Appl. Set-Valued Anal. Optim. 7 (2025), No. 2, pp. 179-189
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.7.2025.2.03

A NOTE ON WEIGHTED PLURICOMPLEX ENERGY CLASS $\mathscr{E}_\chi(\Omega)$

HOANG NHAT QUY

Faculty of Mathematics, The University of Danang
University of Science and Education, 550000, Da Nang, Vietnam

Abstract. The main aim of this note is to study the local property of the weighted pluricomplex energy class $\mathscr{E}_\chi(\Omega)$ and the complex Monge - Ampère measure class $\mathscr{M}_{\chi,loc}(\Omega)$ on this function class. At the same time, we also study the complete pluripolar sets in class $\mathscr{E}_\chi(\Omega)$. As an application, we describe the relations between the class $\mathscr{E}_\chi(\Omega)$ with the classes $\mathscr{B}_0(\Omega)$ and $\overline{\mathscr{B}}_0(\Omega)$.
Keywords. Complex Monge - Ampère operator; Complex Monge - Ampère measure; Pluripotential theory.
2020 Mathematics Subject Classification. 32U05.

1. Introduction

In this note, by Ω we always mean a hyperconvex domain in $\mathbb{C}^n$. The set of plurisubharmonic functions (resp. negative) on Ω is denoted $\mathrm{PSH}(\Omega)$ (resp. $\mathrm{PSH}^-(\Omega)$). The local property is also mentioned when studying some important subclasses of plurisubharmonic functions. Here, a class $\mathscr{K}(\Omega) \subset \mathrm{PSH}(\Omega)$ is said to be a local class if $\varphi \in \mathscr{K}(\Omega)$ if and only if $\varphi|_D \in \mathscr{K}(D)$ for all $D \Subset \Omega$.

In [7], Cegrell introduced the pluricomplex energy class $\mathscr{E}(\Omega)$ of plurisubharmonic functions on which the complex Monge - Ampère operator $(dd^c.)^n$ is well defined and is continuous under decreasing sequences of plurisubharmonic functions. Blocki (by [5, Theorem 1.1]) shown that $\mathscr{E}(\Omega)$ is the local class. In [4], the authors introduced the weighted pluricomplex energy class $\mathscr{E}_\chi(\Omega)$ as follows: Assume that $\chi : \mathbb{R}^- \to \mathbb{R}^+$ is a decreasing function. Then, as in [4], we define

$$\mathscr{E}_\chi(\Omega) = \left\{ \varphi \in PSH^-(\Omega) : \exists \mathscr{E}_0(\Omega) \ni \varphi_j \downarrow \varphi, \sup_{j\geq 1} \int_\Omega \chi(\varphi_j)(dd^c\varphi_j)^n < +\infty \right\},$$

where $\mathscr{E}_0(\Omega)$ is the cone of all bounded plurisubharmonic functions φ defined on Ω with finite total Monge - Ampère mass and $\lim_{z\to\xi} \varphi(z) = 0$ for all $\xi \in \partial\Omega$. Then, from the proof of [1, Theorem 1.4 and Corollary 1.5], it follows that if $\varphi \in \mathscr{E}_\chi(\Omega)$, then $\overline{\lim}_{z\to\xi} \varphi(z) = 0$ for all $\xi \in \partial\Omega$. Hence if $\varphi \in \mathscr{E}_\chi(\Omega)$, then $\varphi \notin \mathscr{E}_\chi(D)$ with D is a relatively compact hyperconvex

*Corresponding author.
E-mail address: hoangnhatquy@gmail.com
Received 5 September 2023; Accepted 23 April 2024; Published online 25 March 2025.

subdomain in Ω. Thus the class $\mathscr{E}_\chi(\Omega)$ is not the local one. In [12], the authors introduced the new weighted pluricomplex energy class as follows

$$\mathscr{E}_{\chi,loc}(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \exists\ \psi_D \in \mathscr{E}_\chi(\Omega) \text{ such that } \varphi = \psi_D \text{ on } D,\ \forall\ D \Subset \Omega\}.$$

Then, under some extra assumptions on χ, $\mathscr{E}_{\chi,loc}(\Omega)$ is the local one ([12, Theorem 3.4]).

In this note, we first give the definitions of the local class. Next, we use these concepts to study the local property of the weighted pluricomplex energy class $\mathscr{E}_\chi(\Omega)$ and the complex Monge - Ampère measure class $\mathscr{M}_{\chi,loc}(\Omega)$ on this function class. Then, we prove that if $E \subset \Omega$ is a complete pluripolar set, then there exists $\varphi \in \mathscr{E}_\chi(\Omega)$ such that $E = \{\varphi = -\infty\}$. As an application, we study the relations between the class $\mathscr{E}_\chi(\Omega)$ with the classes $\mathscr{B}_0(\Omega)$ and $\overline{\mathscr{B}_0}(\Omega)$.

2. Preliminaries

2.1. **Definition of local properties.** Let X be a topological space. $\mathscr{K}$ is said to be a class of functions on open sets in X if $\mathscr{K}(D)$ is a class of functions on D for all open set D in X. We introduce some notions of the local classes as follows.

Definition 2.1. Let $\mathscr{K}$ be a class of functions on open sets in a topological space X.
i) $\mathscr{K}$ is said to be a local class of type 1 if

$$\mathscr{K}(U)|_V = \{f|_V : f \in \mathscr{K}(U)\} \subset \mathscr{K}(V),$$

for all open sets $V \subset U \subset X$.
ii) $\mathscr{K}$ is said to be a local class of type 2 if a function f on U such that, for all $x \in U$, there exists a neighbourhood V of x with $f|_V \in \mathscr{K}(V)$, then $f \in \mathscr{K}(U)$.
iii) $\mathscr{K}$ is said to be a local class if $\mathscr{K}$ is a local class of type 1 and 2.

Definition 2.2. Let $\mathscr{K}$ be a class of functions on open sets in a topological space X.
i) We denote $\mathscr{K}^1_{loc}$ by a class of functions on open sets in a topological space X as follows: $\mathscr{K}^1_{loc}(X)$ is the class of functions on X such that, for all $x \in X$, there exists a neighbourhood U of x with $f|_U \in \mathscr{K}(X)|_U$.
ii) We denote $\mathscr{K}^2_{loc}$ by a class of functions on open sets in a topological space X as follows: $\mathscr{K}^2_{loc}(X)$ is the class of functions on X such that, for all $x \in X$, there exists a neighbourhood U of x with $f|_U \in \mathscr{K}(U)$.

It is known that classes of all continuous functions, upper semi-continuous functions, smooth functions, holmorphic functions, convex functions, subharmonic functions, and plurisuharmonic functions are the local classes.

We next state some basic results.

Proposition 2.1. *Let $\mathscr{K}$ be a class of functions on open sets in a topological space X. Then:*
i) $\mathscr{K} \subset \mathscr{K}^1_{loc}$.
ii) If $\mathscr{K}$ is the local class of type 1, then $\mathscr{K}^1_{loc} \subset \mathscr{K}^2_{loc}$.
iii) $\mathscr{K}$ is the local class of type 2 if and only if $\mathscr{K}^2_{loc} \subset \mathscr{K}$.
iv) $\mathscr{K}$ is the local class if and only if $\mathscr{K}^1_{loc} = \mathscr{K}^2_{loc} = \mathscr{K}$.

Proof. Let D be an open subset of X.

i) We prove that $\mathscr{K}(D) \subset \mathscr{K}^1_{loc}(D)$. Indeed, Let f be in $\mathscr{K}(D)$ and $x \in D$. Then we can choose $U = D$ as a neighbourhood of x and we have $f|_U = f|_D = f \in \mathscr{K}(D)$. Thus $f \in \mathscr{K}^1_{loc}(D)$.

ii) Let f be in $\mathcal{K}^1_{loc}(D)$ and $x \in D$. Then there exist a neighbourhood U of x and a function $g \in \mathcal{K}(D)$ such that $f|_U = g|_U$. By the local property of type 1 of $\mathcal{K}$, we infer $g|_U \in \mathcal{K}(U)$. Thus $f|_U \in \mathcal{K}(U)$ and this means that $f \in \mathcal{K}^2_{loc}(D)$.

iii) Suppose that $\mathcal{K}$ is the local class of type 2. Let f be in $\mathcal{K}^2_{loc}(D)$. Then, for all $x \in D$, there exists a neighbourhood U of x such that $f|_U \in \mathcal{K}(U)$. By the local property of type 2 of $\mathcal{K}$, we have $f \in \mathcal{K}(D)$.

On the other hand, suppose that $\mathcal{K}^2_{loc} \subset \mathcal{K}$. Let f be a function on D such that, for all $x \in D$, there exists a neighbourhood U of x with $f|_U \in K(U)$. Then $f \in \mathcal{K}^2_{loc}(D)$ and then $f \in \mathcal{K}(D)$.

iv) It follows from i), ii) and iii) immediately. □

2.2. **Some (weighted) pluricomplex energy classes.** We recall some pluricomplex energy classes that was introduced in [6] and [7]:

$$\mathcal{E}_0(\Omega) = \{\varphi \in \mathrm{L}^\infty(\Omega) : \lim_{z\to\xi} \varphi(z) = 0,\ \forall \xi \in \partial\Omega \text{ and } \int_\Omega (dd^c\varphi)^n < +\infty\},$$

$$\mathcal{F}(\Omega) = \left\{\varphi \in \mathrm{PSH}^-(\Omega) : \exists \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi, \sup_{j\geq 1} \int_\Omega (dd^c\varphi_j)^n < +\infty\right\},$$

and

$$\mathcal{E}(\Omega) = \big\{\varphi \in \mathrm{PSH}^-(\Omega) : \forall z_0 \in \Omega, \exists \text{ a neighbourhood } \omega \ni z_0,$$
$$\left.\mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi \text{ on } \omega, \sup_{j\geq 1} \int_\Omega (dd^c\varphi_j)^n < +\infty\right\}.$$

We recall some weighted pluricomplex energy classes in [4, 12]:

$$\mathcal{E}_\chi(\Omega) = \left\{\varphi \in PSH^-(\Omega) : \exists \mathcal{E}_0(\Omega) \ni \varphi_j \downarrow \varphi, \sup_{j\geq 1} \int_\Omega \chi(\varphi_j)(dd^c\varphi_j)^n < +\infty\right\}$$

and

$$\mathcal{E}_{\chi,loc}(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \exists\ \psi_D \in \mathcal{E}_\chi(\Omega) \text{ such that } \varphi = \psi_D \text{ on } D,\ \forall\ D \Subset \Omega\}.$$

Remark 2.1. i) By the Remark from [7, Definition 4.6] we have, for every $u \in \mathcal{E}(\Omega)$ and every $\omega \Subset \Omega$, there exists a $u_\omega \in \mathcal{F}(\Omega)$ sucth that $u = u_\omega$ on ω.

ii) It is known from [3] and [4] that if χ is bounded and $\chi(0) > 0$, then $\mathcal{E}_\chi(\Omega) = \mathcal{F}(\Omega)$. iii) By [11, Theorem 3.1], if $\chi \not\equiv 0$, then $\mathcal{E}_\chi(\Omega) \subset \mathcal{E}(\Omega)$.

iv) By [11, Remark 3.4], if $\chi(-\infty) = +\infty$, then $\mathcal{E}_\chi(\Omega) \subset \mathcal{E}^a(\Omega)$, where

$$\mathcal{E}^a(\Omega) := \{\varphi \in \mathcal{E}(\Omega) : (dd^c\varphi)^n(E) = 0,\ \forall E \subset \Omega \text{ is the pluripolar set}\}.$$

v) By [11, Theorem 3.1], if $\chi(t) > 0$ for all $t < 0$, then $\mathcal{E}_\chi(\Omega) \subset \mathcal{N}(\Omega)$, where the class $\mathcal{N}(\Omega)$ has introduced in [8].

vi) If the function χ such that $\chi(2t) \leq a\chi(t)$ for some $a > 1$, then $\mathcal{E}_\chi(\Omega)$ is a cone.

We set

$$\mathcal{A}_a = \{\chi : \mathbb{R}^- \to \mathbb{R}^+ : \chi \text{ is decreasing and satisfies } \chi(2t) \leq a\chi(t)\}.$$

We have the following characterizations for the weighted pluricomplex energy class when the weighted functions are the linear combination, max, min of the functions in $\mathcal{A}_a$.

Proposition 2.2. *Let $\chi, \eta \in \mathscr{A}_a$. Then*
i) $\mathscr{E}_{a\chi+b\eta}(\Omega) = \mathscr{E}_\chi(\Omega) \cap \mathscr{E}_\eta(\Omega), \ \forall a, b > 0.$
ii) $\mathscr{E}_{\max(\chi,\eta)}(\Omega) = \mathscr{E}_\chi(\Omega) \cap \mathscr{E}_\eta(\Omega).$
iii) The function $\varphi \in \mathscr{E}_{\min(\chi,\eta)}(\Omega)$ if and only if there exists $u \in \mathscr{E}_\chi(\Omega)$ and $v \in \mathscr{E}_\eta(\Omega)$ such that $u \geq \varphi, v \geq \varphi$ and $\varphi \geq u + v$.

Proof. i) $\varphi \in \mathscr{E}_{a\chi+b\eta}(\Omega)$ if and only if there exists the decreasing sequence $\{\varphi_j\} \subset \mathscr{E}_0(\Omega)$ such that $\varphi_j \searrow \varphi$ on Ω and

$$\sup_j \int_\Omega [a\chi(\varphi_j) + b\eta(\varphi_j)](dd^c\varphi_j)^n < +\infty$$
$$\Leftrightarrow a \sup_j \int_\Omega \chi(\varphi_j)(dd^c\varphi_j)^n + b \sup_j \int_\Omega \eta(\varphi_j)(dd^c\varphi_j)^n < +\infty$$
$$\Leftrightarrow \sup_j \int_\Omega \chi(\varphi_j)(dd^c\varphi_j)^n < +\infty \text{ and } \sup_j \int_\Omega \eta(\varphi_j)(dd^c\varphi_j)^n < +\infty.$$

Thus $\varphi \in \mathscr{E}_\chi(\Omega) \cap \mathscr{E}_\eta(\Omega)$.
ii) Since $\max(\chi, \eta) \geq \chi, \eta$, we have $\mathscr{E}_{\max(\chi,\eta)}(\Omega) \subset \mathscr{E}_\chi(\Omega), \mathscr{E}_\eta(\Omega)$. This imply that $\mathscr{E}_{\max(\chi,\eta)}(\Omega) \subset \mathscr{E}_\chi(\Omega) \cap \mathscr{E}_\eta(\Omega)$.

On the other hand, we take $\varphi \in \mathscr{E}_\chi(\Omega) \cap \mathscr{E}_\eta(\Omega)$. Then there exists $\mathscr{E}_0(\Omega) \ni u_j \searrow \varphi$ and $\mathscr{E}_0(\Omega) \ni v_j \searrow \varphi$ such that

$$\sup_{j\geq 1} \int_\Omega \chi(u_j)(dd^c u_j)^n < +\infty \text{ and } \sup_{j\geq 1} \int_\Omega \eta(v_j)(dd^c v_j)^n < +\infty.$$

Setting $\varphi_j = \max(u_j, v_j)$, we have $\mathscr{E}_0 \ni \varphi_j \searrow \varphi$. It follows that

$$\begin{aligned}
\max[\chi(\varphi_j), \eta(\varphi_j)] &= \max[\chi(\max(u_j, v_j)), \eta(\max(u_j, v_j))] \\
&= \max[\min(\chi(u_j), \chi(v_j)), \min(\eta(u_j), \eta(v_j))] \\
&\leq \min(\chi(u_j), \chi(v_j)) + \min(\eta(u_j), \eta(v_j)) \\
&\leq \chi(u_j) + \eta(v_j).
\end{aligned}$$

Thus

$$\begin{aligned}
\int_\Omega \max[\chi(\varphi_j), \eta(\varphi_j)](dd^c\varphi_j)^n &\leq \int_\Omega \chi(u_j)(dd^c\varphi_j)^n + \int_\Omega \eta(v_j)(dd^c\varphi_j)^n \\
&\leq \int_\Omega \chi(u_j)(dd^c u_j)^n + \int_\Omega \eta(v_j)(dd^c v_j)^n.
\end{aligned}$$

This infers that $\sup_{j\geq 1} \int_\Omega \max[\chi(\varphi_j), \eta(\varphi_j)](dd^c\varphi_j)^n < +\infty$. Hence, $\varphi \in \mathscr{E}_{\max(\chi,\eta)}(\Omega)$.
iii) Set $\theta = \min\{\chi, \eta\}$. We prove the necessity of the condition. Take $\varphi \in \mathscr{E}_\theta(\Omega)$. We next prove that there exist $u \in \mathscr{E}_\chi(\Omega)$ and $v \in \mathscr{E}_\eta(\Omega)$ such that $u \geq \varphi, v \geq \varphi$ and $\varphi \geq u + v$. Indeed, let $\varphi_j \in \mathscr{E}_0(\Omega)$ such that $\varphi_j \searrow \varphi$ and $\sup\limits_{j\geq 1} \int\limits_\Omega \theta(\varphi_j)(dd^c\varphi_j)^n < +\infty$. By [10, Theorem 2.2] (see also [14, Theorem 5.9]), there exists the functions $u_j \in \mathscr{E}_0(\Omega)$ and $v_j \in \mathscr{E}_0(\Omega)$ such that

$$(dd^c u_j)^n = 1_{\{\chi(\varphi_j) < \eta(v_j)\}}(dd^c\varphi_j)^n$$

and

$$(dd^c v_j)^n = 1_{\{\chi(\varphi_j) \geq \eta(v_j)\}}(dd^c\varphi_j)^n.$$

However, it is plain that

$$(dd^c\varphi_j)^n = (dd^c u_j)^n + (dd^c v_j)^n \le \big(dd^c(u_j+v_j)\big)^n.$$

By the Comparison principle ([9, Corollary 2.19]), it follows that

$$u_j + v_j \le \varphi_j \le u_j \text{ and } u_j + v_j \le \varphi_j \le v_j \quad \text{on } \Omega. \tag{2.1}$$

By the decreasing of weighted functions, we obtain the chain of inequalities

$$\begin{aligned}
\int_\Omega \chi(u_j)(dd^c u_j)^n &= \int_{\{\chi(\varphi_j)<\eta(v_j)\}} \chi(u_j)(dd^c\varphi_j)^n \\
&\le \int_{\{\chi(\varphi_j)<\eta(v_j)\}} \chi(\varphi_j)(dd^c\varphi_j)^n \\
&= \int_{\{\chi(\varphi_j)<\eta(v_j)\}} \theta(\varphi_j)(dd^c\varphi_j)^n \\
&\le \int_\Omega \theta(\varphi_j)(dd^c\varphi_j)^n.
\end{aligned}$$

Hence, by the hypothesis $\varphi \in \mathscr{E}_\theta(\Omega)$, we have the following estimate

$$\sup_{j\ge 1}\int_\Omega \chi(u_j)(dd^c u_j)^n \le \sup_{j\ge 1}\int_\Omega \theta(\varphi_j)(dd^c\varphi_j)^n < +\infty. \tag{2.2}$$

Following the argument analogous to the previous one, we have the following inequalities

$$\begin{aligned}
\int_\Omega \eta(v_j)(dd^c v_j)^n &= \int_{\{\chi(\varphi_j)\ge\eta(v_j)\}} \eta(v_j)(dd^c\varphi_j)^n \\
&\le \int_{\{\chi(\varphi_j)\ge\eta(v_j)\}} \eta(\varphi_j)(dd^c\varphi_j)^n \\
&= \int_{\{\chi(\varphi_j)\ge\eta(v_j)\}} \theta(\varphi_j)(dd^c\varphi_j)^n \\
&\le \int_\Omega \theta(\varphi_j)(dd^c\varphi_j)^n,
\end{aligned}$$

and then

$$\sup_{j\ge 1}\int_\Omega \eta(v_j)(dd^c v_j)^n \le \sup_{\Omega}\int_{j\ge 1} \theta(\varphi_j)(dd^c\varphi_j)^n < +\infty. \tag{2.3}$$

Combining (2.2) and (2.3), we obtain $u = (\overline{\lim} u_j)^* \in \mathscr{E}_\chi(\Omega)$ and $v = (\overline{\lim} v_j)^* \in \mathscr{E}_\eta(\Omega)$.

Now, from the inequalities in (2.1), it is clear that $u \ge \varphi, v \ge \varphi$ and $\varphi \ge u+v$. We now turn to the proof of the sufficiency. Assume $\varphi \in \mathrm{PSH}^-(\Omega)$ such that there exists $u \in \mathscr{E}_\chi(\Omega), v \in \mathscr{E}_\eta(\Omega)$ with $u \ge \varphi, v \ge \varphi$ and $\varphi \ge u+v$. We are going to prove that $\varphi \in \mathscr{E}_\theta(\Omega)$. Indeed, we can check that $\mathscr{E}_\chi(\Omega) \subset \mathscr{E}_\theta(\Omega)$ and $\mathscr{E}_\eta(\Omega) \subset \mathscr{E}_\theta(\Omega)$. Since $\mathscr{E}_\theta(\Omega)$ is a cone, we infer that

$$\mathscr{E}_\chi(\Omega) + \mathscr{E}_\eta(\Omega) \subset \mathscr{E}_\theta(\Omega) + \mathscr{E}_\theta(\Omega) \subset \mathscr{E}_\theta(\Omega).$$

Thus $u+v\in\mathcal{E}_\theta(\Omega)$. Since $\mathcal{E}_\theta(\Omega)$ is a cone, we derive $\varphi\in\mathcal{E}_\theta(\Omega)$. □

3. Local Properties Relate to the Class $\mathcal{E}_\chi(\Omega)$

Proposition 3.1. *$\mathcal{E}_0(\Omega)$ is only the local class of type 2.*

Proof. By the maximum principle of plurisubharmonic functions, we have $\mathcal{E}^2_{0,loc}(\Omega)=\{0\}$. By Proposition 2.1 (iii), we see that $\mathcal{E}_0(\Omega)$ is the local class of type 2.

Now we prove that
$$\mathcal{E}^1_{0,loc}(\Omega)=PSH^-\cap L^\infty(\Omega).$$
Indeed, for $f\in PSH^-\cap L^\infty(\Omega)$ and $x\in\Omega$, we can choose $r>0$ small enough such that $B=B(x,r)\Subset\Omega$. We set
$$h^f_{B,\Omega}=\sup\{\psi\in \mathrm{PSH}^-(\Omega):\psi\le f \text{ on } B\}.$$
By [15], we deduce that $h^f_{B,\Omega}\in\mathcal{E}_0(\Omega)$. Obviously, we also have $h^f_{B,\Omega}=f$ on B. Thus $f\in\mathcal{E}^1_{0,loc}(\Omega)$.

Conversely, if $f\in\mathcal{E}^1_{0,loc}(\Omega)$, then we obtain by the local property of the class of $PSH^-(\Omega)$ that $f\in PSH^-(\Omega)$. Using the definition of $\mathcal{E}^1_{0,loc}(\Omega)$, we have $f\in L^\infty(\Omega)$. Since $f\in PSH^-\cap L^\infty(\Omega)$, the desired conclusion follows. This, by Proposition 2.1 (ii), implies that $\mathcal{E}_0(\Omega)$ is not the local class of type 1. □

Remark 3.1. We set $t_0=\sup\{t\in\mathbb{R}^-:\chi(t)>0\}$. Then $\varphi\in\mathcal{E}_\chi(\Omega)$ for all $\varphi\in PSH^-(\Omega)$ with $\varphi\ge -t_0$. Take $\varphi\in\mathcal{E}_\chi(\Omega)$. By the proof [11, Theorem 3.1], we have
$$\varliminf_{z\to w}\varphi(z)\ge -t_0,\ \forall w\in\partial\Omega.$$

Now we state some local properties of the class $\mathcal{E}_\chi(\Omega)$.

Theorem 3.1. *Let $\chi:\mathbb{R}^-\longrightarrow\mathbb{R}^+$ be a decreasing function and $t_0=\sup\{t\in\mathbb{R}^-:\chi(t)>0\}$. Then*
i) $(\mathcal{E}_\chi)^1_{loc}(\Omega)=\mathcal{E}_{\chi,loc}(\Omega)$ and $(\mathcal{E}_\chi)^2_{loc}(\Omega)=\{\varphi\in PSH^-(\Omega):\varphi\ge -t_0\}$.
ii) $\mathcal{E}_\chi(\Omega)$ is only the local class of type 2.

Proof. i) From the main result in [12], the class $\mathcal{E}_{\chi,loc}(\Omega)$ is the local one, so $\mathcal{E}_{\chi,loc}(\Omega)=(\mathcal{E}_\chi)^1_{loc}(\Omega)$. Take $\varphi\in(\mathcal{E}_\chi)^2_{loc}(\Omega)$ and let $w\in\Omega$. For every $D\Subset\Omega$ such that $w\in\partial D$, we have $\varphi|_D\in\mathcal{E}_\chi(D)$. It follows that
$$\varphi(w)\ge\varlimsup_{z\to w}\varphi(z)\ge\varlimsup_{z\in D,z\to w}\varphi(z)\ge\varliminf_{z\to w}\varphi(z)\ge -t_0.$$
Conversely, let $\varphi\in PSH^-(\Omega)$ be such that $\varphi\ge -t_0$. Let $D\Subset\Omega$. Then $\varphi\in PSH^-(D)$. It follows from [7, Theorem 2.1] that $\varphi\in\mathcal{E}_\chi(D)$. Thus $\varphi\in(\mathcal{E}_\chi)^2_{loc}(\Omega)$.

ii) It follows from i) and Remark 3.1 immediately. □

Corollary 3.1. *$\mathcal{F}(\Omega)$ is only local class of type 2.*

Proof. Assume that χ is bounded and $\chi(0)>0$. Then $\mathcal{E}_\chi(\Omega)=\mathcal{F}(\Omega)$ and
$$t_0=\sup\{t\in\mathbb{R}^-:\chi(t)>0\}=0.$$
Using Theorem 3.1(i), we have $\mathcal{F}^2_{loc}(\Omega)=(\mathcal{E}_\chi)^2_{loc}(\Omega)=\{0\}$. Hence $\mathcal{F}(\Omega)$ is the local class of type 2.

On the other hand, we have $\mathscr{F}^1_{loc}(\Omega)=\mathscr{E}(\Omega)$. By Proposition 2.1(ii), $\mathscr{F}(\Omega)$ is not the local class of type 1. $\square$

Remark 3.2. i) We define $\mathscr{P}$ by a class of functions on open sets in X as follows: $\mathscr{P}(X)=\{0\}$ and $\mathscr{P}(U)=\{0,-1\}$ for all open U of X with $U\neq X$. We can check that $\mathscr{P}^1_{loc}=\mathscr{P}$, $\mathscr{P}^2_{loc}=\{0,-1\}\neq\mathscr{P}$, and $\mathscr{P}$ is only the local class of type 1.
ii) We define $\mathscr{P}$ by a class of functions on open sets in X as follows: $\mathscr{P}(X)=\{0\}$ and $\mathscr{P}(U)=\{-1\}$ for all open U of X with $U\neq X$. We can check that $\mathscr{P}^1_{loc}=\mathscr{P}$, $\mathscr{P}^2_{loc}=\{-1\}$, and $\mathscr{P}$ is not the local class of type 1 and 2.

Now, we are going to investigate the local property on complex Monge - Ampère measure classes. Let $\mathscr{M}(\Omega)$ be the set of all nonnegative Radon measure on Ω. We set

$$\begin{aligned}\mathscr{M}^a(\Omega)&=\{\mu\in\mathscr{M}(\Omega):\mu(E)=0\text{ for all the Borel pluripolar set in }\Omega\},\\ \mathscr{M}_\chi(\Omega)&=\{\mu\in\mathscr{M}(\Omega):\exists\varphi\in\mathscr{E}_\chi(\Omega)\text{ such that }\mu=(dd^c\varphi)^n\},\\ \mathscr{M}_{\chi,loc}(\Omega)&=\{\mu\in\mathscr{M}(\Omega):\forall K\Subset\Omega,\exists\varphi\in\mathscr{E}_{\chi,loc}(\Omega)\text{ such that }\mu|_K=(dd^c\varphi)^n\}.\end{aligned}$$

Remark 3.3. i) We have a characterization for the class $\mathscr{M}_{\chi,loc}(\Omega)$ as follows:

$$\mathscr{M}_{\chi,loc}(\Omega)=\{\mu\in\mathscr{M}(\Omega):\mu|_K\in\mathscr{M}_\chi(\Omega)\text{ for all }K\Subset\Omega\}.$$

Indeed, we take $K\Subset D\Subset\Omega$. There exists $\varphi\in\mathscr{E}_{\chi,loc}(\Omega)$ such that $\mu|_K=(dd^c\varphi)^n$. Setting

$$h^\varphi_{D,\Omega}=\sup\{\psi\in\mathrm{PSH}^-(\Omega):\psi\leq\varphi\text{ on }D\},$$

we see that $h^\varphi_{D,\Omega}\in\mathscr{E}_\chi(\Omega)$ and $(dd^ch^\varphi_{D,\Omega})^n\geq 1_D(dd^c\varphi)^n\geq\mu|_K$. By the main theorem in[2], we can find $u\in\mathscr{E}(\Omega)$ such that $u\geq h^\varphi_{D,\Omega}$ and $\mu|_K=(dd^cu)^n$.
 ii) We can check that $\mathscr{M}_\chi(\Omega)\subset\mathscr{M}_{\chi,loc}(\Omega)$.
 iii) By the main theorem in [2], we have

$$\begin{aligned}\mathscr{M}_\chi(\Omega)&=\{\mu\in\mathscr{M}(\Omega):\exists\varphi\in\mathscr{E}_\chi(\Omega)\text{ such that }\mu\leq(dd^c\varphi)^n\}\\ \mathscr{M}_{\chi,loc}(\Omega)&=\{\mu\in\mathscr{M}(\Omega):\forall K\Subset\Omega,\exists\varphi_K\in\mathscr{E}_{\chi,loc}(\Omega)\\ &\qquad\text{such that }\mu|_K\leq(dd^c\varphi_K)^n\}.\end{aligned}$$

Proposition 3.2. *i) If $\mu,\nu\in\mathscr{M}_\chi(\Omega)$, then $a\mu+b\nu\in\mathscr{M}_\chi(\Omega)$ for all $a,b\geq 0$. ii) If $\mu,\nu\in\mathscr{M}_{\chi,loc}(\Omega)$, then $a\mu+b\nu\in\mathscr{M}_{\chi,loc}(\Omega)$ for all $a,b\geq 0$. iii) If $\mu\in\mathscr{M}(\Omega)$, $\mu\leq\nu$ and $\nu\in\mathscr{M}_\chi(\Omega)$, then $\mu\in\mathscr{M}_\chi(\Omega)$. iv) If $\mu\in\mathscr{M}(\Omega)$, $\mu\leq\nu$ and $\nu\in\mathscr{M}_{\chi,loc}(\Omega)$, then $\mu\in\mathscr{M}_{\chi,loc}(\Omega)$.*

Proof. i) Since $\mu,\nu\in\mathscr{M}_\chi(\Omega)$, then there exists $\varphi,\psi\in\mathscr{E}_\chi(\Omega)$ such that $\mu=(dd^c\varphi)^n$ and $\nu=(dd^c\nu)^n$. For $a,b\geq 0$, we have

$$\begin{aligned}a\mu+b\nu=a(dd^c\varphi)^n+b(dd^c\psi)^n&=\left(dd^c(\sqrt[n]{a}\varphi)\right)^n+\left(dd^c(\sqrt[n]{b}\psi)\right)^n\\ &\leq\left(dd^c(\sqrt[n]{a}\varphi+\sqrt[n]{b}\psi)\right)^n.\end{aligned}$$

From the main theorem in [2], one sees there exists $\phi\in\mathscr{E}(\Omega)$ such that

$$a\mu+b\nu=(dd^c\phi)^n\quad\text{ and }\quad\phi\geq\sqrt[n]{a}\varphi+\sqrt[n]{b}\psi.$$

Since $\phi\geq\sqrt[n]{a}\varphi+\sqrt[n]{b}\psi$, then $\phi\in\mathscr{E}_\chi(\Omega)$.
 ii) By using arguments similar to i), we have ii) directly.

iii) Since $\nu \in \mathscr{M}_\chi(\Omega)$, then there exists $\psi \in \mathscr{E}_\chi(\Omega)$ such that $\nu = (dd^c\psi)^n$. Thus $\mu \leq (dd^c\psi)^n$. By the main theorem in [2], there exists $\varphi \in \mathscr{E}(\Omega)$ such that $\mu = (dd^c\varphi)^n$ and $\varphi \geq \psi$. Since $\varphi \geq \psi$, then $\varphi \in \mathscr{E}_\chi(\Omega)$.

iv) By using arguments similar to iii), we obtain iv) immediately. □

Theorem 3.2. *$\mathscr{M}_{\chi,loc}(\Omega)$ have local property.*

Proof. • Take $\mu \in \mathscr{M}_{\chi,loc}(\Omega)$. We next prove that $\mu|_D \in \mathscr{M}_{\chi,loc}(D)$ for all $D \Subset \Omega$. Indeed, let $D' \Subset D$. Since $\mu \in \mathscr{M}_{\chi,loc}(\Omega)$, then there exists $\varphi \in \mathscr{E}_\chi(\Omega)$ such that $\mu|_D = (dd^c\varphi)^n$. Set

$$\varphi_{D'} = \sup\{\psi \in \mathrm{PSH}^-(D) : \psi \leq \varphi \text{ on } D'\}.$$

Then $\varphi_{D'} \in \mathscr{E}_\chi(D)$, $\varphi_{D'} = \varphi$ on D', and $\mu|_{D'} = (dd^c\varphi_{D'})^n$.

• Let $\Omega_j,\ j \in I$ be the sets such that $\Omega_j \Subset \Omega$ and $\Omega = \cup_{j\in I}\Omega_j$. Take $\mu \in \mathscr{M}(\Omega)$ such that $\mu|_{\Omega_j} \in \mathscr{M}_{\chi,loc}(\Omega_j)$ and let $D_j \Subset \Omega_j$ be the sets $\mu|_{D_j} = (dd^c\varphi_j)^n$ for some $\varphi_j \in \mathscr{E}_\chi(\Omega_j)$ and $\cup_{j\in I}D_j = \Omega$. For all $K \Subset \Omega$, we can find $\{j_1,...,j_k\} \subset I$ such that $K \subset D_{j_1} \cup ... \cup D_{j_k}$. Since $\mathscr{E}_\chi(\Omega_j)$ has subextension on Ω, then

$$\psi_j := \sup\{\psi \in \mathrm{PSH}^-(\Omega) : \psi \leq \varphi_j \text{ on } \Omega_j\} \in \mathscr{E}_\chi(\Omega),\ \forall j \in I.$$

Thus

$$\mu|_K \leq \sum_{i=1}^k \mu|_{D_{j_i}} \leq \sum_{i=1}^k (dd^c\psi_{j_i})^n \leq \left(dd^c\left(\sum_{i=1}^k \psi_{j_i}\right)\right)^n$$

By the main theorem in [2], one see that there exists $\psi \in \mathscr{E}(\Omega)$ such that

$$\mu|_K = (dd^c\psi)^n \quad \text{and} \quad \psi \geq \psi_{j_1} + ... + \psi_{j_k}.$$

Since $\psi \geq \psi_{j_1} + ... + \psi_{j_k}$, then $\psi \in \mathscr{E}_\chi(\Omega)$. □

Theorem 3.3. *Let $\mu \in \mathscr{M}(\Omega)$ be a measure such that $\int_\Omega -hd\mu < +\infty$ for some $h \in PSH^-(\Omega)$. Assume that, for all $z \in \Omega$, there exists a neighbourhood U of z such that $\mu|_U \in \mathscr{M}_{\chi,loc}(U)$. Then there exists $\varphi \in \mathscr{E}_{\chi,loc}(\Omega)$ such that $\mu = (dd^c\varphi)^n$.*

Proof. By Proposition 3.2, we have $\mu \in \mathscr{M}_{\chi,loc}(\Omega)$. Let

$$\Omega_1 \Subset \Omega_2 \Subset ... \Subset \Omega, \text{ and } \cup_{j=1}^\infty \Omega_j = \Omega.$$

Then there exists $\varphi_j \in \mathscr{E}_\chi(\Omega_j), j = 1,2,\cdots$ such that $\mu|_{\Omega_j} = (dd^c\varphi_j)^n,\ j = 1,2,\cdots$. Since $\mathscr{E}_\chi(\Omega) \subset \mathscr{N}(\Omega)$ and $\mu|_{\Omega_j} \leq \mu|_{\Omega_{j+1}}$, then $\varphi_1 \geq \varphi_2 \geq ...$ on Ω. This infer that $\varphi_j \searrow \varphi$, with $\varphi \in \mathrm{PSH}^-(\Omega)$ or $\varphi \equiv -\infty$. For $D \Subset \Omega$, we set

$$h^{\varphi_j}_{\Omega_j,\Omega} = \sup\{\psi \in \mathrm{PSH}^-(\Omega) : \psi \leq \varphi_j \text{ on } \Omega_j\}.$$

Then we have $h^{\varphi_j}_{\Omega_j,\Omega} \in \mathscr{F}(\Omega) \cap \mathscr{E}_\chi(\Omega)$ and

$$h^{\varphi_j}_{\Omega_j,\Omega} \searrow h^{\varphi}_{D,\Omega} = \sup\{\psi \in \mathrm{PSH}^-(\Omega) : \psi \leq \varphi \text{ on } D\}.$$

For all $h \in \mathrm{PSH}^-(\Omega)$, we have

$$\int_\Omega -h(dd^c h^{\varphi_j}_{\Omega_j,\Omega})^n \leq \int_\Omega -h(dd^c\varphi_j)^n = \int_\Omega -hd\mu|_{\Omega_j} = \int_{\Omega_j} -hd\mu.$$

This infers that

$$\sup_{j\geq 1} \int_\Omega -h(dd^c h^{\varphi_j}_{\Omega_j,\Omega})^n \leq \int_\Omega -hd\mu < +\infty,$$

so

$$\sup_{j\geq 1}\int_\Omega (dd^c h^{\varphi_j}_{\Omega_j,\Omega})^n \leq \frac{\int_\Omega -hd\mu}{-\sup_D h} < +\infty.$$

Hence $h^{\varphi_j}_{\Omega_j,\Omega} \searrow h^{\varphi}_{D,\Omega} \in \mathscr{F}(\Omega)$. This derives $\varphi \in \mathscr{E}(\Omega)$. Now, by Lemma 3.3 and the proof of [12, Theorem 3.4 (ii)], we have $h^{\varphi}_{D,\Omega} \in \mathscr{E}_\chi(\Omega)$. This derives $\varphi \in \mathscr{E}_{\chi,loc}(\Omega)$. □

Remark 3.4. The condition

$$\int_\Omega -hd\mu < +\infty, \qquad \text{for some } h \in \mathrm{PSH}^-(\Omega)$$

in the Theorem 3.3 is sharp.

Proof. Let $\Omega_1 \Subset \Omega_2 \Subset ... \Subset \Omega$ and $\Omega = \cup_{j=1}^\infty \Omega_j$. Set

$$h_{\Omega_j,\Omega} = \sup\{\psi \in \mathrm{PSH}^-(\Omega) : \psi \leq -1 \ \text{ on } \ \Omega_j\}.$$

Then $\mathrm{supp}(dd^c h_{\Omega_j,\Omega}) \subset \partial\Omega_j$. Setting $\mu = \sum_{j=1}^\infty (dd^c h_{\Omega_j,\Omega})^n$, one has $\mathrm{supp}\mu \subset \cup_{j=1}^\infty \partial\Omega_j$. Taking $K \Subset \Omega$, one sees that there exists $k \geq 1$ such that $K \Subset \Omega_j$ for all $j \geq k$. Thus

$$\mu|_K = \sum_{j=1}^k (dd^c h_{\Omega_j,\Omega})^n \leq \left(dd^c\left(\sum_{j=1}^k h_{\Omega_j,\Omega}\right)\right)^n,$$

which implies that $\mu \in \mathscr{M}_{\chi,loc}(\Omega)$.

Now, we are in a position to prove that there does not exist $\varphi \in \mathscr{E}(\Omega)$ such that $(dd^c\varphi)^n = \mu$. Indeed, we assume that there exists $\varphi \in \mathscr{E}(\Omega)$ such that $\mu = (dd^c\varphi)^n$. Since $(dd^c\varphi)^n \geq (dd^c h_{\Omega_j,\Omega})^n$ for all $j \geq 1$, we have $\varphi \leq h_{\Omega_j,\Omega}$ on Ω for all $j \geq 1$. This infers that $\varphi \leq -1$ on Ω. Thus $u+1 \in \mathrm{PSH}^-(\Omega)$ and $\mu = (dd^c(\varphi+1))^n$. Repeating above arguments, we obtain $\varphi \leq -k$ for all $k \geq 1$. This derives $\varphi \equiv -\infty$ on Ω. □

4. Pluripolar Sets in the Class $\mathscr{E}_\chi(\Omega)$

In [7], the author proved that for every pluripolar set E in Ω we can find $u \in \mathscr{F}_1(\Omega)$ such that $E = \{u = -\infty\}$. In [13], the author used the ideas given in [6] to state the pluripolar sets in the class $\mathscr{F}_\infty$. Here, we establish the pluripolar sets in the class $\mathscr{E}_\chi(\Omega)$.

Theorem 4.1. *Let $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ be a decreasing function ($\chi \not\equiv 0$). Let E be a complete pluripolar set in Ω. Then, there exists a function $\varphi \in \mathscr{E}_\chi(\Omega)$ such that $E = \{\varphi = -\infty\}$.*

Proof. We can choose a convex, strictly decreasing, and smooth function $\theta : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ such that $\theta(t) \geq \chi(t) + |t|$ for all $t \in \mathbb{R}^-$. By [13, Theorem 3.1], we can find a function $u \in \mathscr{F}_\infty(\Omega)$ such that $E = \{u = -\infty\}$. We set $\varphi = \max((-\theta)^{-1} \circ [u - \theta(0)], u)$. Then $\varphi \in \mathscr{F}_\infty(\Omega)$ and $E = \{\varphi = -\infty\}$. Now, we only need to prove that $\varphi \in \mathscr{E}_\chi(\Omega)$. Indeed, by [7, Theorem 2.1], we can find $u_j \in \mathscr{E}_0(\Omega)$ such that $u_j \searrow u$. Set $\varphi_j = \max((-\theta)^{-1} \circ [u_j - \theta(0)], u_j)$. Then $\varphi_j \in \mathscr{E}_0(\Omega)$ and $\varphi_j \searrow v$. Using [2, Lemma 3.3], we have

$$\int_\Omega \chi(\varphi_j)(dd^c\varphi_j)^n \leq \int_\Omega [-u_j + \theta(0)](dd^c\varphi_j)^n \leq \int_\Omega [-u_j + \theta(0)](dd^c u_j)^n.$$

Thus we deduce that

$$\sup_{j\geq 1}\int_\Omega \chi(\varphi_j)(dd^c\varphi_j)^n \leq \sup_{j\geq 1}\int_\Omega [-u_j + \theta(0)](dd^c u_j)^n < +\infty.$$

□

Using Theorem 4.1, we can describe more clear the range of the class $\mathscr{E}_\chi(\Omega)$ as follows.

Theorem 4.2. *Let $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ be a decreasing function such that $\chi(t) > 0$ for all $t \in \mathbb{R}^-$. Then $\mathscr{E}_\chi(\Omega) \subset \overline{\mathscr{B}}_0(\Omega)$ and $\mathscr{E}_\chi(\Omega) \not\subset \mathscr{B}_0(\Omega)$, where*

$$\mathscr{B}_0(\Omega) = \{\varphi \in PSH^-(\Omega) : \lim_{z\to w} \varphi(z) = 0 \ \forall w \in \partial\Omega\}$$

and

$$\overline{\mathscr{B}}_0(\Omega) = \{\varphi \in PSH^-(\Omega) : \overline{\lim_{z\to w}}\, \varphi(z) = 0 \ \forall w \in \partial\Omega\}.$$

Proof. Let $\{D_j\}_{j\geq 1}$ be a sequence of subsets in Ω such that $D_j \Subset D_{j+1}$ and $\Omega = \cup_{j\geq 1} D_j$. Take $\varphi \in \mathscr{E}_\chi(\Omega)$. Using [11, Theorem 3.1], we have $h^{\varphi}_{D_j,\Omega} \nearrow 0$ as $j \to \infty$ a.e in dV_{2n}, which implies that $\overline{\lim_{z\to w}}\, \varphi(z) = 0$ for all $w \in \partial\Omega$. Hence $\varphi \in \overline{\mathscr{B}}_0(\Omega)$.

To show that $\mathscr{E}_\chi(\Omega) \not\subset \mathscr{B}_0(\Omega)$, we Set $E = \Omega \cap \mathbb{Q}^n$. Since E is a pluripolar set, we find from Theorem 4.1 a function $\varphi \in \mathscr{E}_\chi(\Omega)$ and $E \subset \{z \in \Omega : \varphi(z) = -\infty\}$. Thus

$$\underline{\lim_{z\to w}}\, \varphi(z) \leq \underline{\lim_{z\in\Omega\cap\mathbb{Q}^n, z\to w}}\, \varphi(z) = -\infty.$$

for all $w \in \partial\Omega$. □

Acknowledgments

I am grateful to Professor Pham Hoang Hiep for valuable comments during the preparation of this work. The author is also indebted to the referee for carefully reading this paper.

References

[1] P. Åhag, A Dirichlet problem for the complex Monge-Ampère operator in $\mathscr{F}(f)$, Michigan Math. J. 55 (2007), 123-138.
[2] P. Åhag, U. Cegrell, R. Czyz, P.H. Hiep, Monge-Ampère measures on pluripolar sets, J. Math. Pures Appl. 92 (2009), 613-627.
[3] S. Benelkourchi, Weighted Pluricomplex Energy, Potential Anal. 31 (2009), 1-20.
[4] S. Benelkourchi, V.Guedj, A.Zeriahi, Plurisubharmonic functions with weak singularities, In: Proceedings from the Kiselmanfest, pp. 57–74, Uppsala University, Västra Aros, 2009.
[5] Z. Blocki, The domain of definition of the complex Monge-Ampère operator, Amer. J. Math. 128 (2006), 519-530.
[6] U. Cegrell, Pluricomplex energy, Acta. Math. 180 (1998), 187-217.
[7] U. Cegrell, The general definition of the complex Monge-Ampère operator, Ann. Inst. Fourier (Grenoble) 54 (2004), 159-179.
[8] U. Cegrell, A general Dirichlet problem for the complex Monge-Ampère operator, Ann. Polon. Math. 94 (2008), 131-147.
[9] L. H. Chinh, On Cegrell's classes of m-subharmonic functions, arXiv 1301.6502v1, 2013.
[10] N. N. Cuong, Subsolution theorem for the complex Hessian equation, Universitatis Iagellonicae Acta Math. 50 (2012), 69-88.
[11] L. M. Hai, P. H. Hiep, Some Weighted Energy Classes of Plurisubharmonic Functions, Potential Anal. 34 (2011), 43-56.
[12] L. M. Hai, P. H. Hiep, H. N. Quy, Local property of the class $\mathscr{E}_{\chi,loc}$, J. Math. Anal. Appli. 402 (2013), 440-445.

[13] P. H. Hiep, Pluripolar sets and the subextension in Cegrell's classes, Complex Var. Elliptic Equations 53 (2008), 675-684.
[14] V. V. Hung and N. V. Phu, Hessian measures on m-polar sets and applications to the complex Hessian equations, Com. Var. Elliptic Equations, 62(2017), 1135-1164.
[15] J. B. Walsh, Continuity of envelopes of plurisubharmonic functions, J. Math. Mech. 18 (1968), 143-148.