uniquely which solves $V_a(x) = \max_{y \in X} \psi_a(x, y)$. Thus $V_a(x) = \psi_a(x, y_a(x)) \geq \psi_a(x, P_{X^*}(x)) \geq \gamma d(x, X^*)$ due to (4.2). Finally, by using Theorem 3.1, we see that $X^*$ fulfills the weak sharpness condition. $\square$

**Remark 4.1.** The authors in [19] considered an equilibrium problem with a bifunction $F$. According to [19, Remark 3], the monotonicity of $F$ is required in order to ensure that $F$ is linear conditioned if the solution set fulfills the weak sharpness condition. However, in Theorem 4.1, we proved that linear conditioning of regularized NI function $\psi_a$ is equivalent to the weak sharpness property for the set of NNE, where $\psi_a$ is not necessarily monotone. In fact, the regularized NI function $\psi_a$ is not even pseudomonotone on $X$. According to Lemma 2.1, it is negatively pseudomonotone over $X^*$ w.r.t $X$ (see [23] for more information on the various monotonicity properties of bifunctions).

## 5. Finite Termination of an Iterative Algorithm

Now, we demonstrate the termination of proximal point algorithm [18, 21] after finite number of steps if the set of NNE fulfills the weak sharpness property (or equivalently the regularized NI function is linearly conditioned). For this purpose, we apply the results derived in previous sections. From (4.1), one can employ the proximal point algorithm (PPA) to determine the set of NNE. Let $x_0 \in X$ be chosen arbitrarily. $\{x_k\}_{k\in\mathbb{N}}$ generated by PPA is given as [18, 21]

$$\psi_a(x_{k+1},z) - \frac{1}{r_k}\langle z - x_{k+1}, x_{k+1} - x_k\rangle \le 0, \text{ for each } z \in X, \tag{5.1}$$

where the sequence $\{r_k\}_{k\in\mathbb{N}}$ fulfills $r_k > 0$ for each $k$ and $\liminf_{k\to\infty} r_k = r > 0$.

**Theorem 5.1.** *Let $\psi_a(.,y)$ be a convex function for any $y \in X$. Then, for each sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (5.1), some $k_\circ \in \mathbb{N}$ can be obtained which satisfies $x_k \in X^*$ for all $k > k_\circ$ if one of the following condition holds,*

(a) *the regularized NI function $\psi_a$ is linearly conditioned;*
(b) *the set of normalized Nash equilibria $X^*$ fulfills the weak sharpness property.*

*Proof.* Let hypothesis (a) hold. Since $\psi_a$ is linearly conditioned, we obtain some $\gamma > 0$ satisfying

$$\psi_a(x, P_{X^*}(x)) \ge \gamma d(x, X^*). \tag{5.2}$$

In order to derive the stated result, we first prove that any sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by PPA (5.1) satisfies $\|x_k - x_{k+1}\| \to 0$ as $k \to \infty$. Suppose that $z \in X$ is arbitrary, Then the expression in (5.1) can be written as

$$\psi_a(x_{k+1},z) + \frac{1}{2r_k}\|x_k - x_{k+1}\|^2 + \frac{1}{2r_k}\|z - x_{k+1}\|^2 - \frac{1}{2r_k}\|z - x_k\|^2 \le 0.$$

Suppose that $x^* \in X^*$ is arbitrary. By considering $z = x^*$, we have

$$\psi_a(x_{k+1},x^*) + \frac{1}{2r_k}\|x_k - x_{k+1}\|^2 + \frac{1}{2r_k}\left\|x^* - x_{k+1}\right\|^2 - \frac{1}{2r_k}\left\|x^* - x_k\right\|^2 \le 0. \tag{5.3}$$

Since $x^* \in X^*$ is a NNE, we observe $\psi_a(x^*,z) \le 0$ for any $z$ in $X$ by virtue of (4.1). Finally, we obtain $\psi_a(z,x^*) \ge 0$ for any $z$ in $X$ due to Lemma 2.1. Therefore, inequality (5.3) yields

$$\|x_{k+1} - x_k\|^2 \le \left\|x^* - x_k\right\|^2 - \left\|x^* - x_{k+1}\right\|^2. \tag{5.4}$$

This further obtains $\sum_{k=0}^{\infty}\|x_{k+1} - x_k\|^2 \le \|x^* - x_0\|^2 < \infty$. Hence, $\|x_{k+1} - x_k\| \to 0$ as $k \to \infty$. We observe that $\{\frac{1}{r_k}\}_{k\in\mathbb{N}}$ is a bounded sequence as $\liminf_{k\to\infty} r_k = r > 0$. Hence, $\frac{1}{r_k}\|x_{k+1} - x_k\| \to 0$

as $k \to \infty$. Equivalently, for $\gamma > 0$, some $k_\circ \in \mathbb{N}$ exists with

$$\frac{1}{r_k}\|x_{k+1} - x_k\| < \gamma \text{ for each } k \geq k_\circ. \tag{5.5}$$

We aim to show that $x_k \in X^*$ for each $k > k_\circ$. Suppose, on contrary that, $x_j \notin X^*$ for some $j > k_\circ$. Then $d(x_j, X^*) \neq 0$ (as $X^* \subset X$ is closed convex). Hence, it follows from (5.1) that

$$\psi_a(x_j, P_{X^*}(x_j)) \leq \frac{1}{r_{j-1}} d(x_j, X^*)\left\|x_j - x_{j-1}\right\|. \tag{5.6}$$

We obtain the following inequality on combining (5.2) and (5.6) $\left\|x_j - x_{j-1}\right\| \geq \gamma r_{j-1}$. This contradicts (5.5) as $j-1 \geq k_\circ$. Thus $x_k$ is in $X^*$ for all $k > k_\circ$. According to Theorem 4.1, the regularized NI function $\psi_a$ is linearly conditioned iff $X^*$ fulfills the weak sharpness property. Hence, the proof follows provided that hypothesis (b) holds. □

**Remark 5.1.** By following the proof of Theorem 5.1, we observe that finite termination of PPA holds under condition (a) even when $a = 0$ in regularized NI function $\psi_a$. On the other hand, to prove the finite termination of PPA under condition (b), we need $a > 0$ so that $V_a$ is well defined and the assumptions of Theorem 4.1 hold true.

5.1. **Application to the games with quadratic loss functions.** Let us consider a jointly convex GNEP in which the loss function for each player is defined with the help of matrices. We apply the results obtained in previous sections to obtain an estimate for the number of iterations required to determine the normalized Nash equilibria for such games. We prove the below-stated result by following some arguments presented in [19].

**Theorem 5.2.** *Let $\theta_i(x) = \frac{1}{2}(x_i)^T A_{ii} x_i + \sum_{l=1, l\neq i}^N (x_l)^T A_{li} x_i$ for all $i = 1, 2, \cdots N$, where $x = (x_i)_{i=1}^N \in \mathbb{R}^n$, $A_{li}$ are matrices in $\mathbb{R}^{n_l \times n_i}$, and $A_{ii}$ are symmetric matrices with $\sum_{i=1}^N n_i = n$. Let $C$ be a positive definite matrix defined by*

$$C = \begin{bmatrix} \frac{1}{2}A_{11} & A_{12} & \cdots A_{1N} \\ A_{21} & \frac{1}{2}A_{22} & \cdots A_{2N} \\ \vdots & & \\ A_{N1} & A_{N2} & \cdots \frac{1}{2}A_{NN} \end{bmatrix}.$$

*Let $\delta > 0$ denote the smallest eigenvalue for $C + C^T$. If the set of normalized Nash equilibria fulfills the weak sharpness condition, then by considering $a \in (0, \delta]$ and $\frac{1}{r_k} \in (0, \varepsilon)$ for some $\varepsilon > 0$, the PPA in (5.1) converges in $k_\circ + 1$ iterations for some initial point $x_\circ$ in $X$, where $k_\circ \leq \frac{d(x_\circ, X^*)^2 \varepsilon^2}{\gamma^2}$.*

*Proof.* According to [27, Proposition 3.1.2], one can observe that the regularized NI function $\psi_a(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ corresponding to the above-mentioned GNEP is convex w.r.t. $x$ for any $y \in X$. Since $X^*$ is assumed to satisfy the weak sharpness property, we observe that $\psi_a$ is linearly conditioned by Theorem 4.1. Now, according to Theorem 5.1, we obtain some $k_\circ \in \mathbb{N}$ satisfying $x_k \in X^*$ for each $k > k_\circ$, where $\{x_k\}$ is a sequence generated by PPA in (5.1). By following the proof of Theorem 5.1, one has $\|x_{k+1} - x_k\| \to 0$ as $k \to \infty$. Hence, we obtain some least positive $k_\circ \in \mathbb{N}$ satisfying

$$\|x_{k+1} - x_k\| < \frac{\gamma}{\varepsilon} \text{ for all } k \geq k_\circ. \tag{5.7}$$

We claim that $x_k \in X^*$ for each $k > k_\circ$. Suppose, on contrary, $x_j \notin X^*$ for some $j > k_\circ$. Let $P_{X^*}(x_j) = y_j$. By linear conditioning and (5.1) we obtain

$$\gamma\|x_j - y_j\| = \gamma d(x_j, X^*) \leq \psi_a(x_j, y_j) \leq \frac{1}{r_j}\langle y_j - x_j, x_j - x_{j-1}\rangle < \gamma\|y_j - x_j\|.$$

Clearly, this contradiction yields $x_k \in X^*$ for any $k > k_\circ$. Now, one can easily obtain following from inequality (5.4) that

$$\begin{aligned}\|x^* - x_\circ\|^2 &\geq \|x^* - x_1\|^2 + \|x_1 - x_\circ\|^2 \\ &\geq \cdots \\ &\geq \|x^* - x_{k_\circ}\|^2 + \sum_{v=0}^{k_\circ - 1}\|x_{v+1} - x_v\|^2 \\ &\geq \sum_{v=0}^{k_\circ - 1}\|x_{v+1} - x_v\|^2\end{aligned}$$

Consequently, we have the following by employing inequality (5.7)

$$d(x_\circ, X^*)^2 = \inf_{x^* \in X^*}\|x^* - x_\circ\|^2 \geq \sum_{v=0}^{k_\circ - 1}\|x_{v+1} - x_v\|^2 \geq \frac{(k_\circ)\gamma^2}{\varepsilon^2},$$

which gives us the estimate, $k_\circ \leq \frac{d(x_\circ, X^*)^2\varepsilon^2}{\gamma^2}$. □

5.2. **The application to the saddle point problems with joint constraint sets.** We apply the results in previous sections to obtain an estimate for the number of iterations required to determine solutions of saddle point problem (SPP) (or two player game) with joint constraint set motivated by [9, 18, 21]. Suppose that the Lagrangian $L : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is convex-concave continuous function. Let $X \subset \mathbb{R}^n$ be non-empty, closed, and convex joint constraint set, where $n = n_1 + n_2$. We consider $K_i(x_{-i}) = \{x_i \in \mathbb{R}^{n_i} | (x_{-i}, x_i) \in X\}$ for $i = 1, 2$ and $K(x) = K_1(x_2) \times K_2(x_1)$. Then $x^* = (x_1^*, x_2^*) \in X$ solves the saddle point problem (SPP) with joint constraint set if

$$L(x_1^*, y_2) \leq L(x_1^*, x_2^*) \leq L(y_1, x_2^*) \text{ for all } (y_1, y_2) \in K(x^*). \tag{5.8}$$

Note that problem (5.8) reduces to classical saddle point problem studied in [9, 18, 21] if $X = X_1 \times X_2$. We observe that $x^* \in X$ solves SPP with joint constraint set if and only if it solves the jointly convex GNEP with

$$\theta_1(x_1, x_2) = L(x_1, x_2) \text{ and } \theta_2(x_1, x_2) = -L(x_1, x_2).$$

From (2), one sees that the corresponding NI function $\psi : X \times X \to \mathbb{R}$ becomes

$$\psi(x, y) = L(x_1, y_2) - L(y_1, x_2). \tag{5.9}$$

Let $X^* = \{x^* \in X | \psi(x^*, y) \leq 0 \text{ for all } y \in X\}$ indicate the normalized solution set of SPP (5.8). By assuming $a = 0$ and $\psi$ as (5.9), the PPA in (5.1) can be rewritten as follows. Let $x_\circ \in X$ be arbitrary. Then the sequence $\{x_k\}_{k\in\mathbb{N}} = \{(x_1^k, x_2^k)\}_{k\in\mathbb{N}} \subset X$ is given by

$$L_1(x_1^{k+1}, y_2) - L(y_1, x_2^{k+1}) - \frac{1}{r_k}\langle z - x_{k+1}, x_{k+1} - x_k\rangle \leq 0, \text{ for each } z \in X, \tag{5.10}$$

where $\{r_k\}_{k\in\mathbb{N}}$ fulfills $r_k > 0$ for each $k$ and $\liminf_{k\to\infty} r_k = r > 0$. We observe that the NI function $\psi$ of SPP with joint constraint set, defined in (5.9), is linearly conditioned if it satisfies the following condition,

$$\psi(x, P_{X^*}(x)) \geq \gamma d(x, X^*). \tag{5.11}$$

In the view of Remark 5.1, one can easily derive the following result by following the proof of Theorem 5.1 and Theorem 5.2.

**Theorem 5.3.** *Consider a continuous function $L : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ which is convex w.r.t. $x_1 \in \mathbb{R}^{n_1}$ and concave w.r.t. $x_2 \in \mathbb{R}^{n_2}$. Suppose that the NI function $\psi$ of SPP satisfy condition (5.11) for some $\gamma > 0$ and $\frac{1}{r_k} \in (0, \varepsilon)$ for some $\varepsilon > 0$. Then, the PPA in (5.10) converges in $k_\circ + 1$ iterations for some initial point $x_\circ$ in $X$, where $k_\circ \leq \frac{d(x_\circ, X^*)^2 \varepsilon^2}{\gamma^2}$.*

**Remark 5.2.** We observe that the concept of linearly conditioning for SPP given in (5.9) reduces to the one introduced by Lehdili in [28, Definition 4.2] if $X = X_1 \times X_2$ (see [18, Section 3.2]). Hence, our result, Theorem 5.3, extends [28, Theorem 5.2] to the case of joint constraint sets.

## 6. Conclusion

We studied two important aspects for the finite convergence analysis of jointly convex GNEP, which includes the weak sharpness property for normalized Nash equilibria and the linear conditioning technique for Nikaido-Isoda functions. We demonstrated the equivalence of both the properties under suitable conditions. For this purpose, we first provided the characterization of the weak sharpness property for normalized Nash equilibria in terms of regularized gap functions. We applied these results to show the finite termination of a proximal point algorithm for the jointly convex generalized Nash game with quadratic objective functions and saddle point problems with joint constraint sets. In future, we aim to study the finite termination of iterative algorithms to solve Arrow-Debreu generalized Nash equilibrium problems.

**Acknowledgments**

The first author acknowledges Science and Engineering Research Board (SERB), India for the financial support under (MTR/2021/000164). The second author is grateful to the University Grants Commission (UGC), New Delhi, India for the financial assistance provided by them throughout this research work under the grant (1313/(CSIRNETJUNE2019)).

## References

[1] J. Nash, Equilibrium points in n-person games, Proc. Nat. Acad. Sci. 36 (1950), 48-49.
[2] K.J. Arrow, G. Debreu, Existence of an equilibrium for a competitive economy, Econometrica. 22 (1954), 265-290.
[3] D. Aussel, A. Sultana, V. Vetrivel, On the existence of projected solutions of quasi-variational inequalities and generalized Nash equilibrium problems, J. Optim. Theory Appl. 170 (2016), 818-837.
[4] J. Cotrina, J. Zúñiga: Time-dependent generalized Nash equilibrium problem. J. Opt. Theory Appl. 179 (2018), 1054-1064.
[5] D. Salas, K.C. Van, D. Aussel, L. Montastruc, Optimal design of exchange networks with blind inputs and its application to Eco-industrial parks, Comput. Chem. Eng. 143 (2020), 1-18.
[6] F. Facchinei, C. Kanzow, Generalized Nash equilibrium problems, Ann. Oper. Res. 175 (2010), 177-211.
[7] J.B. Rosen, Existence and uniqueness of equilibrium points for concave n-person games, Econometrica. 33 (1965), 520-534.

[8] F. Facchinei, A. Fischer, V. Piccialli, On generalized Nash games and variational inequalities, Oper. Res. Lett. 35 (2007), 159-164.

[9] F. Facchinei, J.S. Pang, Finite-dimensional Variational Inequalities and Complementarity Problems, Vol.1, Springer, New York, 2003.

[10] A. Von Heusinger, C. Kanzow, Optimization reformulations of the generalized Nash equilibrium problem using Nikaido-Isoda-type functions, Comp. Optim. Appl. 43 (2009), 353-377.

[11] B. Panicucci, M. Pappalardo, M. Passacantando, On solving generalized Nash equilibrium problems via optimization, Optim. Lett. 3 (2009), 419-435.

[12] M. Fukushima, Restricted generalized Nash equilibria and controlled penalty algorithm, Comput. Manag. Sci. 8 (2011) 201-218.

[13] F. Al-Khayyal, J. Kyparisis, Finite convergence of algorithms for nonlinear programs and variational inequalities, J. Optim. Theo. Appl. 70 (1991), 319-332.

[14] J.V. Burke, M.C. Ferris, Weak sharp minima in mathematical programming, SIAM J. Control Optim. 31 (1993), 1340-1359.

[15] B.T. Polyak, Sharp minima. Institute of control sciences lecture notes, Moscow, USSR, IIASA workshop on generalized Lagrangians and their applications, IIASA, Laxenburg, Austria, 1979.

[16] M. Patriksson, A unified framework of descent algorithms for nonlinear programs and variational inequalities, Linköping University, Sweden, 1993.

[17] P. Marcotte, D. Zhu, Weak sharp solutions of variational inequalities, SIAM J. Optim. 9 (1998), 179-189.

[18] A. Moudafi, On finite and strong convergence of a proximal method for equilibrium problems, Num. Func. Anal. Optim. 28 (2007), 1347-1354.

[19] L. Van Nguyen, Q.H. Ansari, X. Qin, Linear conditioning, weak sharpness and finite convergence for equilibrium problems, J. Glob. Optim. 77 (2020), 1-20.

[20] G.C. Bento, J.X. Cruz Neto, A. Soubeyran, A proximal point-type method for multicriteria optimization, Set-Valued Var. Anal. 22 (2014), 557–573.

[21] S.D. Flåm, A.S. Antipin, Equilibrium programming using proximal-like algorithms, Math. Program. 78 (1996), 29-41.

[22] R.T. Rockafellar, Convex Analysis, Princeton University Press, New Jersey, 2015.

[23] G. Bigi, M. Passacantando, Twelve monotonicity conditions arising from algorithms for equilibrium problems, Optim. Meth. Softw. 30 (2015), 323-337.

[24] H. Nikaido, K. Isoda, Note on noncooperative convex games, Pac. J. Math. 5 (1955), 807-815.

[25] J. Zhou, C. Wang, A note on finite termination of iterative algorithms in mathematical programming, Oper. Res. Lett. 36 (2008), 715-717.

[26] Y. Liu, Z. Wu, Characterization of weakly sharp solutions of a variational inequality by its primal gap function, Optim. Lett. 10 (2016), 563-576.

[27] A. Von Heusinger, Numerical methods for the solution of the generalized Nash equilibrium problem, PhD Thesis, Universität Würzburg, 2009.

[28] N. Lehdili, Méthode proximales de sélection et de décomposition pour les inclusions monotones. Thèse Université de Montpellier, 1996.